B. Homotopy groups

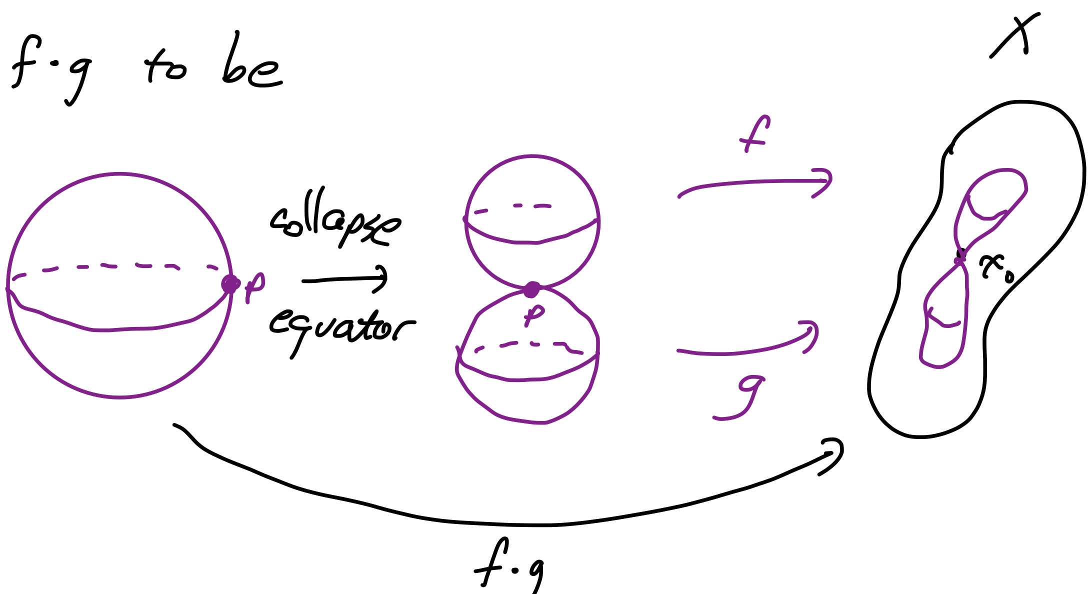
recall $\pi_n(X) = [S^n, X]_0$

and since S^n is an H'-space there is a product on $\pi_n(X)$

but what is the product?

given $f, g : (S^n, p) \to (X, x_0)$

define $f \cdot g$ to be



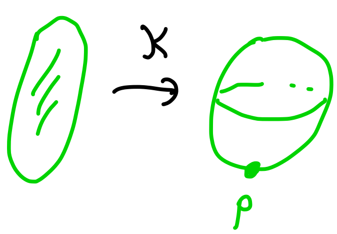
easy to check that $[f \cdot g] = [f] \cdot [g]$

sometimes it is useful to see $\pi_n(X)$ as

$$[(D^n, \partial D^n), (X, x_0)]$$

i.e. homotopy classes of maps $D^n \to X$
sending $\partial D^n \to x_0$

Indeed if $\kappa : D^n \to S^n$ collapses ∂D^n to $p \in S^n$



then $\pi_n(X) \longrightarrow [(D^n, \partial D^n), (X, x_0)]$ is

$$[f] \longmapsto [f \circ \chi]$$

this is clearly well-defined and injective and onto since any $f : (D^n, \partial D^n) \longrightarrow (X, x_0)$ factors through (S^n, p)

i.e. given

$$\begin{tikzcd} (D^n, \partial D^n) \arrow[r, "f"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\circ"] & (X, x_0) \\ (S^n, p) \arrow[ur, dashed, "\exists \bar{f}"'] & \end{tikzcd}$$

similarly any homotopy in $[(D^n, \partial D^n), (X, x_0)]$ factors through (S^n, p) so the map is also surjective

What is the product structure on $[(D^n, \partial D^n), (X, x_0)]$?

