VII. Surfaces and Dehn Fillings

The main question we want to address is

> When can essential surfaces be created or distroyed by Dehn surgery?

Th^{m} 1:

Suppose $T^2 \subset M^3$ is a torus in a 3-manifold, K is a knot on T^2 and $\mathcal{F}$ is the framing on K coming from T^2

1) if T^2 is separating, so $M \setminus T^2 = M_1' \cup M_2'$, then

$$M_K(\mathcal{F}) = M_1 \# M_2$$

where $M_i = M_i' \cup \underbrace{\text{2-handle} \cup \text{3-handle}}_{\text{solid torus}}$

and attaching sphere of 2-handle is $K \subset \partial M_i'$

2) if T^2 is non-separating, so $M \setminus T^2 = \hat{M}$, then

$$M_K(\mathcal{F}) = M' \# S^1 \times S^2$$

where $M' = \hat{M} \cup \underbrace{\text{2-h} \cup \text{3-h}}_{} \cup \underbrace{\text{2-h} \cup \text{3-h}}_{}$ solid tori

and attaching spheres of 2-handles is K on each boundary component of $\hat{M}$

exercise:

What is $M_K(\mathcal{F}+\frac{1}{n})$?

example:

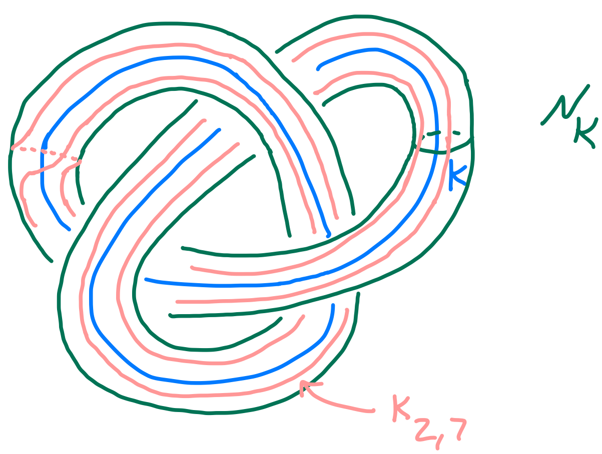
given a knot $K \subset M$

let $N(K)$ be a neighborhood of K

choose the longitude-meridian basis for $H_1(\partial N)$

so curves on $\partial N(K)$ correspond to an elt of $\mathbb{Q} \cup \{\infty\}$

the <u>(p,q)-cable of K</u> is the curve $K_{p,q}$ on $\partial N(K)$ realizing the homology class $p\lambda + q\mu$



exercise:

If K is null-homologous (so has a Seifert framing) the the framing on $K_{p,q}$ given by $\partial N(K)$ is pq

so from Th$^{\underline{m}}$ 1 we see

$$M_{K_{p,q}}(pq) = M_K(q/p) \# -L(p,q)$$

this is because, $M \setminus \partial N(K) = (M - N(K)) \cup N(K)$

so we Dehn fill $M - N(K)$ and $N(K) = S^3 - N(U)$

note: if $K \subset S^3$ then we see a surgery on $K_{(p,q)}$ gives a reducible manifold!

i.e. [an essential 2-sphere is created by surgery!]

Conjecture:

(Cabling conjeture of González-Acuña and Short)

If $K \subset S^3$ and $S^3_K(r)$ is a non-trivial connected sum, then K is a (p,q)-cable of some knot and $r = pq$

Gordon-Lueche, 1987:

if $S^3_K(r)$ is a non-trivial connected sum then $r \in \mathbb{Z}$ and one of the summands is a lens space

Greene, 2015:

if $S^3_K(r)$ is a connect sum of lens spaces then i) K is either a (p,q)-torus knot or a (p,q)-cable of an (r,s)-torus knot with $p = qrs \pm 1$

2) $r = pq$

3) $S^3_K(r) = -(L(p,q) \# L(q,p))$ or $-(L(p,ps^2) \# L(q,\pm 1))$

respectively

Remark: The cabling conjecture is true for

1) alternating knots Menasco-Thistlethwaite, 1992

2) satellite knots Scharlemann, 1990

(and other families)

so Cabling conjecture can be formulated as "Surgery on a hyperbolic knot is irreducible"

exercise:

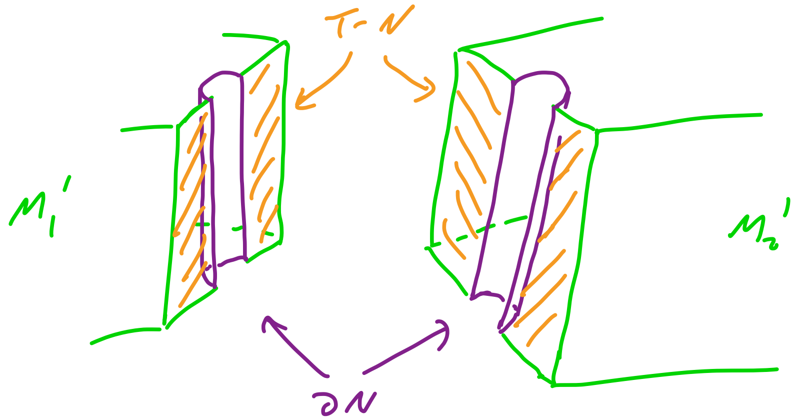
Show $M^3_{K_{p,q}}(pq \pm 1) = M^3_K\left(\frac{pq \mp 1}{p^2}\right)$

Proof of Th$^{\underline{m}}$ 1:

let N = nbhd of K

glue $D^2 \times \{0\}$ to $D^2 \times \{2\}$ and both $D^2 \times \{1\}$'s

$$M_{\mathcal{F}}(K) = \overline{M - N} \cup S^1 \times D^2/_\sim$$
$$= \left\{ \left[\overline{M_1' - N} \cup \overline{M_2' - N} \right] \cup \left[[0,1] \times D^2 \cup [1,2] \times D^2 \right] \right\} /$$



now $T-N$ = annulus A

meridional disks in $S^1\times D^2$ glued to ∂A $\quad S^1\times D^2 = [0,2]\times D^2/\sim$

let $D_1=\{0\}\times D^2$ $\quad D_2=\{1\}\times D^2 \subset [0,2]\times D^2/\sim$

$A\cup D_1\cup D_2 = S^2$

and S^2 splits $M_{\mathcal{F}}(K)$ into 2 pieces

one is $\overline{M_1'-N}\cup[0,1]\times D^2$

where $[0,1]\times D^2$ is glued along $[0,1]\times\partial D^2$

i.e. as a 2-handle and $\{1/2\}\times\partial D^2$ is

glued to $\partial(\overline{M_1'-N})$ along $K\subset T$

now $\partial[(\overline{M_1'-N})\cup[0,1]\times D^2/\sim] = S^2$

glue in B^3 to get M_1

similarly get M_2 and $M_{\mathcal{F}}(K) = M_1 \# M_2$

<u>exercise</u>: prove part 2

let T be a torus

the distance between two slopes r_1, r_2 on T is

$$\Delta(r_1, r_2) = |\gamma_1 \cdot \gamma_2|$$

where γ_i is a simple closed curve on T representing r_i

exercise:

If $r_i = a_i/b_i$, then $\Delta(r_1, r_2) = |a_1 b_2 - a_2 b_1|$

Thm 2 (Gordon-Litherland 1984):

let K be a knot in a 3-manifold with $M \setminus K$ irreducible.
If $M_K(r)$ and $M_K(s)$ are reducible, then

$$\Delta(r, s) \leq 4$$

Corollary 3:

if K and M as above, then there are at most 6 distinct r such that $M_r(K)$ is reducible

Remarks:

1) Gordon-Luecke 1995:

improved Thm 2 to $\Delta(r, s) \leq 1$

$\therefore$ at most 3 reducible surgeries

2) this bound is optimal:

let $K_0 = K_1 \# K_2$ in $M = M_1 \# M_2$

where K_i are notrivial knots in non-simply connected irreducible homology spheres M_i

let K = (p,q)-cable of K_0

exercise: $M - K$ is irreducible

note: $M_K(\infty) = M = M_1 \# M_2$

$M_K(pq) = M_{K_0}(q/p) \# -L(p,q)$

and $\Delta(\infty, pq) = \left| \frac{1}{0} \cdot \frac{pq}{1} \right| = 1$

there are non-cable examples, but harder to describe

question: is there a $K \subset M$ with $M \setminus K$ irreducible s.t. there are 3 reducible surgeries?

