1) obstruct k-frames over varios skeleta

2) say when we can reduce the structure group

e.g. $w_1(E)=0 \Leftrightarrow$ structure group reduces from $O(n)$ to $SO(n)$

3) differentiate bundles

4) obstruct immersions and embeddings

5) obstruct bounding a compact mfd.

there are many more applications and similarly for $c_i(E)$ and $p_i(E)$

example:

if Pontrjagin numbers not 0 then

1) no orientation reversing diffeomorphism and
2) does not bound an compact oriented mfd

e.g. $\mathbb{C}P^{2n}$ has no or$^{\underline{n}}$ reversing diffeo. and does not bound an oriented mfd

note: can prove both these with intersection pairings (e.g. Poincaré duality)

C. Classifying Spaces

Grassmannian

Recall $G_{n,m}$ = all n-dim'l subspaces in $\mathbb{R}^m$

we have $G_{n,m} \subset G_{n,m+1}$

so $G_n = \bigcup_m G_{n,m}$

or $G_n =$ all n-dim'l subspaces of $\mathbb{R}$

let $E_n = \{(\ell, v) \in G_n \times \mathbb{R}^\infty : v \in \ell\}$ so $p(\ell, v) = \ell$

exercise: $E_n \xrightarrow{p} G_n : (\ell, v) \to \ell$ is an n-diml vector bundle

Hint: if $\ell \in G_n$ let $\pi_\ell : \mathbb{R}^\infty \to \ell$ be orthogonal proj

let $U_\ell = \{\ell' \in G_n : \pi_\ell(\ell')$ has dim $n\}$

Show: U_ℓ open and

$$h : p^{-1}(U_\ell) \longrightarrow U_\ell \times \ell \quad \text{is a local triv.}$$
$$(\ell', v) \longmapsto (\ell', \pi_\ell(v))$$

Th$^{\text{m}}$ 12:

> If X is paracompact, then
> $$[X, G_n] \to Vect^n(X)$$
> $$f \longmapsto f^*(E_n)$$
> is a bijection.

Proof: from Th$^{\text{m}}$ II.1 the above map is well-defined!

to go further we first observe

Claim: for $E \to X$ an $\mathbb{R}^n$-bundle

$E \cong f^*(E_n)$ some $f : X \to G_n$

$\Longleftrightarrow$

$\exists$ a map $E \to \mathbb{R}^\infty$ that is linear injective on each fiber

to see this suppose $f : X \to G_n$ and $\psi : E \xrightarrow{\cong} f^*(E_n)$ so we have

$$\begin{tikzcd}
E \arrow[r, "\psi"] \arrow[dr, "p"] & f^*(E_n) \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & \mathbb{R}^\infty \\
& X \arrow[r, "f"] & G_n &
\end{tikzcd}$$

and top row linear injective on fibers

now if $E \xrightarrow{g} \mathbb{R}^\infty$ is such a map

define $f: X \to G_n: x \longmapsto g(p^{-1}(x))$

and $\tilde{f}: E \to E_n : v \longmapsto g(v)$

exercise: this implies $f^*(E_n) \cong E$

map in th^m is surjective:

given $p: E \to X$ an $\mathbb{R}^n$-bundle

(we assume X compact Hausdorff, but OK for paracompact)

let $\{U_i\}_{i=1}^{k}$ be a cover of X by local triv. of E i.e. $\exists$

$$\begin{tikzcd}
\phi_i : p^{-1}(U_i) \arrow[r] \arrow[rr, bend right, "\tilde{\phi}_i"'] & U_i \times \mathbb{R}^n \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$

set $g: E \to \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k\text{-times}} \subset \mathbb{R}^\infty$

$$v \longmapsto (\psi_1(p(v))\,\tilde{\phi}_1(v), \dots, \phi_k(p(v))\,\tilde{\phi}_1(v))$$

exercise: g is linear on fibers so $g^*(E_n) \cong E$ from above map in th$^{\text{m}}$ is injective:

suppose $g_0^*(E_n) \cong g_1^*(E_n)$

for $g_i : X \to G_n$

from above $\exists\, f_i : X \to \mathbb{R}^\infty$ s.t. f_i linear injective on fibers

from proof above can assume f_0 maps to odd coords in $\mathbb{R}^\infty$

and f_1 maps to even coords in $\mathbb{R}^\infty$

let $f_t = (1-t)f_0 + t f_1$

exercise: f_t is linear injective on each fiber

set $g_t(x) = f_t(p^{-1}(x))$

exercise: this is a homotopy g_0 to g_1 $\blacksquare$

note: Claim from previous section that any line bundle

$p : E \to X$ comes from $f^*(\gamma)$ clearly follows

(ie $\gamma = E_1$)

more generally given a Lie group G one can show there

is a space BG and EG such that

$$\begin{array}{ccc} G & \to & EG \\ & & \downarrow \\ & & BG \end{array}$$

is a principal bundle and EG is weakly contractible

we call BG the classifying space for principal G-bundles

note:

$$\underbrace{H_k(EG)}_{=0} \to H_k(BG) \to H_{k-1}(G) \to \underbrace{H_{k-1}(EG)}_{=0} \qquad k \geq 1$$

so $\quad H_k(BG) \cong H_{k-1}(G) \quad \forall k \geq 1$

Thm 13:

$$\boxed{[X, BG] \xleftrightarrow[\text{correspondace}]{\text{one-to-one}} \text{Principal } G\text{-bundles over } X}$$

Thm 14:

$$\boxed{\text{the homotopy type of } BG \text{ is unique}}$$

examples:

1) G_n is the classifying space of $\mathbb{R}^n$-bundles

(really $\mathcal{F}(E_n)$ is the $SL(n, \mathbb{R})$ bundle

and G_n the $SL(n, \mathbb{R})$ space)

exercise: $\mathcal{F}(E_n)$ is contractible

2) $\mathbb{R} \to S^1$ is a principal $\mathbb{Z}$-bundle so

principal $\mathbb{Z}$-bundles over X

$$\leftrightarrow$$

$$[X, S^1] = [X, K(\mathbb{Z}, 1)] \cong H^1(X; \mathbb{Z})$$

recall $K(\mathbb{Z},1) \simeq S^1$

Brown representation thm see next section

$$\boxed{H^n(X; G) = [X, K(G, n)]}$$

3) $S^\infty \to \mathbb{R}P^\infty$ is a principal $\mathbb{Z}/2$ bundle

exercise: S^∞ contractible

note: $O(1) \cong \mathbb{Z}/2$ so $BO(1) = \mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$ and

check

line bundles over X

$\longleftrightarrow$

principal $O(1)$ bundles

$\longleftrightarrow$

$$[X, BO(1)] = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}/2, 1)] \cong H^1(X; \mathbb{Z}/2)$$

Brown

4) $S^\infty \to S^\infty/S^1 \cong \mathbb{C}P^\infty$ is a principal S^1-bundle

so $BS^1 = BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ and

complex line bundles over X

$\longleftrightarrow$

principal $U(1)$-bundles over X

$\longleftrightarrow$

$$[X, BU(1)] = [X, \mathbb{C}P^\infty] \cong H^2(X; \mathbb{Z})$$

Brown

to prove Thm 13 we need a definition

a GCW-complex is a space X with a G-action that is a union of skeleta

$$X^{(0)} \subset X^{(1)} \subset \dots \subset X^{(n)} \subset \dots$$

where for each $k \geq 0$ there is

1) a collection of k-cells e_i^k

2) subgroup $H_i < G$, and

3) $\phi_i : (\partial e_i^k \times G/H_i) = (S^{k-1} \times G/H_i) \longrightarrow X^{(k-1)}$

s.t. each ϕ_i is G-equivariant and

$$X^{(k)} = X^{(k-1)} \cup (e_i^k \times G/H_i)/\sim$$ ← glue with ϕ_i

we can take EG to be a GCW-complex

<u>exercise:</u>

1) if X a GCW-complex then X/G has the structure of a CW complex

2) if G is a compact Lie group then any principal G-bundle over a CW-complex is a GCW-complex

