B Fibrations

if $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a Serre fibration with B a CW-complex

let $B^{(k)} = k$-skeleton of B

set $E^k = p^{-1}(B^{(k)})$

this is a filtration of E

$$\emptyset = E^{-1} \subset E^0 \subset \dots \subset E^n = E$$

and induces a filtration of $C_*(E)$

$$F_s C_*(E) = C_*(E^s)$$

and the homology

$$F_s H_*(E) = \operatorname{im}\left(H_*(E^s) \to H_*(E)\right)$$

so by Th$^{\underline{m}}$2 we have an E^1-spectral sequence with

- $E^1_{s,t} = H_{s+t}\left(C_*(E^s) / C_*(E^{s-1})\right)$
- d^1 connecting homomorphis in long exact sequence for (E^s, E^{s-1}, E^{s-2})
- $G(H_*(X))_{s,t} = E^\infty_{s,t}$

lemma 3:

$$\boxed{\begin{array}{l} \text{if } \pi_1(B) = 0 \text{ and } B \text{ connected then} \\ \quad E^1_{s,t} = H_{s+t}\left(C_*(E^s) / C_*(E^{s-1})\right) = C^{CW}_s(B; H_t(F)) \\ \text{where } F \text{ is the homotopy fiber of } p: E \to B \end{array}}$$

Proof: we consider the case of a locally trivial fibration (this just makes things easier)

$$H_{s+t}\left(C_*(E^s)/C_*(E^{s-1})\right) \underset{\text{by def}^{\underline{n}}}{=} H_{s+t}\left(C_*(E^s, E^{s-1})\right) \underset{\text{by def}^{\underline{n}}}{=} H_{s+t}(E^s, E^{s-1})$$

$$\underset{\text{excision}}{\cong} H_{s+t}\left(E^s/E^{s-1}, E^{s-1}/E^{s-1}\right) \cong \tilde{H}_{s+t}\left(E^s/E^{s-1}\right)$$

let σ_i^s be an s-cell of B

$$E^s/E^{s-1} = p^{-1}(B^{(s)})/p^{-1}(B^{(s-1)}) = \bigvee_i p^{-1}(\sigma_i^s)/p^{-1}(\partial\sigma_i^s)$$

Claim: $p^{-1}(\sigma_i^s)/p^{-1}(\partial\sigma_i^s) \cong D^s \times F / S^{s-1}\times F$

Pf:

$$\begin{array}{ccccccccc} S^{s-1}\times F & \subset & D^s\times F = j^*E & \xrightarrow{\tilde{j}} & E|_{\sigma_i^s} & \subset & E \\ & & \downarrow & & \downarrow & & \downarrow \\ S^{s-1} & \subset & D^s & \xrightarrow[j]{} & \sigma_i^s & \subset & B \end{array}$$

$j|_{\text{int } D^s}$ is a homeo onto $\text{int }\sigma_i^s$

and $j|_{S^{s-1}}$ is onto $\partial\sigma_i^s$

so $\tilde{j}$ induces a homeo

$$D^s\times F/S^{k-1}\times F \to E|_{\sigma^s}/E|_{\partial\sigma^s} = p^{-1}(\sigma_i^s)/p^{-1}(\partial\sigma_i^s) \checkmark$$

now $C_s^{CW}(B; H_t(F)) \underset{\text{def}^{\underline{n}}}{=} H_s(B^{(s)}, B^{(s-1)}; H_t(F))$

$$= \bigoplus_{s\text{-cells}} H_t(F)$$

also
$$H_{s+t}\left(E^s / E^{s-1} \right) = H_{s+t}\left(\bigvee_i p^{-1}(\sigma_i^s) / p^{-1}(\partial \sigma_i^s) \right)$$
$$= \bigoplus_{s\text{-cells}} H_{s+t}\left(p^{-1}(\sigma_i^s) / p^{-1}(\partial\sigma_i^s) \right)$$
$$= \bigoplus_{s\text{-cells}} H_{s+t}\left(D^s \times F / S^{s-1} \times F \right)$$

so we will be done if
$$H_{s+t}\left(D^s \times F / S^{s-1} \times F \right) \cong H_t(F)$$

exercise: for $p>0$ $H_p(Y) \cong H_{p+1}(\Sigma Y)$
where ΣY is the suspension of Y