Proof of Cor 3:

let $\mathcal{S}$ be a set of slopes on T with $\Delta(r,s) \le n \ \forall r,s \in \mathcal{S}$

Claim: we can choose coordinates on T such that

$$\mathcal{S} \subset \mathcal{S}_n$$

where $\mathcal{S}_n = \{ a/b : 0 \le a < b \le 1 \} \cup \{\infty\}$

given this note

$\mathcal{S}_4 = \{ \frac{0}{1}, \frac{1}{1}, \cancel{\frac{0}{2}}, \frac{1}{2}, \cancel{\frac{0}{3}}, \frac{1}{3}, \frac{2}{3}, \cancel{\frac{0}{4}}, \frac{1}{4}, \cancel{\frac{2}{4}}, \frac{3}{4}, \infty \}$

has 8 elements

but $\Delta(\frac{1}{3}, \frac{3}{4}), \Delta(\frac{2}{3}, \frac{1}{4}), \Delta(\frac{1}{4}, \frac{3}{4}) > 4$

so $\mathcal{S} \subset \mathcal{S}_4$ must omit at least 2 of $1/3, 2/3, 1/4, 3/4$

$\therefore$ corollary true!

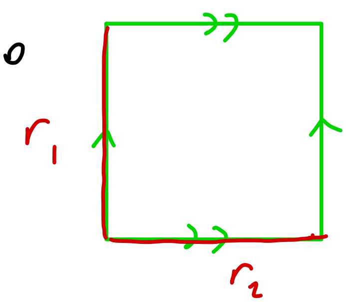
for the claim consider the $n=1$ case

given $r_1, r_2 \in \mathcal{S}$

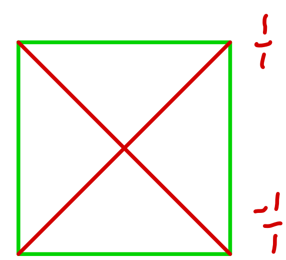
exercise: $\exists$ coordinates on T s.t. $r_1 = \frac{0}{1}$, $r_2 = \frac{1}{0}$

(i.e. r_1, r_2 form a basis for $H_1(T)$)

so



the only other curves that intersect r_1, r_2 one time are $\frac{1}{1}, \frac{-1}{1}$



but $\Delta(\frac{1}{1}, \frac{-1}{1}) = 2$ so can only have one of these

if $\frac{1}{1} \in \mathcal{S}$ then $\mathcal{S} \subseteq \{ a/b : 0 \le a \le b \le 1 \} \cup \{\infty\}$

if $r_3 = \frac{-1}{1} \in \mathcal{S}$, then note $\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so if we take r_2 and r_3 as a basis for $H_1(T)$ then

$$r_1 = 1/1 \quad r_2 = 1/0 \quad r_3 = 0/1$$

so again $\mathcal{S} \subseteq \mathcal{S}_1$

now consider $n \geq 2$ choose $r_1, r_2 \in \mathcal{S}$ with $\Delta(r_1, r_2) = n$

exercise: there are coordinates on T s.t.

$$r_1 = \frac{1}{0} \text{ and } r_2 = \frac{a}{b} \text{ with}$$

$$0 \leq a < b$$

Hint: r_1 is a basis vector for $H_1(T)$ there are lots of choices for r_1' s.t. r_1, r_1' is a basis, choose the right r_1'.

now $\Delta(r_1, r_2) = 1 \cdot b - 0 \cdot a = n$ so $b = n$

(and $a > 0$ since $n \geq 2$)

if $r_3 = \frac{c}{d} \in \mathcal{S}$ then we can assume $d \geq 0$

and so $\Delta(r_1, r_3) = d \leq n$

$$\Delta(r_2, r_3) = |ad - nc| \leq n$$

$$\therefore \; ad - nc \leq n \quad \text{and} \quad nc - ad \leq n$$

positive

so $-c \leq \frac{n-ad}{n} \Rightarrow c \geq \frac{ad}{n} - 1 > -1$

and $c \leq 1 + \frac{ad}{n} < d+1$

$\therefore \quad 0 \leq c \leq d \leq n$ and $r_3 \in \mathcal{S}_n$ $\boxplus$

now let's give a slick generalization of the corollary

lemma (Agol 2000):

> let $\mathcal{S}$ be a collection of slopes on T^2 with distance bounded by n. let p be any prime greater than n. Then $|\mathcal{S}| \leq p+1$

Proof:

fix a basis for T^2 so slopes correspond to pairs of relatively prime integers, up to sign.

each slope $\pm(a,b)$ gives a point in the projective line $P\mathbb{F}_p^1$ over the field $\mathbb{F}_p$ by sending

$$\begin{array}{ccc} \mathbb{Z}^2/\pm & \longrightarrow & P\mathbb{F}_p^1 \\ \Psi & & \\ \pm(a,b) & \longmapsto & [a:b] \bmod p \end{array}$$

given $(a,b), (c,d) \in \mathcal{S}$ distinct points we know

$$0 < \left| \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right| < p$$

and so they map to distinct points of $\mathbb{P}\mathbb{F}_p^1$

<u>exercise:</u> Show $|\mathbb{P}\mathbb{F}_p^1| = p+1$

so $|\mathcal{S}| \leq p+1$ ▦

Suppose M and K are as in Th$^{\underline{m}}$ 2

if $M_K(r)$ is reducible, then the is an essential embedded $S^2 \subset M_K(r)$

this S^2 must intersect the surgery torus (or else $M_K = M - N(K)$ would be reducible)