think of D^n as $D^{n-1} \times [0,1]$

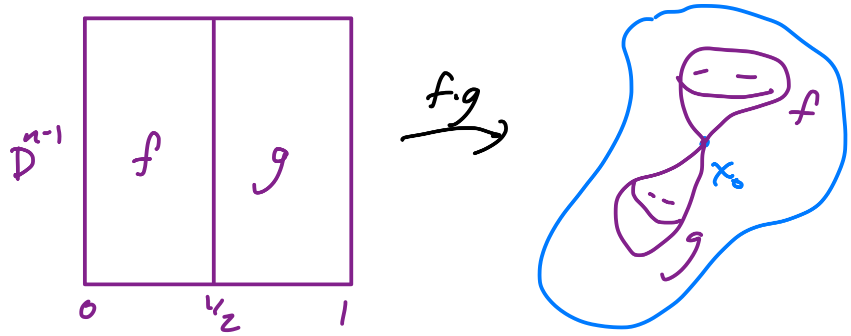
so given $f : D^{n-1} \times [0,1] \longrightarrow X$

$g : D^{n-1} \times [0,1] \longrightarrow X$

then define $f \cdot g : D^{n-1} \times [0,1] \to X$ by

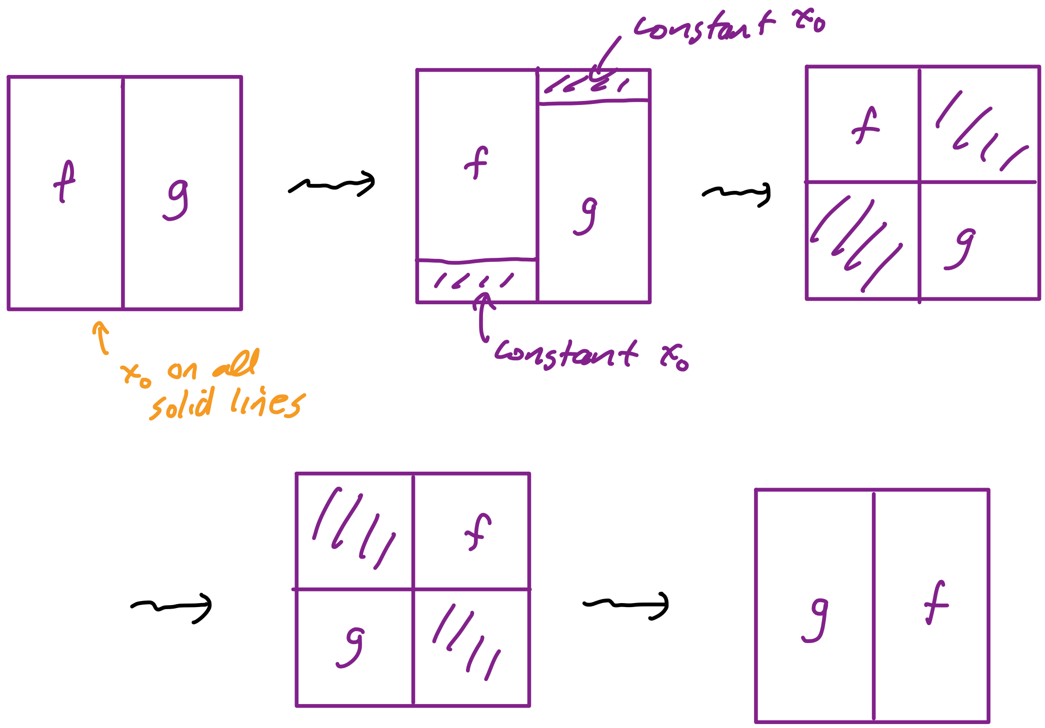
$$(x,t) \longmapsto \begin{cases} f(x, 2t) & 0 \le t \le 1/2 \\ g(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