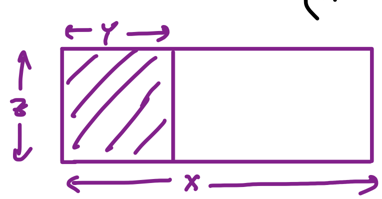
recall $\Sigma Y = S^1 \wedge Y = S^1 \times Y / S^1 \vee Y$

$$S^p = \underbrace{S^1 \wedge \dots \wedge S^1}_{p \text{ times}}$$

so $\Sigma^s Y = S^1 \wedge \dots \wedge S^1 \wedge Y = S^s \wedge Y$

note: $\exists$ an S^s in $D^s \times F / S^{s-1} \times F$
just take $D^s \times \{pt\} / S^{s-1} \times \{pt\} = S^s$

exercise: for any spaces X, Y, Z with $Y \subset X$
$$\frac{X \times Z}{(Y \times Z) \cup (X \times \{pt\})} \cong \frac{X/Y \times Z}{(\{pt\} \times Z) \vee \left(\frac{X \times \{pt\}}{Y \times \{pt\}} \right)}$$



from this $\left(\frac{D^s\times F}{S^{s-1}\times F}\right)\Big/ S^s = \frac{D^s\times F}{(S^{s-1}\times F)\cup(D^s\times\{pt\})}$

from above

$$\cong \frac{S^s\times F}{(\{pt\}\times F)\vee S^s\times\{pt\}}$$

$$= S^s\wedge F$$

so for $t>1$, the long exact sequence of a pair gives

$$\underbrace{H_{s+t}(S^s)}_{=0}\to H_{s+t}\left(D^s\times F\big/ S^{s-1}\times F\right)\xrightarrow{\cong} H_{s+t}\left(D^s\times F\big/ S^{s-1}\times F,\ S^s\right)\to \underbrace{H_{s+t-1}(S^s)}_{=0}$$

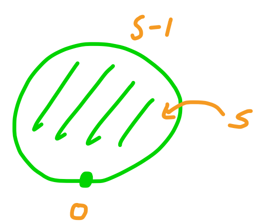
$$H_{s+t}\left(D^s\times F\big/ S^{s-1}\times F,\ S^s\right)\cong H_{s+t}(\Sigma^s\wedge F)\cong H_t(F)$$

exercise: think about $t=0,1$

this finishes the proof, but note if F a CW complex

then $(s+t)$-cells of $D^s\times F$ come in 3-types

$D^s = (0\text{-cell})\cup((s-1)\text{-cell})\cup(s\text{-cell})$



so $(s+t)$-cells of $D^s\times F$ are

(1) $(0\text{-cell of } D^s)\times((s+t)\text{-cells of } F)$

(2) $((s-1)\text{-cell of } D^s)\times((t+1)\text{-cell of } F)$

(3) $(s\text{-cell of } D^s)\times(t\text{-cell of } F)$

and $H_{s+t}\left(D^s\times F\big/ S^{s-1}\times F\right) = H_{s+t}\left(\frac{C_*(D^s\times F)}{C_*(S^{s-1}\times F)}\right)$

$= H_{s+t}(C_*(\text{cells of type }(3)))$

$= H_t(F)$

Remark: why do we need $\pi_1(B)=0$?

to get $H_{s+t}\left(C_*(E^s)/C_*(E^{s-1})\right) \cong C_s(B; H_t(F))$

we needed to identify $E|_{\sigma^s} / E|_{\partial\sigma^s}$ with $D^s \times F / S^{s-1}\times F$

there are potentially many such identifications

but if we fix one identification of $p^{-1}(pt)$ with $\{pt\}\times F$

then the identification if fixed

so if we fix an identification of $p^{-1}(pt)$ for a point in interior of each s-cell in B then we are OK, but there are many such choices.