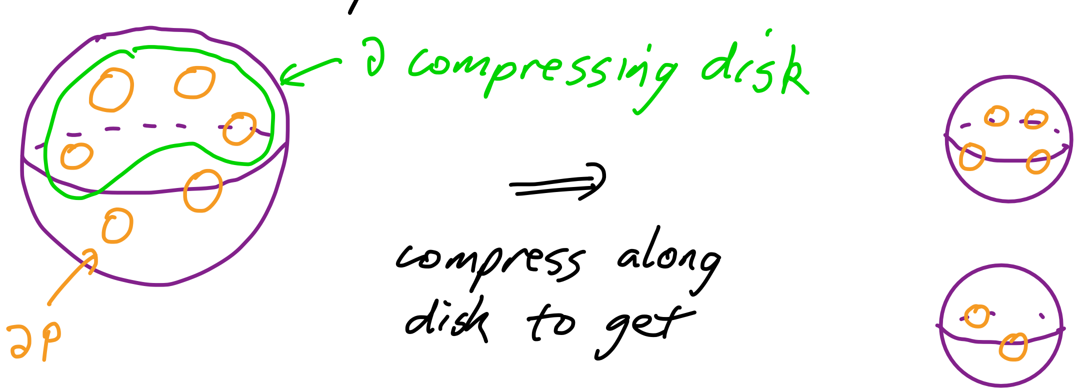
let $P = S^2 \cap M_K$

assume S^2 intersected the surgery torus minimally

<u>lemma 4:</u>

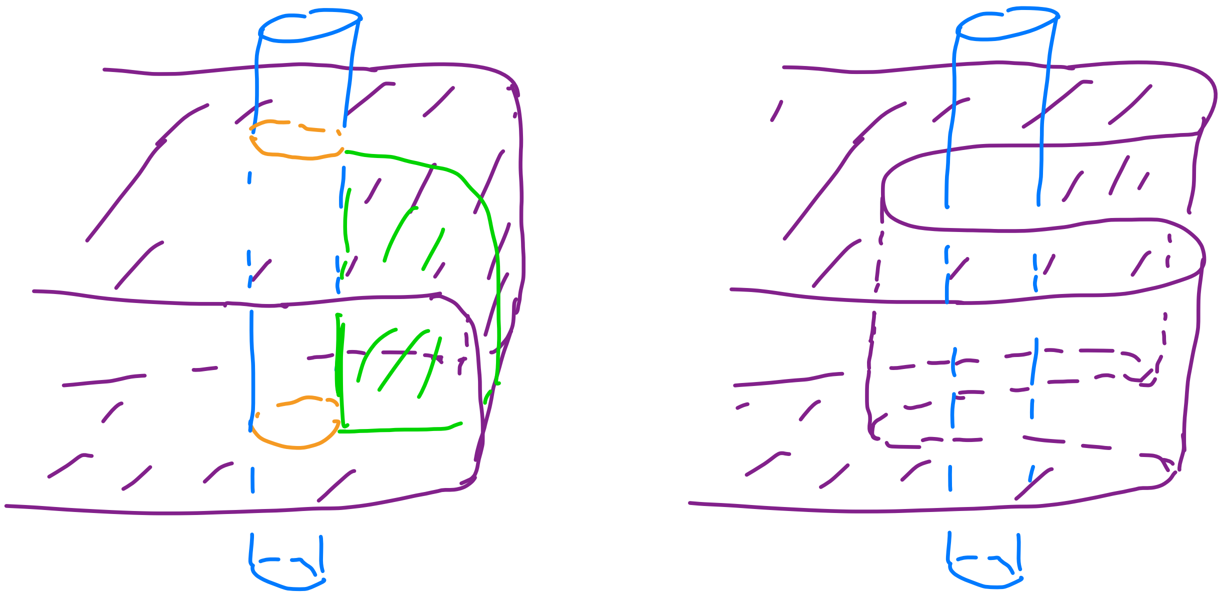
$(P, \partial P) \subset (M_K, \partial M_K)$ is an incompressible and boundary incompressible surface

<u>Proof:</u> if P is compressible we see



this contradicts minimality of intersections with surgery torus!

if P is boundary compressible we see



so we again find an S^2 with fewer intersections

$\therefore$ P is incompressible and boundary incompressible

so $\boxed{\text{essential } S^2\text{'s in } M_K(r) \text{ yield incompressible, boundary incompressible planar surfaces in } M_K}$

<u>note</u>: all components of ∂P have the same slope on ∂M_K

call the slope of one of these the <u>boundary slope</u> of P

let $P\mathcal{S}(M_K, \partial M_K)$ = set of boundary slopes of incomp., ∂-incomp. planar surfaces in M_K

Th$^{\underline{m}}$ 5:

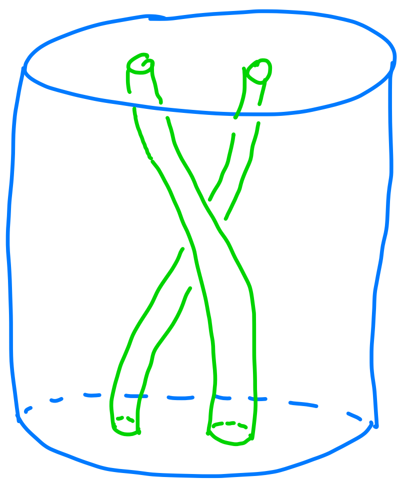
$$\boxed{\text{If } M \text{ is oriented and } T \text{ is a component of } \partial M \text{ then } \forall\, r,s \in \mathcal{PS}(M,T),\ \Delta(r,s) \le 4}$$

clearly Th$^{\underline{m}}$ 2 follows from lemma 4 and Th$^{\underline{m}}$ 5

let V be a solid torus and $V' \subset V$ a sub torus (with V' isotopic to V)

let $K_{p,q}$ be a q/p curve on V' $p \geq 2$

let $C_{p,q} = \overline{V - nbhd(K_{p,q})}$



$C_{2,1}$

$C_{p,q}$ is called the standard (p,q)-cable space

exercise:

1) $C_{p,q}$ is a Seifert fibered space over an annulus with one singular fiber of order p

2) $C_{p,q}$ is homeomorphic to $C_{p',q'}$

$$\Longleftrightarrow$$

$$p = p' \text{ and } q \equiv \pm q' \bmod p$$

if T is a torus boundary component of M

then we say (M,T) is <u>(p,q) cabled</u> if

M contains a submanifold C homeomorphic

to $C_{p,q}$ such that $C \cap \partial M = T$

<u>lemma 6:</u>

> if there are $r,s \in \mathcal{PS}(M,T)$ s.t. $\Delta(r,s) \geq 5$
>
> then (M,T) is cabled

<u>lemma 7:</u>

> if (M,T) is cabled, then $\Delta(r,s) \leq 1$ for
>
> all $r,s \in \mathcal{PS}(M,T)$

Clearly $\text{Th}^{\underline{m}}5$ (and hence $\text{Th}^{\underline{m}}2$) follow from 6 and 7

<u>Proof of lemma 6:</u>

let M be an oriented 3-manifold and $T^2 \subset \partial M$

let $(P_i, \partial P_i) \subset (M,T)$ be an incompressible

∂-incompressible planar surface for $i = 0,1$

with boundary slope r_i for P_i

(assume P_i connected)

isotop so 1) P_0 is transverse to P_1

2) each component of ∂P_0 intersects

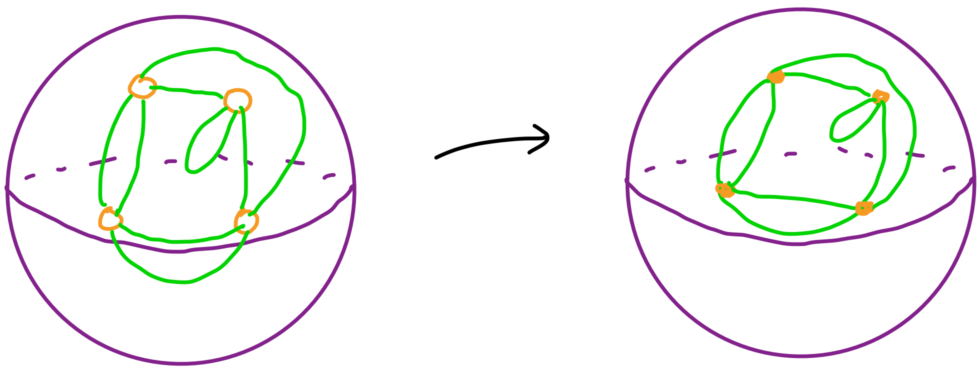
each component of ∂P_1, $\Delta(r_0, r_1)$ times

so $P_0 \cap P_1 = A \amalg S$

where A = disjoint union of properly embedded arcs

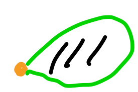
S = disjoint union of embedded circles

to each P_i we get a graph Γ_i in S^2 by looking at A in P_i and collapsing ∂-components of P_i



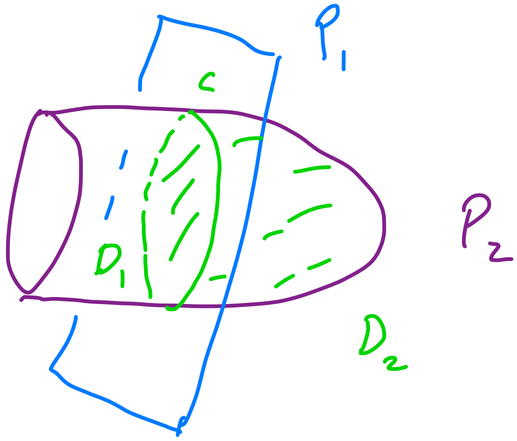
lemma 8:

we can assume

1. no component of $P_0 \cap P_1$ bounds a disk in P_i
2. no edge in Γ_i is an edge of a 1-gone i.e. no 