so



Note: it is easy to see $\pi_n(X)$ is abelian for $n \geq 2$

here the homotopy from $f \cdot g$ to $g \cdot f$ is



using these definitions it is also easy to define

<u>relative homotopy groups</u>

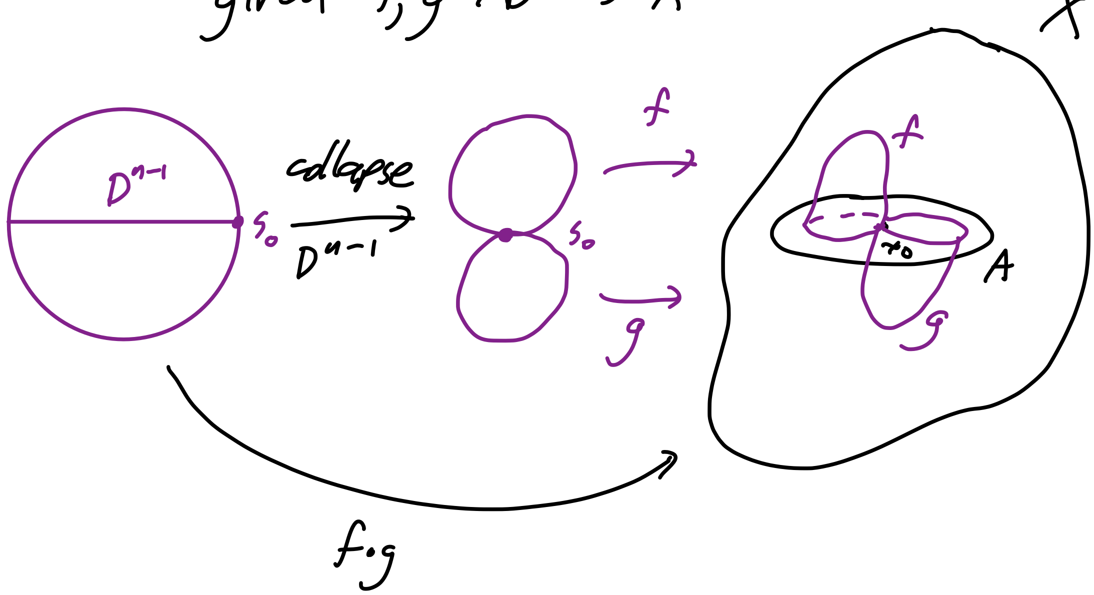
given a space X a subspace A and $x_0 \in A$

let $\pi_n(X, A) = [(D^n, \partial D^n, s_0), (X, A, x_0)]$ where $s_0 \in \partial D^n$

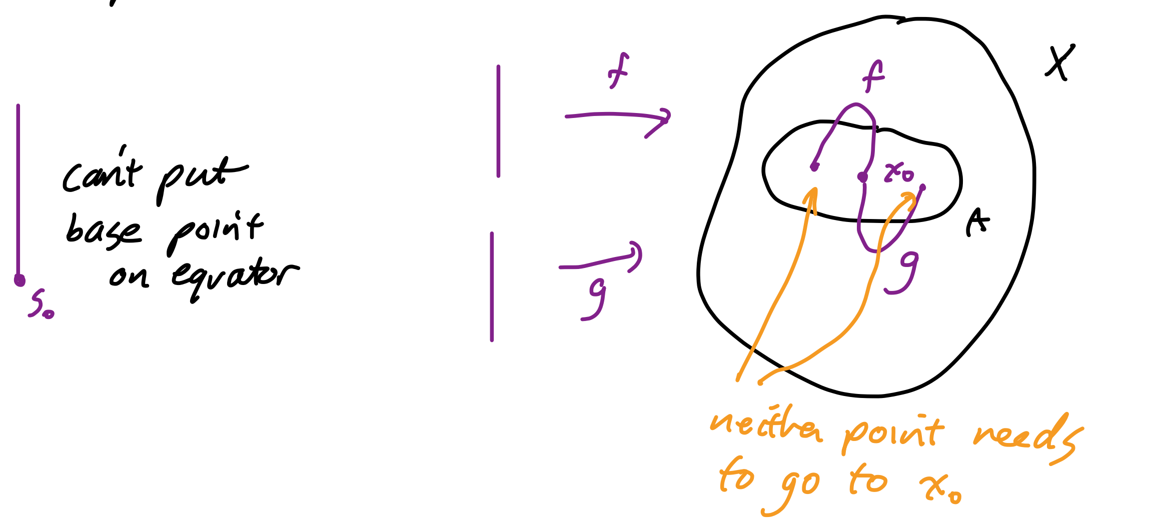
the set of relative homotopy classes of $D^n \to X$ sending ∂D^n to A and s_0 to x_0

the product structure is

given $f, g : D^n \to X$



note: this product does not make sense on $\pi_1(X, A)$



to prove $\pi_n(X, A)$ is a group it is helpful to give a different definition

let $D^n = [0,1]^n$ and

$$J = \overline{\partial D^n - (D^{n-1} \times \{1\})} = (D^{n-1} \times \{0\}) \cup \partial D^{n-1} \times [0,1]$$

exercise: $[(D^n, \partial D^n, s_0), (X, A, x_0)]_0$ is in one-to-one correspondence with

$$[(D^n, \partial D^n, J), (X, A, x_0)]_0$$

$$\left(\text{note } (D^n, \partial D^n, J)/_J \cong (D^n, \partial D^n, s_0)\right)$$

now given $f, g \in \pi_n(X, A)$ define

$$f \cdot g(x_1 \ldots x_n) = \begin{cases} f(2x_1, x_2 \ldots x_n) & x_1 \in [0, 1/2] \\ g(2x_1 - 1, x_2 \ldots x_n) & x_1 \in [1/2, 1] \end{cases}$$

and $f^{-1}(x_1, \ldots x_n) = f(1 - x_1, x_2, \ldots x_n)$

exercise:

1) show $\pi_n(X, A)$ a group with identity the constant map, if $n \geq 2$
2) $\pi_n(X, A)$ abelian for $n \geq 3$
3) $\pi_1(X, A)$ is just a set (ie. product doesn't make sense)
4) $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$

the following lemma will be useful

lemma 16:

$f:(D^n, \partial D^n, s_0) \to (X, A, x_0)$ is 0 in $\pi_n(X,A)$

$\Longleftrightarrow$

it is homotopic rel ∂D^n and s_0 to a map whose image is in A

Proof: ($\Leftarrow$) suppose we have such a homotopy f to g

we know D^n deformation retracts to s_0

$$H: D^n \times [0,1] \to D^n$$

$$H(x,0) = x \quad H(x,1) = s_0 \quad H(s_0,t) = s_0$$

now $g \circ H$ is a homotopy from g to constant map

$\therefore f$ is trivial in $\pi_n(X,A)$

($\Rightarrow$) if $[f] = 0$ in $\pi_n(X,A)$ then

$\exists\ H: D^n \times [0,1] \to X$ such that

$$H(x,0) = f(x)$$
$$H(x,1) = x_0$$
$$H(x,t) \in A \quad \forall\, x \in \partial D^n$$

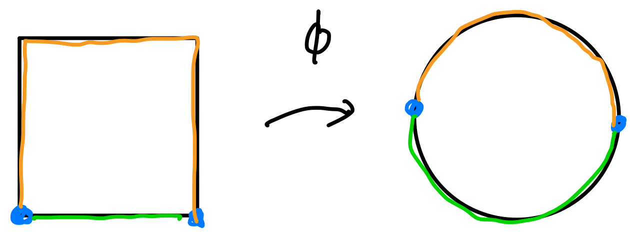
now $H\big|_{D^n \times \{1\} \cup (\partial D^n \cup [0,1])}$

is a map of $D^n \to A$ with $\partial D^n \to A$ and $s_0 \mapsto x_0$

so we can use H on $D^n \times [0,1]$ to give a homotopy f to a map with image in A

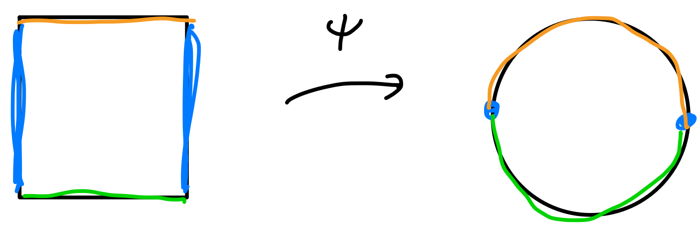
you can write this out explicitly but here is the idea

note: $\exists$ a homeomorphism $D^n \times [0,1] \xrightarrow{\phi} D^{n+1}$



define ϕ on ∂ and cone

there is also a map $D^n \times [0,1] \xrightarrow{\psi} D^{n+1}$ that collapses $\partial D^n \times [0,1]$ to equator



define ψ on ∂ and cone

now $H \circ \phi^{-1} \circ \psi$ is the homotopy

note: ① $(A, x_0) \underset{i}{\subseteq} (X, x_0) \underset{j}{\subseteq} (X, A)$ inclusions

then i, j induce maps

$$\pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A)$$

② If $f: (D^n, \partial D^n, J) \to (X, A, x_0)$

then define $\partial f: (\partial D^n, J) \to (A, x_0)$ to be $f|_{\partial D^n}$

this induces a map

$$\pi_n(X, A) \to \pi_{n-1}(A)$$

note: $\pi_{n-1}(A) = [(\partial D^n, J), (A, x_0)]_0$

exercise: show this is well-defined.