if $\pi_1(B)=0$, or $\pi_1(B)$ acts trivially on $H_t(F)$

then we can make one choice once and for all

if not, then lemma still OK but need to use "twisted coefficients"

now $d^1: E^1_{s,t} \to E^1_{s-1,t}$ is connecting homomorphism in L.E.S. of (E^s, E^{s-1}, E^{s-2})

$$\begin{array}{ccc} H_{s+t}(E^s, E^{s-1}) & \xrightarrow{\partial} & H_{s+t-1}(E^{s-1}, E^{s-2}) \\ \wr\vert & & \wr\vert \\ C_s(B; H_t(F)) & & C_{s-1}(B; H_t(F)) \end{array}$$

exercise: show $d' =$ boundary map for chain complex $C_*(B; H_t(F))$

$\therefore$ E^2-term of spectral sequence is

$$E^2_{s,t}(B; H_t(F))$$

so we have proved

Thm 4 (Leray-Serre spectral sequence):

if $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a Serre fibration and B is a simply connected CW-complex (or $\pi_1(B)$ acts trivially on $H_*(F)$)

then there is a spectral sequence converging to $G(H_*(E))$ with

$$E^2_{s,t} = H_s(B; H_t(F))$$

here are some consequences

Thm 5:

$$H_q(\Omega S^k) = \begin{cases} \mathbb{Z} & q = a(k-1) \quad a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$k \geq 2$

Proof: recall we have

$$\begin{tikzcd} \Omega S^k \arrow[r] & PS^k \simeq * \arrow[d] \\ & S^k \end{tikzcd}$$

path space

$\pi_1(S^k)=0$ for $k\geq 2$

$$E^2_{s,t} = H_s(S^k; H_t(\Omega S^k)) = \begin{cases} H_t(\Omega S^k) & s=0,k \\ 0 & s\neq 0,k \end{cases}$$

if we write H_t for $H_t(\Omega S^k)$ we have

$$\begin{array}{c|ccccc} & & \vdots & & \vdots & \\ & & H_3 & & H_3 & \\ & & H_2 & & H_2 & \\ & 0 & H_1 & 0 & H_1 & 0 \\ \hline & & H_0 & & H_0 & \\ & & & & s=k & \\ & & & 0 & & \end{array}$$

only nontrivial differential is d^k

so $E^2 = E^3 = \dots = E^k$

and $E^{k+1} = E^{k+2} = \dots = E^\infty$

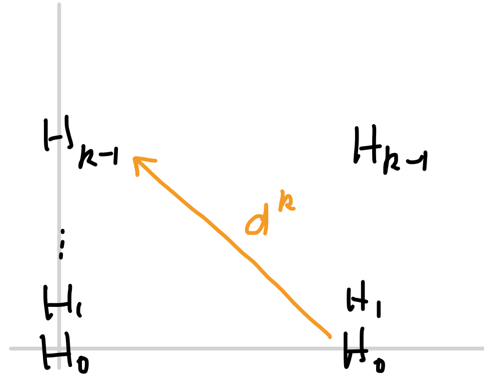
$E^\infty_{s,t} = G(H_*(PS^k))_{s,t}$

but $H_p(PS^k) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & p\neq 0 \end{cases}$

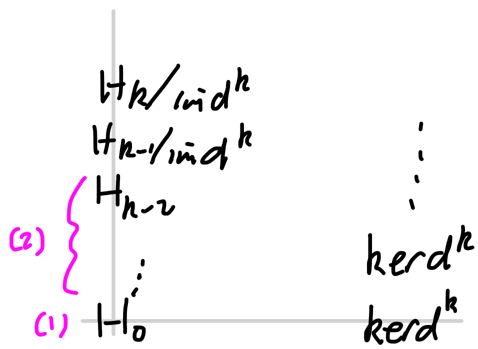
so $E^\infty_{s,t}$ is

$$\begin{array}{|cccc} \vdots & & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \cdots \\ \hline \mathbb{Z} & 0 & 0 & \end{array}$$

now look at E^k



so $E^\infty = E^{k+1}$ is



so H_0 by (1)