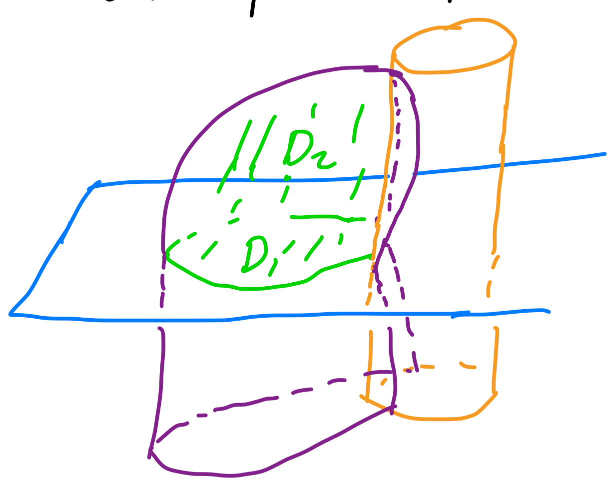
Proof:

1. if $c \subset P_0 \cap P_1$ bounds a disk D_0 in P_0 then since P_1 is incompressible it bounds a disk D_1 in P_1 too
we can replace D_1 in P_1 by D_0 and then push P_1 off P_0 to remove c



② if a is an arc in P_0 bounding a 1-gon D_0 in P_0 then D_0 is a ∂-compressing disk for P_1

since P_1 is ∂-incompressible a bounds a disk D_1 in P_1



we can again swap D_1 in P_1 for D_0 to elliminate the arc a. ▦

let n_i = number of boundary components of P_i

so Γ_i has n_i verticies

and each vertex has valance Δn_{i+1}

where $\Delta = \Delta(r_0, r_1)$, ($i+1$ is taken mod 2)

note: can assume $n_i > 1$ since if not

then P_i is a disk

$\therefore$ Γ_i has no edges (if there were any edges there would be a 1-gon)

and hence $\Delta = 0$ and $r_0 = r_1$

lemma 9:

let Γ be a graph in S^2 with no 1-gons

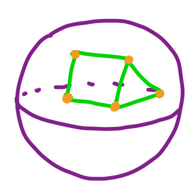
suppose that for some $n \geq 2$ every vertex has order greater than $5(n-1)$

then Γ has n mutually parallel edges

Proof: one can assume Γ is connected (add edges)

consider $n = 2$

assume no parallel edges



so each face has at least 3 sides, so

$$E \geq \frac{3F}{2}$$

(valence $> 1 \Rightarrow$ each edge touches 2 faces)

where $V = \#$ vertices

$E = \#$ edges

$F = \#$ faces

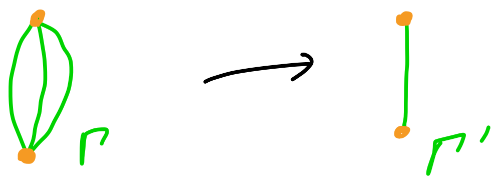
all verticies have ≥ 6 edges touching them so

$$E \geq \frac{6V}{2}$$

so $2 = V - E + F \leq \frac{1}{3}E - E - \frac{2}{3}E = 0$ $\cancel{\circ}$

$\therefore$ there are parallel edges

in general, form Γ' by identifying parallel edges



so Γ' has no parallel edges and no 1-gons

$\therefore$ by above some vertex v has ≤ 5 edges

but v has valance $> 5(n-1)$ in Γ

so v in Γ must have at least n mutually parallel edges

now assume $\Delta \geq 5$

so each vertex of Γ_i has order

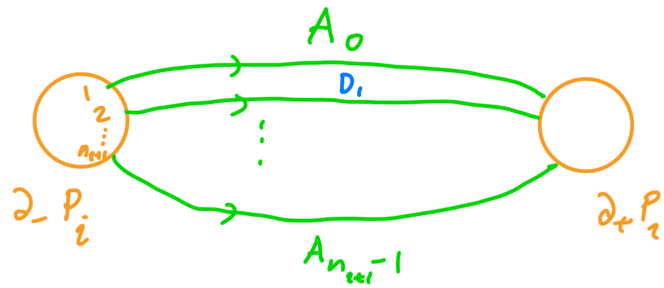
$$\Delta n_{i+1} \geq 5 n_{i+1} > 5(n_{i+1} - 1)$$

$\therefore$ Γ_i has n_{i+1} mutually parallel edges

denote the parallel edges $A_0, \ldots, A_{n_{i+1}-1}$

where A_j and A_{j+1} cobound a disk D_j

note: D_j are disjoint from S ← circles in $P_1 \cap P_2$

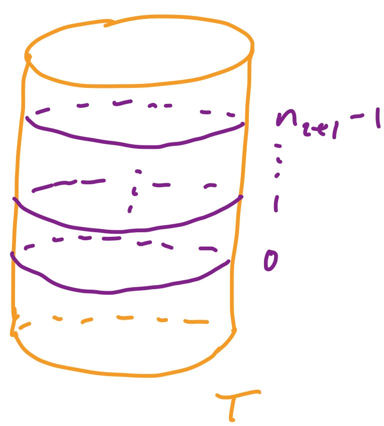


note: all oriented in same direction

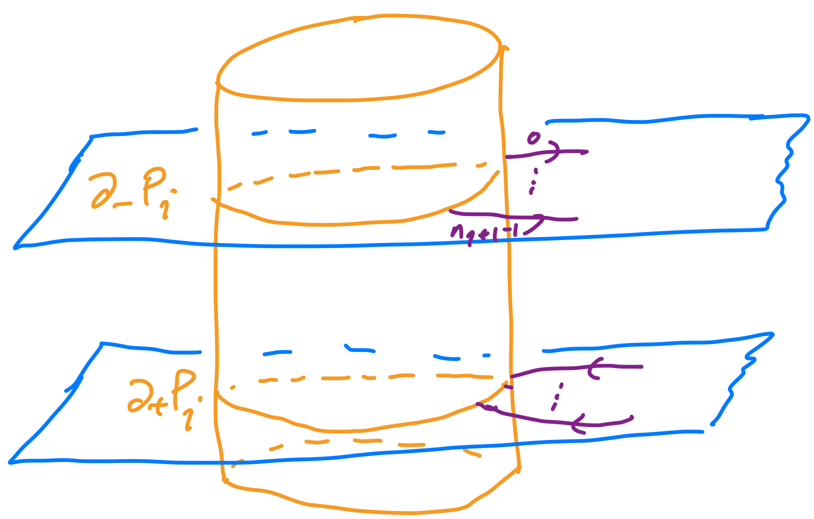
the A_j are oriented and go from one ∂ component $\partial_- P_i$ to another $\partial_+ P_i$