Thm 17:

given (X, A, x_0) we have a long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

(and it is equivariant under $\pi_1(A)$ action)

Proof: clearly $j_* \circ i_* = 0$ by lemma 16

now if $[f] \in \ker j_*$, then

$f : (D^n, \partial D^n) \to (X, A)$ and

$\exists$ homotopy $H : D^n \times [0,1] \to X$ s.t.

1) $H(x,0) = f(x)$

2) $H(x,1) \in A$

3) $H(x,t) \in A \quad \forall t$ if $x \in \partial D^n$

4) $H(s_0, t) = x_0$

note: $D' = D^n \times \{1\} \cup \partial D^n \times [0,1]$ is a disk and

$H|_{D'} : D' \to A$ st. $H(\partial D') = x_0$

so $g = H|_{D'} : D^n \to A$ is in $\pi_n(A)$

and as in proof of lemma 16 H gives a homotopy

from f to g in $\pi_n(X)$

$\therefore [f] \in \operatorname{im} i_*$ and we have $\operatorname{im} i_* = \ker j_*$

Suppose $[f] \in \pi_n(X,A)$ and $\partial[f] = [0]$

so $\exists$ homotopy $H: S^{n-1} \times [0,1] \to A$ s.t.

$$H(x,0) = f(x)$$
$$H(x,1) = x_0$$
$$H(s_0,t) = x_0$$

$D' = D^n \cup S^{n-1} \times [0,1]$ is a disk and

$$f': D' \to X : x \longmapsto \begin{cases} f(x) & x \in D^n \\ H(x) & x \in S^{n-1} \times [0,1] \end{cases}$$

and $f'(\partial D') = x_0$ so $[f'] \in \pi_n(X)$

easy to check $f' \sim f$ in $\pi_n(X,A)$ so

$j_*([f']) = [f]$ and $\ker \partial \subset \operatorname{im} j_*$

now if $[f] \in \pi_n(X)$ then $f(\partial D^n) = x_0$

so $\partial[j_*([f])] = [\text{constant}] = [0]$ in $\pi_{n-1}(A)$

so $\ker \partial = \operatorname{im} j_*$

exercise: show $\operatorname{im} \partial = \ker i_*$ ▦

Thm 18:

let $p: \tilde{X} \to X$ be a connected covering space
Then $\pi_n(\tilde{X}, \tilde{x}_0) \cong \pi_n(X, p(\tilde{x}_0))$ for all $n \geq 2$ and $\tilde{x}_0 \in X$

Proof: $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, p(\tilde{x}_0))$ is a homomorphism

recall given $f: Y \to X$ with $y_0 \in Y$ st. $f(y_0) = p(x_0)$

lifting criterion $\left\{ \begin{array}{c} \text{then } \exists\ \tilde{f}: Y \to \tilde{X} \text{ s.t. } f(y_0) = \tilde{x}_0 \text{ and } p\circ\tilde{f} = f \\ \Longleftrightarrow \\ \tilde{f}_*(\pi_1(Y, y_0)) < p_*(\pi_1(\tilde{X}, x_0)) \end{array} \right.$

p_* is surjective for $n \geq 2$

indeed, given $[f] \in \pi_n(X, p(\tilde{x}_0))$

$$f_*(\pi_1(S^n, s_0)) = \{e\} < p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

so $\exists\ \tilde{f}: S^n \to \tilde{X}$ s.t. $\tilde{f}(s_0) = \tilde{x}_0$ and $p\circ\tilde{f} = f$

$\therefore p_*([\tilde{f}]) = [f]$

p_* is injective for $n \geq 2$

indeed suppose $p_*([f]) = [0]$ in $\pi_n(X, p(\tilde{x}_0))$

then $\exists$ a homotopy $H: S^n \times [0,1] \to X$

s.t. $H(x, 0) = p\circ f(x)$

$H(x, 1) = p(\tilde{x}_0)$

$H(s_0, t) = p(\tilde{x}_0)$

recall covering spaces satisfy homotopy lifting

so $\exists\ \tilde{H}: S^n \times [0,1] \to X$ st. $\tilde{H}(x, 0) = f(x)$

since $\tilde{H}(S^n, 1) \subset \underbrace{p^{-1}(p(\tilde{x}_0))}_{\text{discreat space}}$ $\longrightarrow$ $\tilde{H}(x, 1) = \tilde{x}_0$

$\tilde{H}(s_0, t) = \tilde{x}_0$

$\therefore [f] = [0]$ in $\pi_n(\tilde{X}, x_0)$

$\#$