$H_1 = \cdots = H_{k-2} = 0$ by (2)

and $d^k : H_0 \to H_{k-1}$ is an isomorphism

since $\ker d^k = 0$ and

$H_{k-1}/\mathrm{im}\, d^k = 0$

actually $d^k : H_\ell \to H_{\ell+k-1}$ is an isomorphism $\forall \ell \geq 0$

$$\therefore H_q(\Omega S^k) = \begin{cases} \mathbb{Z} & q = a(k-1) \\ 0 & \text{otherwise} \end{cases}$$

Th^{m} 6 (Gysin Sequence)

let $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ be a fibration with fiber $F = S^n$

assume $\pi_1(B)$ acts trivially on $H_*(F)$

then $\exists$ an exact sequence

$$H_r(E) \xrightarrow{p_*} H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \xrightarrow{p_*} H_{r-1}(B) \to \dots$$

Proof:

we have a spectral sequence converging to $G(H(E))_{s,t}$ with

$$E^2_{s,t} = H_s(B; H_t(F)) = \begin{cases} H_s(B) & t = 0, n \\ 0 & t \neq 0, n \end{cases}$$

$$\begin{array}{r|llll} t=n & H_0 & H_1 & H_2 & H_3 \ \dots \\ & & & & \\ t=0 & H_0 & H_1 & H_2 & H_3 \ \dots \\ \hline \end{array}$$

all other entries 0

so the only non-trivil differential is

$$d^{n+1} : E^{n+1}_{s,0} \to E^{n+1}_{s-n-1,n}$$

$$(\text{so } E^2 = E^3 = \dots = E^{n+1}$$
$$\text{and } E^{n+2} = \dots = E^\infty)$$

so we have

$E^\infty = E^{n+2}$

$$\begin{array}{c|cccccc} & H_0/\operatorname{im} d^{n+1} & H_1/\operatorname{im} d^{n+1} & \cdots & & & \\ & & & & & \cdots & \\ \hline & H_0 & H_1 & \cdots & H_n & \ker d^{n+1} & \ker d^{n+1} \cdots \end{array}$$

so from lemma 1 we have

for $k \geq n+1$

$$0 \to \frac{H_{k-n}}{\operatorname{im} d^{n+1}} \to H_k(E) \to \ker d^{n+1} \to 0$$

$$\frac{H_{k-n}}{\operatorname{im} d^{n+1}} = \frac{H_{k-n}(B)}{\operatorname{im}\left(d^{n+1} : H_{k+1}(B) \to H_{k-n}(B)\right)}, \qquad \ker d^{n+1} \subset H_k(B)$$

so

$$H_{k+1}(B) \xrightarrow{d^{n+1}} H_{k-n}(B) \to H_k(E) \to H_k(B) \xrightarrow{d^{n+1}} H_{k-n-1}(B)$$

is exact

so we are done except for identifying the map $H_k(E) \to H_k(B)$ as p_*

exercise: Check this (not so easy)

note: we also know $H_k(E) \cong H_k(B)$ for $k = 0, \dots, n-1$

exercise: if $\begin{array}{c} F \to E \\ \downarrow \\ S^n \end{array}$ is a fibration show

there is an exact sequence

$$\cdots \to H_r(F) \to H_r(E) \to H_{r-n}(F) \to H_{r-1}(F) \to \cdots$$

this is called the Wang exact sequence

Thm 7 (Leray-Serre for cohomology):

let $\begin{array}{c} F \to E \\ \downarrow p \\ B \end{array}$ be a Serre fibration with B a CW-complex s.t.