denote $\partial_\pm A_j \in \partial_\pm P_i$

label components of ∂P_{i+1} cyclically along T



so we see

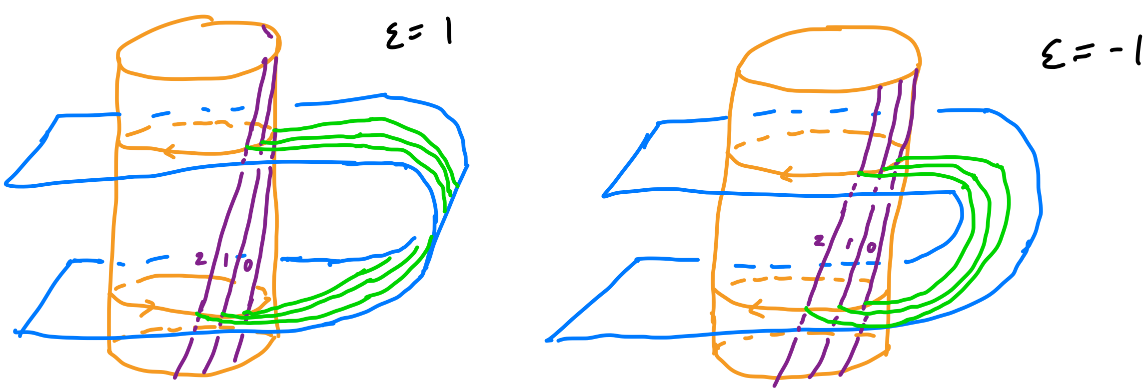


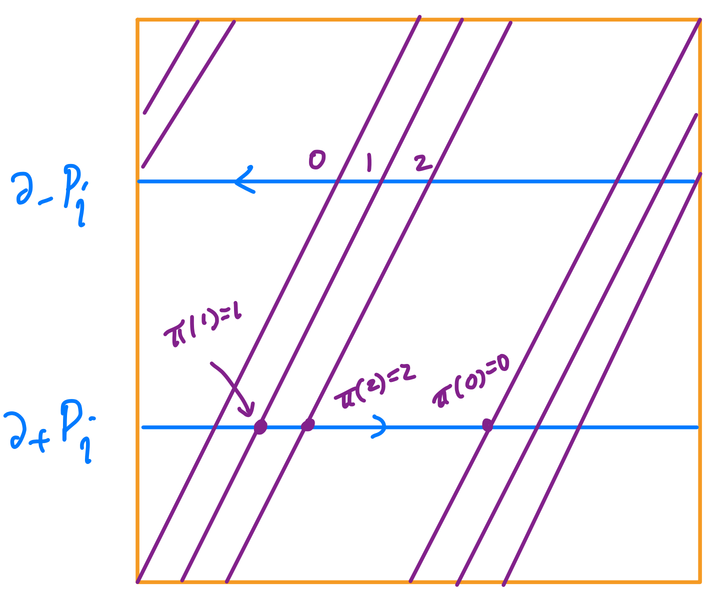
so on $\partial_+ P_i$ we get $\partial_+ A_j = \pi(j)$ component of ∂P_{i+1} for some permutation π of $\{0, \dots, n_{i+1}-1\}$ where π is of the form

$$\pi(j) = \varepsilon j + s \mod n_{i+1}$$

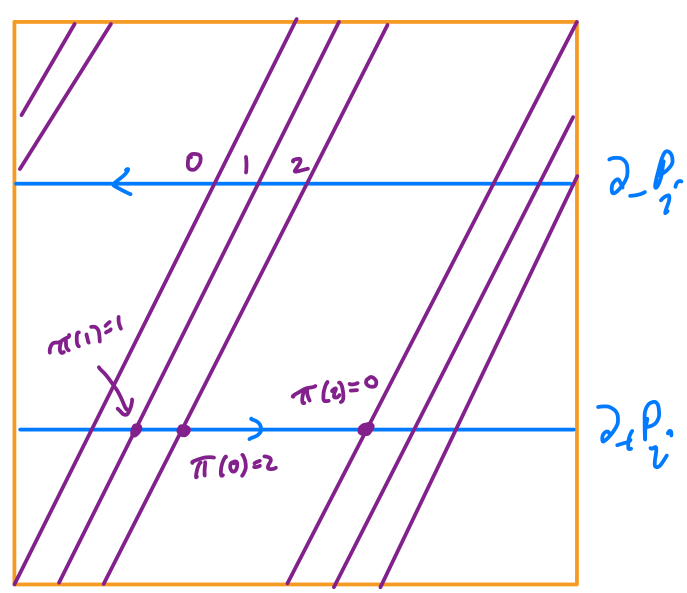
for $\varepsilon = \pm 1$ and some s

so we see





$j \mapsto j$

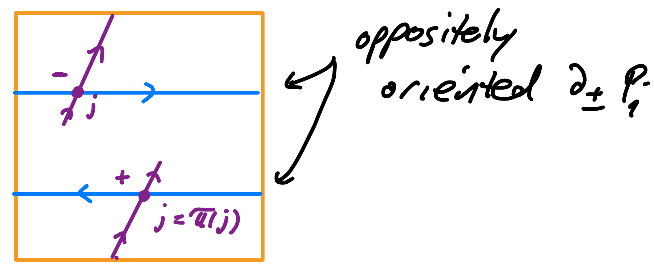


$j \mapsto -j+2$

claim: π has no fixed points

Proof: note: $\partial A_j = \bullet_+ \ \bullet_-$

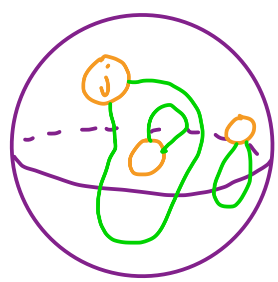
so we see



so must have $\varepsilon = +1$

$\therefore \ s = 0$ and $\pi(j) = j \ \forall j$

so each A_j as seen on P_{i+1} is



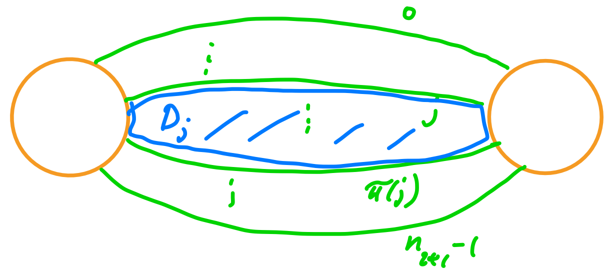
an "inner most" one is a 1-gon $\emptyset$

Case 1. $\varepsilon = -1$

$\pi^2(j) = \pi(-j+s) = j - s + s = j$

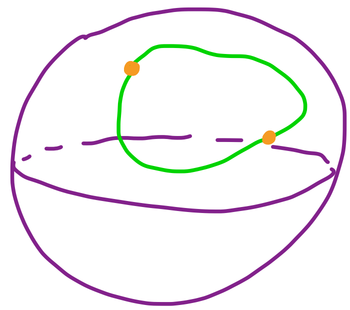
so $\pi^2 = id$ and thus $\{0, 1, \dots n_{i+1}-1\}$ is grouped in pairs $\{j, \pi(j)\}$

each pair gives 2 arcs A_j and $A_{\pi(j)}$ that bound a disk D in P_i.



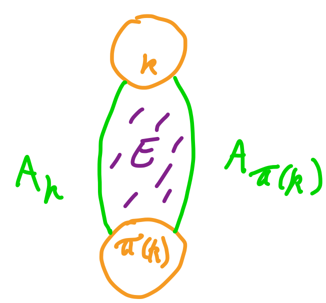
$\subset P_q$.

consider A_j, $A_{\pi(j)}$ in P_{q+1}: get a circle c_j.

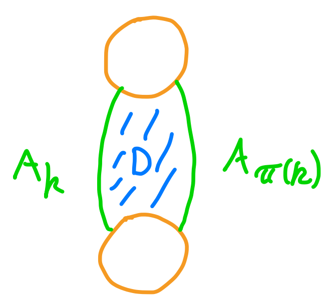


for each j get a circle c_j in P_{q+1} and they are all disjoint, so an innermost c_k bounds a disk E whose interior is disjoint from other verticies