<u>Proof of Thm 13:</u> let M be a CW-complex

clearly
$$\Psi : [M, BG] \longrightarrow \mathrm{Princ}^G(M), \quad f \longmapsto f^* EG$$
← here we consider only $\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}$ with P a GCW-complex

is well-defined by Thm II.1

<u>Claim:</u> Ψ is surjective

indeed, given $\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}$ we construct $h : P \to EG$ skeleta-by-skeleta

s.t. h is G-equivariant and maps G orbits is P homeomorphically onto image

since G action on P is free all strata are attachments of $e^k \times G$ to $P^{(k-1)}$

now $P^{(0)} = \{pt\} \times G$ and we can clearly map this to a fiber of EG

now if $h_{k-1} : P^{(k-1)} \to EG$ defined we try to extend over $e^k \times G$

consider the disk $e^k \times \{1\} \subset e^k \times G$

h_{k-1} is defined on $\partial e^k \times \{1\} \subset P^{(k-1)}$

and $h_{k-1}|_{\partial e^k \times \{1\}}$ extends to $e^k \times \{1\}$

if it is null-homotopic

since EG is weakly contractible $h_{k-1}|_{\partial e^k \times \{1\}}$ extends to $h : (e^k \times \{1\}) \longrightarrow EG$

now define h_k on $e^k \times G$ by extending h on $e^k \times \{1\}$ G-equivariantly

i.e. $h_k : e^k \times G \to EG : (p,g) \longmapsto h(p) \cdot g$

clearly $h_k : P^{(k)} \longrightarrow EG$ is G-equivariant and a homeo. on orbits

so we have $h : P \to EG$

this induces a map $f : M = P/G \to BG = EG/G$

$$\begin{array}{ccc} P & \xrightarrow{h} & EG \\ \downarrow q & & \downarrow p \\ M & \xrightarrow{f} & BG \end{array}$$

we now have

$$P \to f^*EG = \{(x,v) \in M \times EG : f(x) = p(v)\}$$
$$w \longmapsto (q(w), h(x))$$

exercise: Show this is an isomorphism

Claim: Ψ is injective

Suppose we have $f_i : M \to BG$ $i=0,1$ and

$$\Phi : f_0^*EG \to f_1^*EG \qquad \text{(assume cellular)}$$

is an isomorphism

we need to construct a homotopy f_0 to f_1

since f_i are cellular and EG a GCW-complex

one can check f_i^*EG are GCW-complexes

define the principal G-bundle $\bar{P} \to M \times [0,1]$ by

$$\bar{P} = \left[(f_0^*EG) \times [0, 1/2]\right] \cup_\Phi \left[(f_1^*EG) \times [1/2, 1]\right]$$

over $M \times \{0,1\}$ be define H by bundle maps covering f_i

$$\begin{array}{ccc} f_i^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ M & \xrightarrow{f_i} & BG \end{array}$$

just as in the surjective case we can extend this to an equivariant map

$$H: \bar{P} \to EG$$

that induces a homeo on fibers

and H induces a map on the quotient spaces

$$F: M \times [0,1] \to BG$$

that is a homotopy f_0 to f_1 ▦

Proof of Th$^{\underline{m}}$ 14:

suppose $\begin{matrix} E_1 \\ \downarrow \\ B_1 \end{matrix}$ and $\begin{matrix} E_2 \\ \downarrow \\ B_2 \end{matrix}$ are both

universal bundles

by Th$^{\underline{m}}$ 13, $\exists$ maps $f: B_1 \to B_2$ and $g: B_2 \to B_1$

such that $E_1 \cong f^* E_2$ and

$$E_2 \cong g^* E_1$$

now $g \circ f: B_1 \to B_1$

$$\begin{aligned} \text{and } (g \circ f)^* E_1 &= f^*(g^*(E_1)) \\ &\cong f^*(E_2) \\ &\cong E_1 \end{aligned}$$