$\pi_1(B)$ acts trivially on $H^*(F)$

then there is a spectral sequence converging to $G(H^*(E))^{s,t}$ with $E_2^{s,t} = H^s(B; H^t(F))$ and

1) $\{E_r^{s,t}\}$ is a bigraded algebra

i.e. $\exists$ product $E_r^{s,t} \times E_r^{p,q} \to E_r^{s+p,t+q}$

2) $d_r : E_r \to E_r$ is a derivation

i.e. if $a \in E_r^{p,q}$, $b \in E_r^{s,t}$ then

$$d_r(a \cdot b) = (d_r a) \cdot b + (-1)^{p+q} a \cdot (d_r b)$$

3) $E_2^{*,0} \cong H^*(B)$ as rings and

$E_2^{0,*} \cong H^*(F)$ " "

what is the product structure on $E_2^{s,t}$?

$$H^p(B; H^q(F)) \times H^s(B; H^t(F)) \to H^{p+s}(B; H^q(F) \otimes H^t(F))$$

$$\begin{array}{ccc} \alpha \in H^p(B; H^q(F)), & \beta \in H^s(B; H^t(F)) & \longmapsto \alpha \cup \beta : C_{p+s}(B) \to H^q(F) \otimes H^t(F) \end{array}$$

now compose with cup product on $H^q(F) \otimes H^t(F)$ to get

$$\alpha \cup \beta : C_{p+s}(B) \to H^{q+t}(F)$$

so $E_2^{q,p} \times E_2^{s,t} \longrightarrow E_2^{q+s,p+t}$

example: let's compute the cohomology ring of $\mathbb{C}P^n$

we will do $\mathbb{C}P^2$, but general case similar

recall $\mathbb{C}P^2$ is simply connected and we have

$$\begin{array}{ccc} S^1 & \to & S^5 \\ & & \downarrow \\ & & \mathbb{C}P^2 \end{array}$$

so the Leray-Serre spectral sequence gives

$$E_2^{s,t} = H^s(\mathbb{C}P^2; H^t(S^1)) = \begin{cases} H^s(\mathbb{C}P^2) & t = 0,1 \\ 0 & t \neq 0,1 \end{cases}$$

let α be the generator of $H^2(\mathbb{C}P^2; H^0(S^1)) \cong H^2(\mathbb{C}P^2)$

β " " of $H^0(\mathbb{C}P^2; H^1(S^1)) \cong H^1(S^1)$

universal coefficients says $\alpha \otimes \beta$ is the generator of $H^2(\mathbb{C}P^2; H^1(S^1))$

so E_2 is

rest $= 0$

$$\begin{array}{c|ccccc} & H^0(\mathbb{C}P^2; H^1(S^1)) & 0 & H^2(\mathbb{C}P^2; H^1(S^1)) & 0 & H^4(\mathbb{C}P^2; H^1(S^1)) \\ & H^0(\mathbb{C}P^2; H^0(S^1)) & 0 & H^2(\mathbb{C}P^2; H^0(S^1)) & 0 & H^4(\mathbb{C}P^2; H^0(S^1)) \\ \hline \end{array}$$

so

$$\begin{tikzcd}
\mathbb{Z}_\beta \arrow[rrd, "d_2"] & 0 & \mathbb{Z}_{\alpha\otimes\beta} \arrow[rrd, "d_2"] & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z}_\alpha & 0 & \mathbb{Z}
\end{tikzcd}$$

note: since $H^n(S^5) \cong \begin{cases} \mathbb{Z} & n = 0, 5 \\ 0 & n \neq 0, 5 \end{cases}$

must have E^∞

$$\begin{tikzcd}
0 & 0 & 0 & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & 0 & 0 & 0
\end{tikzcd}$$

so the two d_2's above must be isomorphisms (since $d_r = 0$ for $r > 2$)