so in P_{q+1} we see

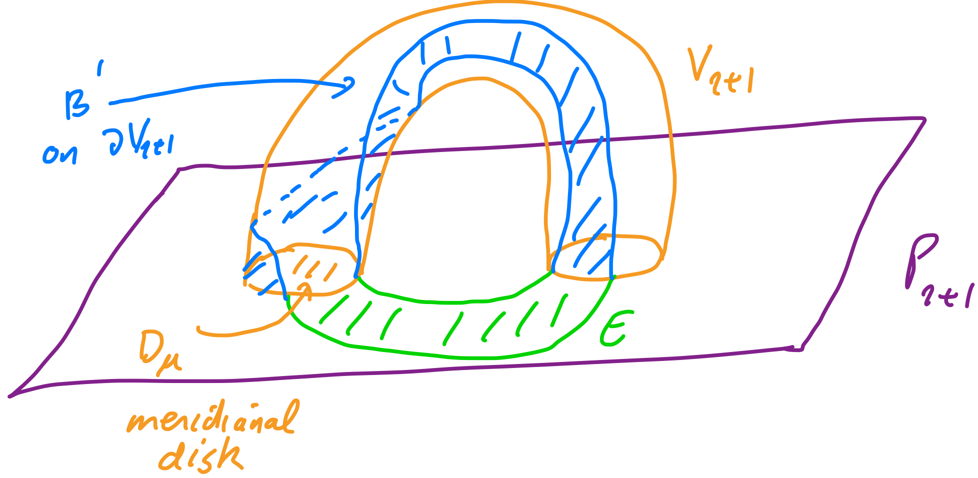


and in P_q we see



in the Dehn filling $M(r_{q+1}) = M \cup V_{q+1}$ ← solid torus

we see a Möbius band



$B = D_\mu \cup E \cup B'$ is a Möbius band

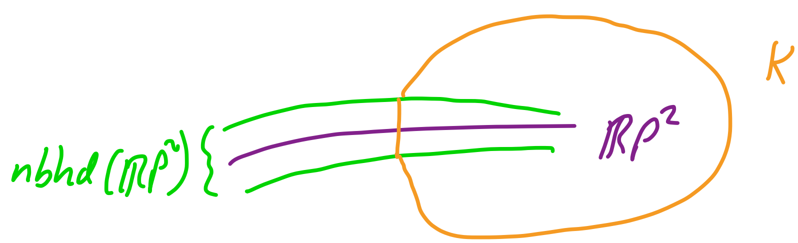
<u>note</u>: $\partial B = \partial D$

so $B \cup D = \mathbb{R}P^2 \subset M(r_{q+1})$

and K_{q+1} = core of V_{q+1} intersects $\mathbb{R}P^2$ once

in $\mathbb{R}P^3$ there is an $\mathbb{R}P^2$ and a knot K that intersects it one time

indeed, $\mathbb{R}P^3 = (\text{nbhd } \mathbb{R}P^2) \cup B^3$



let N = nbhd of $\mathbb{R}P^2$ in $M(r_{q+1})$ and

$M' = \overline{(M(r_{q+1}) - N)} \cup$ 3-ball

set $K = \left(K_{q+1} \cap \overline{(M(r_{q+1}) - N)}\right)$ closed up in M' with an unknotted arc in 3-ball clearly $M(r_{2+1}) = M' \# \mathbb{R}P^3$

and $K_{2+1} = K' \# K$

lemma 10:

$$\boxed{(M,T) \text{ is a } (2,1)\text{-cable}}$$

Proof: $\mathbb{R}P^3 = \bigcirc^{2}$ let K' = core of surgery torus

note: $\mathbb{R}P^3_{K'} = \mathbb{R}P^3 - \text{nbhd}(K') = S^1 \times D^2$

and a meridian of K' on $S^1 \times D^2$ is a curve of slope $1/2$

exercise:

$$\left[(M_1,K_1)\#(M_2,K_2)\right]_{K_1\#K_2} \cong (M_1)_{K_1} \cup_{A_1 = A_2} (M_2)_{K_2}$$

where A_i is a nbhd of the meridian to K_i on $\partial (M_i)_{K_i}$

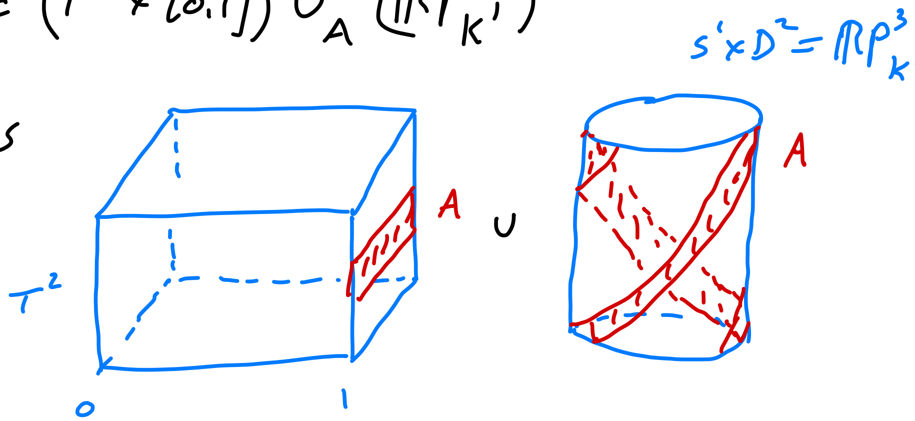
Hint:

so $M = (M(r_{2+1}))_{K_{2+1}} = (M'_{K'}) \cup_A (\mathbb{R}P^3_{K'})$

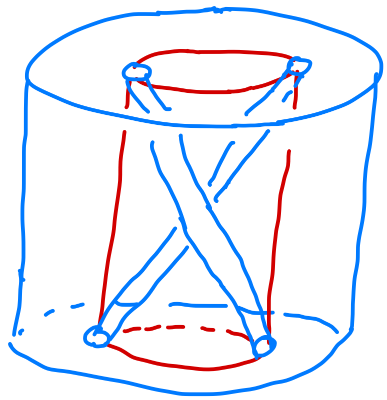
let $T^2 \times [0,1]$ be a nbhd of $\partial(M'_{K'})$

so $C = (T^2 \times [0,1]) \cup_A (\mathbb{R}P^3_{K'})$

is



diffeomorphic to



i.e. $C \cong$ standard $(2,1)$-cable space

<u>Case 2:</u> $\varepsilon = +1$

so $\pi(j) = j + s \mod (n_{i+1})$ some $s \not\equiv 0 \mod (n_{i+1})$

and so π has $d = g.c.d(n_{i+1}, s)$ orbits

each containing $q = \frac{n_{i+1}}{d}$ points

for each orbit θ there is a circle c_θ in Γ_{i+1}

circles are disjoint so $\exists$ an innermost

one that bounds a disk E in P_{i+1}

let $\theta = \{ i_1 < \dots < i_q \}$

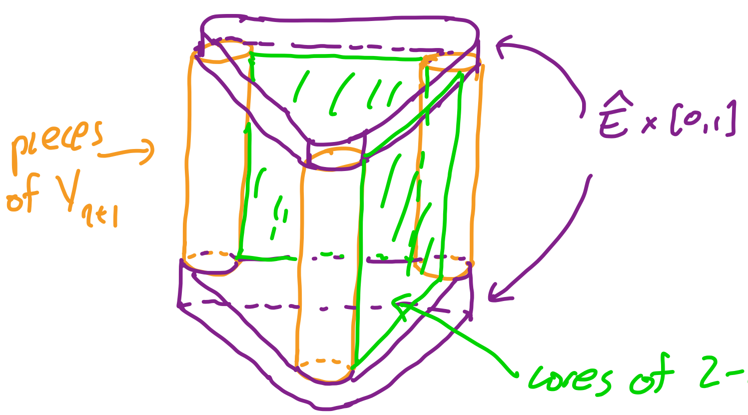
for $j = 1, \dots, q$, let D_j be the disk on P_i between A_{i_j} and $A_{i_{j+1}}$

let $N =$ nbhd of $E \cup (\cup D_j)$ in M

$V = N \cup V_{i+1} \subset M(r_{i+1}) = M \cup V_{i+1}$

$\hat{E} = E \cup$ q-meridional disks in V_{i+1} corresp. to $i_1, \dots, i_q$ boundary components of P_{i+1}

now $V \setminus \hat{E} = 2$ copies of $\hat{E} \times [0,1]$ $\cup \underbrace{q \text{ 1-handles}}_{V_{i+1} \text{ cut along meridional disks}} \cup \underbrace{(q-1) \text{ 2-handles}}_{\text{nbhds of the } D_j}$



note: $V \setminus \hat{E} \cong D^2 \times [0,1]$ and we get V back by gluing $D^2 \times \{0\}$ to $D^2 \times \{1\}$ with a twist

now remove a nbhd of K_{i+1} from V to get a

$C_{q,p}$ cable space

so (M,T) is cabeled

recall $C_{p,q} = (S^1 \times D^2) - \text{nbhd}(K_{p,q})$ where K core of $S^1 \times D^2$

call $\partial\, \text{nbhd}(K_{p,q}) \subset \partial C_{p,q}$ the inner boundary and

$\partial C_{p,q} - \partial\, \text{nbhd}(K_{p,q})$ the outer boundary

lemma 11:

every incompressible, ∂-incompressible, connected planar surface in $C_{p,q}$ is of the following type

(1) an annulus with both boundary components on the inner boundary with slope pq

(2) an annulus with both boundary components on the outer boundary with slope q/p

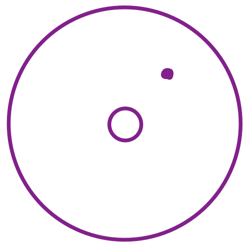
(3) an annulus with one boundary on outer boundary with slope q/p and the other on the inner boundary with slope pq.