thus $(g \circ f)^* E_1 = id_{B_1}^* E_1$

so by Th$^{\underline{m}}$ 13 $g \circ f \simeq id_{B_1}$

similarly $f \circ g \simeq id_{B_2}$

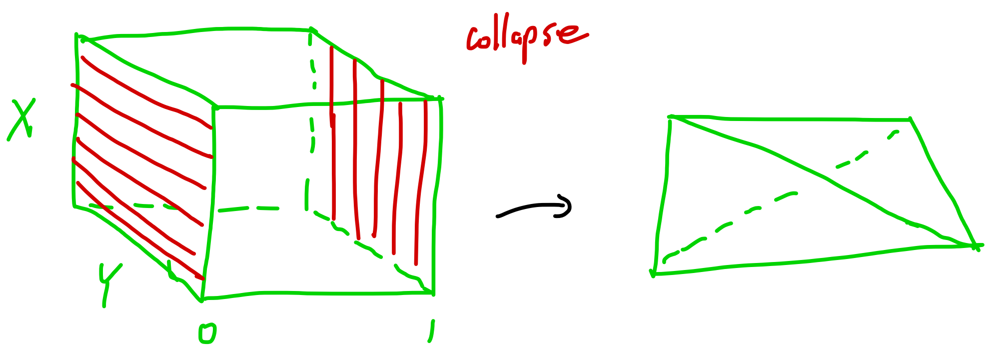
so $B_1 \simeq B_2$ ∎

we are left to see classifying spaces exist, for this we need

if X, Y are two spaces, then their join is

$$X * Y = X \times [0,1] \times Y / \sim$$

where $(x, 0, y_1) \sim (x, 0, y_2) \quad \forall y_1, y_2 \in Y$
and $(x_1, 0, y) \sim (x_2, 0, y) \quad \forall x_1, x_2 \in X$



note: there is an inclusion

$$X \xrightarrow{i} X * Y$$
$$x \longmapsto (x, 0, y) \quad \text{for any } y \in Y$$

and

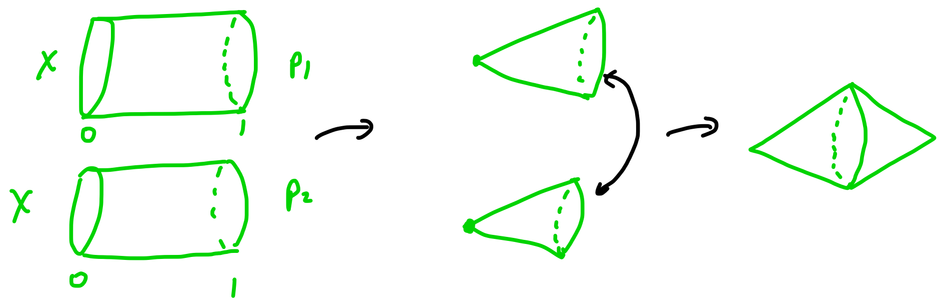
$$Y \xrightarrow{i} X * Y$$
$$y \longmapsto (x, 1, y) \quad \text{for any } x \in X$$

examples:

1) $X * \{p\} \cong CX$

indeed $X * \{p\} = X \times [0,1] / X \times \{0\}$

2) similarly $X * \{p_1, p_2\} \cong \Sigma X$



3) $\{x_0\} * \{x_1\} * \dots * \{x_k\}$ is a k-simplex

4) exercise: $S^n * S^m \cong S^{n+m+1}$

lemma 15:

> the inclusions $i: X \to X * Y$ and
> $j: Y \to X * Y$ from above are null-homotopic

Proof: for any $y_0 \in Y$ not $i: X \to X * Y$ factors through

$$X \hookrightarrow X * \{y_0\} \subset X * Y$$
$$x \mapsto (x, 0, y_0)$$

but $X * \{y_0\} \cong CX$ and hence is contractible

$\therefore i$ null-homotopic

similarly for j

$\boxplus$

now given a topological group G

let $G^{*(k+1)} = \underbrace{G * G * \cdots * G}_{k+1 \text{ times}}$

this has a G action given by

$$(g_0, t_1, g_1, t_2, \ldots t_k, g_k) \cdot g = (g_0 g, t_1, g_1 g, t_2, \ldots, t_k, g_k g)$$

exercise:

1) Prove that $\exists$ a natural G-equivariant map
$$\Delta^k \times G^{k+1} \longrightarrow G^{*(k+1)}$$
that is a homeomorphism when restricted to $\operatorname{int} \Delta^k \times G^{k+1}$
(here G acts trivially on Δ^k and diagonally on G^{k+1})

2) Use above to show $G^{*(k+1)}$ has the structure of a GCW-complex

let $\mathcal{J}(G) = \lim_{k \to \infty} G^{*(k+1)}$

Th^m 16:

$$\boxed{\text{the quotient map } p: \mathcal{J}(G) \to \mathcal{J}(G)/G \text{ is a universal principal } G\text{-bundle}}$$

Proof: prove $p: \mathcal{J}(G) \to \mathcal{J}(G)/_G$ is a principal G-bundle

to show $\mathcal{J}(G)$ is weakly contractible note that

for any map $\alpha: S^n \to \mathcal{J}(G)$

$\exists k$ s.t. $\alpha(S^n) \subset G^{*(k+1)} \subset \mathcal{J}(G)$

and $G^{*(k+1)}$ is null-homotopic in $G^{*(k+2)} \subset \mathcal{J}(G)$ by lemma 15

$\therefore \alpha$ null-homotopic in $\mathcal{J}(G)$ $\blacksquare$

from the construction above note that given

$f: H \to G$ a homomorphism

then we get an induced map

$$\begin{array}{ccc} Ef: EH & \to & EG \\ \shortparallel & & \shortparallel \\ \mathcal{J}(H) & & \mathcal{J}(G) \end{array}$$

and $Bf: BH \to BG$

exercise:

1) Bf is the classifying map for the bundle $BH \times_f G \to BH$

2) If $H < G$ and $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ a principal G-bundle show the structure group of P reduces to H iff the classifying map $f: M \to BG$ (after homotopy) to $M \to BH$

$$\begin{tikzcd} & BH \arrow[d, "Bi"] \\ M \arrow[ur, dashed] \arrow[r, "f"'] & BG \end{tikzcd}$$

$i: H \to G$ inclusion

Here is another view of characteristic classes

Th$^{\underline{m}}$ 17:

$$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2\,[w_1, w_2, \ldots w_n]$$

where w_i has grading i

$$H^*(BU(n); \mathbb{Z}/2) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

where c_i has grading $2i$

Remark: we might prove this later, but we can use Th$^{\underline{m}}$ 17 to define characteristic classes

given an $\mathbb{R}^n$-bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ there is an associated $O(n)$-bundle $\mathcal{F}(E)$ and by Th$^{\underline{m}}$ 13 $\exists$ a map $f: M \to BO(n)$ s.t. $f^* EO(n) \cong \mathcal{F}(E)$

define $w_i(E) = f^* w_i$

similarly for c_i

What about Pontryagin classes or Euler class?

recall given a short exact sequence

$$0 \to H \to G \to K \to 0$$

of abelian groups, we get a long exact sequence

$$\cdots \to H_n(M;H) \to H_n(M;G) \to H_n(M;K) \to H_{n-1}(M;H) \to \cdots$$

so from $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

we get

$$H_n(M) \xrightarrow{\times 2} H_n(M) \to H_n(M;\mathbb{Z}/2) \xrightarrow{\beta} H_{n-1}(M)$$

β is called the Bockstein map

Thm 18:

$$H^*(BSO(2n+1);\mathbb{Z}) = \mathbb{Z}[p_1, \dots, p_n] \oplus \text{Torsion}$$

where $\text{Torsion} = \beta(H^*\underbrace{(BSO(2n+1);\mathbb{Z}/2)}_{\mathbb{Z}/2[w_2, \dots, w_{2n+1}]})$

$$H^*(BSO(2n);\mathbb{Z}) = \mathbb{Z}[p_1, \dots, p_n, e] \big/ \langle e^2 = p_n \rangle \oplus \text{Torsion}$$

where Torsion is as above