$\therefore d_2 \beta = \alpha$ (or $-\alpha$, but then replace α with $-\alpha$)

thus $d_2\, \alpha\otimes\beta = \underbrace{d_2\alpha}_{=0} \otimes \beta + \underbrace{\alpha \otimes \underbrace{d_2\beta}_{=\alpha}}_{(\alpha\otimes 1)\cup(1\otimes\alpha)} = \alpha\cup\alpha$

see comment about product structure above

$\therefore \alpha\cup\alpha$ generates $H^4(\mathbb{C}P^2)$

clearly $\alpha\cup\alpha\cup\alpha = 0$ since $H^6(\mathbb{C}P^2) = 0$

thus $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\langle\alpha^3\rangle$ with degree$(\alpha) = 2$

<u>exercise</u>: show $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/\langle\alpha^{n+1}\rangle$

and $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\alpha]$

Th$^{\underline{m}}$ 8:

$$\boxed{\begin{array}{c} H^*(U(n)) \cong \Lambda(x_1, \dots, x_{2n+1}) \\ \text{with degree } x_i = i \end{array}}$$

exterior algebra

Proof: recall $V_{n,1}(\mathbb{C}) = U(n)/U(n-1) \cong S^{2n-1}$

so we have a bundle

$$\begin{array}{ccc} U(n-1) & \to & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

so we compute cohomology inductively

$U(1) \cong S^1$ so $H^k(U(1)) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\langle x_1 \rangle & k=1 \\ 0 & k \neq 0,1 \end{cases}$

$\cong \Lambda(x_1)$

for $U(2)$ we have

$$\begin{array}{ccc} S^1 & \to & U(2) \\ & & \downarrow \\ & & S^3 \end{array}$$

so the E_2 term in Leray-Serre is

$$E_2^{s,t} = H^s(S^3; H^t(S^1))$$

$$\begin{array}{cccc} H^0(S^3) \otimes H^1(S^1) & 0 & 0 & H^3(S^3) \otimes H^1(S^1) \\ H^0(S^3) & 0 & 0 & H^3(S^3) \end{array}$$

all maps $d_2 = d_3 = \dots = 0$

so E_∞ is same

$$
\begin{array}{c|cccc}
 & \mathbb{Z}_{x_1} & 0 & 0 & \mathbb{Z}_{x_1\cdot x_3} \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z}_{x_3} \\
\hline
\end{array}
$$

$$
\therefore \quad H^k(U(2)) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}_{x_1} & 1 \\ 0 & 2 \\ \mathbb{Z}_{x_3} & 3 \\ \mathbb{Z}_{x_1\cdot x_3} & 4 \\ 0 & \geq 5 \end{cases}
$$

$$
\cong \Lambda(x_1, x_3) \qquad \deg x_i = i
$$

we could continue doing this until $n=5$

there we have
$$
\begin{array}{ccc}
U(4) & \to & U(5) \\
 & & \downarrow \\
 & & S^9
\end{array}
$$

$$
\begin{array}{c|ccccc}
8 & \mathbb{Z}_{x_1\cdot x_7} \oplus \mathbb{Z}_{x_3\cdot x_5} & & & & \\
7 & \mathbb{Z}_{x_7} & & & & \\
6 & \mathbb{Z}_{x_1\cdot x_5} & & & & \\
5 & \mathbb{Z}_{x_5} & & & & \\
4 & \mathbb{Z}_{x_1\cdot x_3} & & & & \\
3 & \mathbb{Z}_{x_3} & & & & \vdots \\
2 & 0 & & & & \\
1 & \mathbb{Z}_{x_1} & & & & \mathbb{Z}_{x_1\cdot x_9} \\
0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z}_{x_9} \\
\hline
 & 0 & 1 & \cdots & 8 & 9
\end{array}
$$