(4) a surface with p inner boundary components of slope $\frac{1+kpq}{k}$ and one outer boundary with slope $\frac{1+kpq}{kp^2}$ (some k)

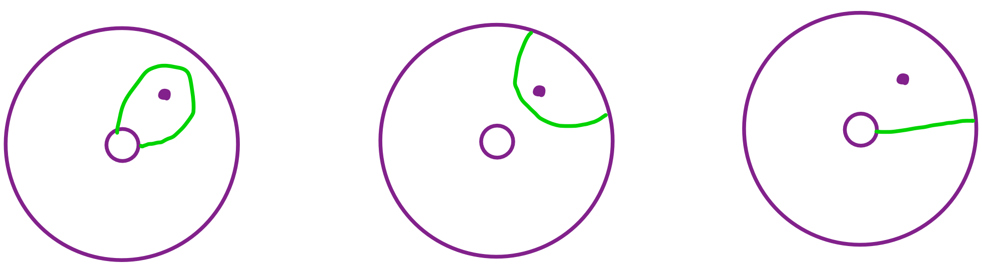
(5) a surface with one inner boundary of slope $\frac{lp^2}{m}$ and p outer boundary components of slope $\frac{l}{m}$ for some l and m s.t. $lp = 1 + mq$

Proof: recall such a surface Σ is either vertical (union of fibers) or horizontal (transverse to fibers)

$C_{p,q}$
$\downarrow$
A



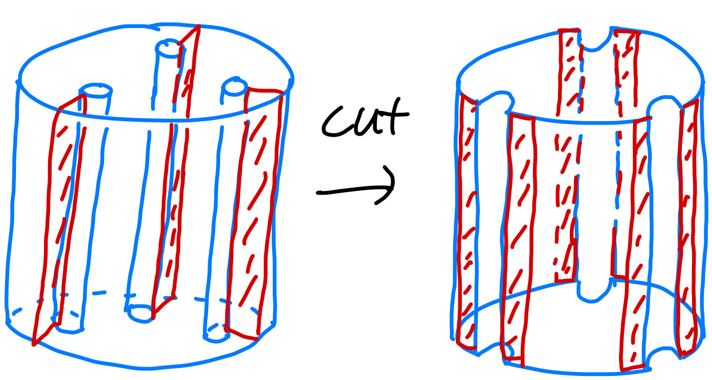
vertical surfaces are



type (1) type (2) type (3)

horizontal surfaces

given Σ a horizontal surface
let A be an annulus of type (3)
$C_{p,q} \setminus A$ = solid torus



A becomes 2 annuli A' and A'' in $\partial(C_{p,q} \setminus A)$

these annuli have slope q/p

to get $C_{p,q}$ back again just reglue A' to A''

we may shift along annuli

now $\Sigma \setminus A \subset C_{p,q} \setminus A$ is a horizontal surface

in $C_{p,q} \setminus A$ ie. a union of n meridional disks

each disk intersects A' (and A'') in p intervals

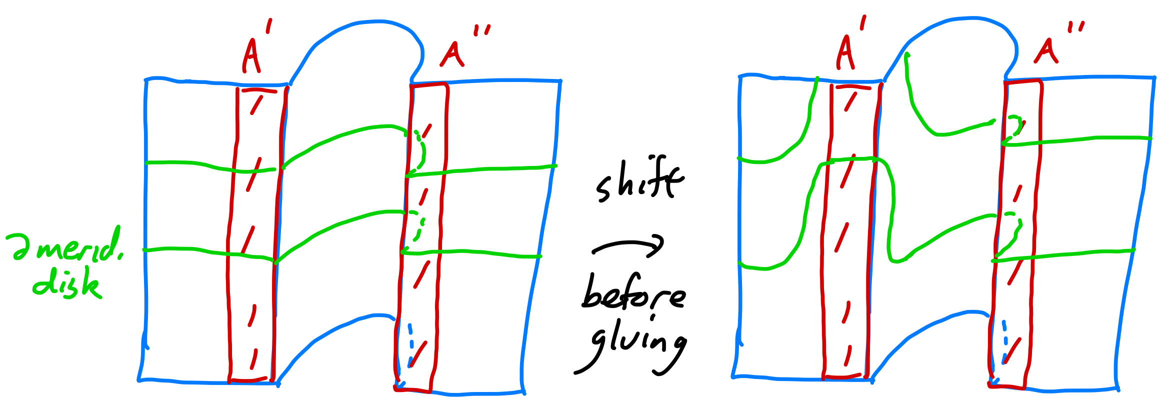
so $\Sigma = n$ 0-handles $\cup$ np 1-handles

so $\chi(\Sigma) = n(1-p)$

$(\Sigma \setminus A) \cap A' = np$ intervals

when gluing A' to A'' can shift so i^{th} interval

is glued to $(i+m)^{th}$ interval



exercise: for Σ to be connected need n, m relatively prime on the inner boundary:

$\partial\Sigma \cdot (\text{fiber of fibration}) = pn$

$\partial\Sigma \cdot (\text{meridian}) = m$

so in this basis slope is $\frac{pn}{m}$

we use framing on inner boundary that differs from fiber framing by $+pq$

so slope is $\frac{p(n+mq)}{m}$ and there are

$$d_1 = \text{g.c.d.}\,(p(n+mq), m) = \text{g.c.d.}\,(p, m)$$

components

alternate computation:

if shift $m=0$ then get $pn(0,1)$

now if shift by m in direction $(1, pq)$

get $pn(0,1) + m(1,pq) = (m, pn+pqm)$

so slope is $\frac{p(n+qm)}{m}$

on the outer boundary:

arguing as in alternate computation above we see

$$n(0,1) + m(p,q) = (mp, n+qm)$$

so slope is $\frac{n+qm}{mp}$

and there are

$$d_2 = \text{g.c.d.}(mp, n+qm) = \text{g.c.d.}(p, n+qm)$$

components

note m and $n+qm$ are relatively prime so

d_1 and d_2 are too

i.e. $p = d_1 d_2 a$ some $a \geq 1$

$\therefore\ d_2 \leq p/d_1$

since Σ is planar we have

$$\begin{aligned} -\chi(\Sigma) = n(p-1) &= d_1 + d_2 - 2 \\ &\leq d_1 + p/d_1 - 2 \\ &\leq p-1 \qquad (1 \leq d_1 \leq p) \end{aligned}$$

$\therefore\ n = 1$ and all inequalities are equalities

so either $d_1 = 1$ and $d_2 = p$

or $d_1 = p$ and $d_2 = 1$

in the first case we have $1 + qm = \ell p$

so we are in case (5)

in the second case we have $m = pk$ some k

so we are in case (4)

Proof of lemma 7:

we need to see that if (M, T) is cabled then

$\Delta(r,s) \leq 1$ for all $r,s \in P\mathcal{S}(M,T)$

first let $C \subset M$ be a cable space with

$$\partial C = \underset{\text{inner}}{T} \amalg \underset{\text{outer}}{T'}$$

and set $M' = \overline{M - C}$

let $(P, \partial P) \subset (M, T)$ be an incompressible, ∂-incompressible, connected, planar surface

choose P so that $P \cap T'$ is minimal among all such surfaces with the same boundary slope

<u>Claim:</u> $P \cap C$ and $P \cap M'$ are incompressible

<u>Proof:</u> let D be a compressing disk in M' for $P \cap M'$

$\partial D = \gamma \subset P$ bounds a disk $D' \subset P$

D' must intersect T' or D would not be a compressing disk for $P \cap M'$

replace D' in P with D and get a new surface P' that intersects T' fewer times $\otimes$ choice of P

same argument for $P \cap C$ ✓

<u>Claim:</u> $P \cap C$ and $P \cap M'$ are boundary incompressible

<u>Proof:</u> we need

lemma 12:

> If $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ is incompressible
> and $\partial\Sigma \subset$ torus component of ∂M
> then it is also ∂-incompressible
> or an annulus
> (will be isotopic into ∂M if M irreducible)

given this we are done since if $P \cap M'$ or $P \cap C$ were not ∂-incompressible then it would be an annulus and we could replace it by one in T' and then isotop to reduce intersection with T' $\otimes$ ✓

we will prove lemma 12 after we finish proof of lemma 7

so $P \cap C$ is a union of pieces from lemma 11

i.e. it can be I) Annulus with both ∂ components on T with slope q/p

II) Annulus with one ∂ on T with slope q/p and other on T' of slope pq

$\cup$ annuli with both boundary components on T'

III) a surface with p ∂ components on T of slope $\frac{1+kpq}{k}$ and one on T' of slope $\frac{1+kpq}{kp^2}$

IV) a surface with one ∂ component on T of slope $\frac{\ell p^2}{m}$ and q on T' with slope $\frac{\ell}{m}$ where $\ell q = 1 + mp$

suppose P is of type IV), then note $P \cap M'$ must be p disks so P is a disk with boundary on T

note: a neighborhood N of $P \cup T$ is a solid torus with a ball removed

$\therefore$ $M = M' \# (S^1 \times D^2)$ and $T = \partial (S^1 \times D^2)$

and the only incompressible surface is the meridional disk so lemma true!

the distance between surfaces of Type I) and II) is 0, and between one of Type I) or II) and III) is 1

so we are left to see the distance between 2 surfaces of type III) is ≤ 1

suppose P_1, P_2 are 2 such surfaces their slopes on T are $r_i = \frac{1 + k_i pq}{k_i}$ and on

T' are $r_2' = \dfrac{1+k_2pq}{k_2p^2}$ for $k_1 \neq k_2$

$\therefore \Delta(r_1, r_2) = |k_1 - k_2|$ and $\Delta(r_1', r_2') = p^2|k_1 - k_2|$

if $p \geq 3$ or $p = 2$ and $|k_1 - k_2| \neq 1$ then

by lemma 6 (M', T') is cabled

so we are done unless (M', T') cabled

so $\exists$ coordinates on T' s.t.

$$r_2' = \frac{1+k_2'p'q'}{k_2'}$$

changing coordinates by $\begin{pmatrix} 1 & -p'q' \\ 0 & 1 \end{pmatrix}$ gives

$$r_2' = \frac{1}{k_2'}$$

but in other coordinates we have $r_2' = \dfrac{1+k_i pq}{k_2p^2}$

$\therefore \exists$ a coordinate change $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in GL(2, \mathbb{Z})$

s.t. $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} k_2p^2 \\ 1+k_2pq \end{bmatrix} = \begin{bmatrix} k_2' \\ 1 \end{bmatrix} \quad \left(= \begin{bmatrix} -k_2' \\ -1 \end{bmatrix} \right)$

$$z k_2 p^2 + w(1+k_2pq) = \pm 1 \qquad (*)$$

$$\| $$

$$w + k_i p(pz + wq) \qquad i = 1,2$$

subtracting

$$(k_1-k_2)p(pz+qw) = 0 \text{ or } \pm 2$$

if $=0$ we get $pz+qw=0$

and pluging into ⊛ gives $w=\pm 1$

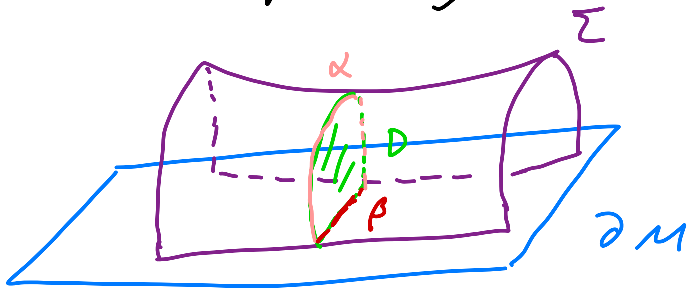
$\therefore\ q = \pm pz$ ⨳ p,q rel. prime

in the other 2 cases we have $p=2$ and

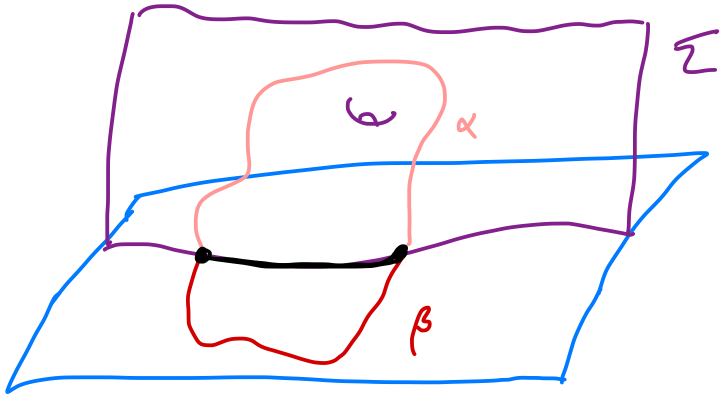
$|k_1-k_2| = 1$ so $\Delta(r_1,r_2) = 1$

Proof of lemma 12:

let D be a ∂-compressing disk for Σ



Case 1: $\partial\beta$ on same component of $\partial\Sigma$



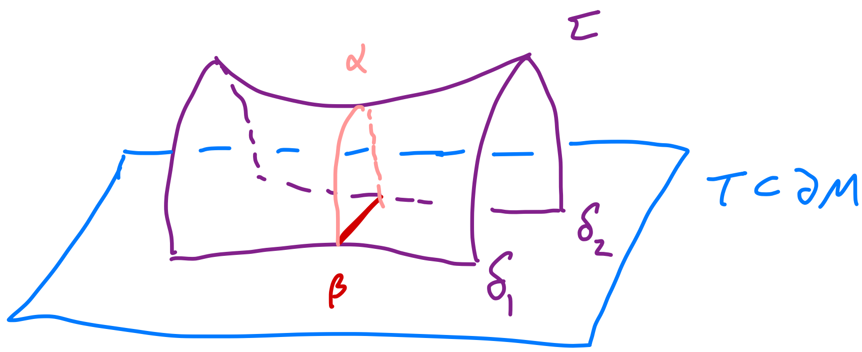
since D is connected a nbhd of $\partial\beta$ in β lies on the same side of $\partial\Sigma$ in $T^2 \subset \partial M$

so $\exists$ an arc $\gamma \subset \partial\Sigma$ such that $\beta \cup \gamma$ bound a disk Δ in T

note $\alpha \cup \gamma$ is a circle in Σ that bounds a disk $D \cup \Delta$

Σ incompressible $\Rightarrow \alpha \cup \gamma$ bound a disk in Σ $\cancel{\Leftarrow}$ D a ∂-compressing disk

<u>Case 2</u>: $\partial\beta$ in distinct components of $\partial\Sigma$



a nbhd $N(D) = D \times [-1, 1]$ s.t.

$N(D) \cap \Sigma = N(\alpha)$ nbhd α in Σ

$N(D) \cap T = N(\beta)$ nbhd β in T

let $D_\pm = D \times \{\pm 1\}$

and $\partial D_\pm = \alpha_\pm \cup \beta_\pm$

<u>note</u>: $\gamma = \Big[(\delta_1 \cup \delta_2) - [(\delta_1 \cup \delta_2) \cap \partial N(\beta)]\Big] \cup (\alpha_+ \cup \alpha_-)$

is a simple closed curve in Σ

and if $\Delta =$ annulus $\delta_1 \cup \delta_2$ bounds minus $N(\beta)$

$N(\beta)$

then γ bounds the disk $D_+ \cup \Delta \cup D_-$

$\therefore$ γ bounds a disk E in Σ since Σ is incompressible

$\therefore$ $\Sigma = E \cup N(\alpha) =$ annulus !

exercise: if M is irreducible show Σ isotopic into ∂M

or more generally $\exists M', M''$ such that $\Sigma \subset M'$ and cobounds a solid torus S with an annulus in $\partial M'$ and $M = M' \# M''$ where connected sum is done in S