d_9

$$
d_9(x_1 x_7) = (d_9 x_1)\cdot x_7 - x_1 \cdot d_9 x_7 = 0
$$

similarly for $d_9(x_3 \cdot x_5) = 0$

so d_q and $d_k = 0 \quad \forall k$

$\therefore E_\infty = E_2$

for each k on the diagonal $s+t=k$
at most 2 non-zero terms

so lemma 1 gives

$$0 \to E_\infty^{0,s+t} \to H^{s+t}(U(n)) \to E_\infty^{q,s+t-q} \to 0$$

free so sequence splits

$$\therefore H^k(U(n)) = E_\infty^{0,k} \oplus E_\infty^{q,k-q}$$

exercise: all products non-zero unless they have to be so get

$$H^*(U(5)) = \Lambda(x_1 \dots x_q)$$

the $n>5$ cases similar

Remark: from this you can compute

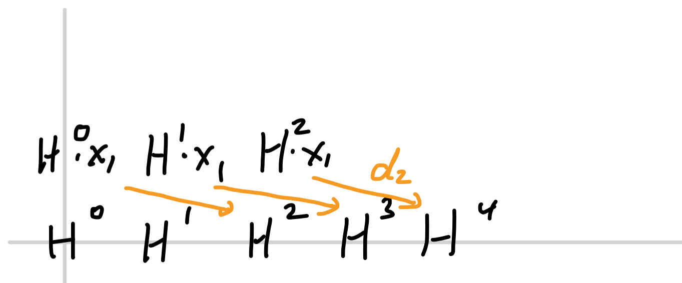
$$H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$$

from Th^{m} 3.17

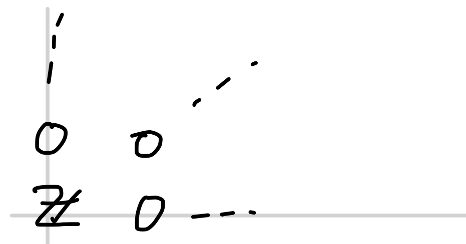
for example: $H^*(BU(1)) \cong \mathbb{Z}[c_1]$

to see this recall
$$\begin{array}{ccc} U(1) & \longrightarrow & EU(1) \simeq \{*\} \\ \parallel & & \downarrow \\ S^1 & & BU(1) \end{array}$$

so $E_2^{s,t} = H^s(BU(1); H^t(S^1)) = \begin{cases} H^s(BU(1)) & t=0 \\ H^s(BU(1))\cdot x_1 & t=1 \\ 0 & t\neq 0,1 \end{cases}$



since $E_\infty^{s,t} = G(H^*(EU(1)))_{st}$ is



and $d^k = 0 \ \forall k > 2$

we must have $H^1 = 0$ and d_2 an isomorphism

$$d_2 : \mathbb{Z}_{\langle x_1 \rangle} \longrightarrow H^2(BU(1)) \quad \cong$$

i.e. $H^2(BU(1)) \cong \mathbb{Z}_{c_1}$ where

$$c_2 = d_2 x_1$$

$H^1\cdot x_1 = 0$ and $d_2 : H^1_{x_1} \to H^3$ $\cong$ so

$H^3 = 0$ similarly $H^{2n+1}(BU(1)) = 0$

$$d_2 : H^2\cdot x_1 \longrightarrow H^4 \quad \cong$$

so $d_2(c_1\cdot x_1) = \cancel{dc_1}^{\,0}\cdot x_1 + c_1\cdot dx_1 = c_1\cdot c_1$

generates $H^4(BU(1))$

similarly $\underbrace{c_1\cdot c_1\cdots c_1}_{n}$ generates $H^{2n}(BU(1))$

ie. $H^*(BU(1)) \cong \mathbb{Z}[c_1]$

now lets try $BU(2)$

$$\begin{array}{ccc} U(2) \to & EU(2) \simeq \{*\} \\ & \downarrow \\ & BU(2) \end{array}$$

we know $H^*(U(2)) \cong \Lambda(x_1, x_2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}_{x_1} & 1 \\ 0 & 2 \\ \mathbb{Z}_{x_3} & 3 \\ \mathbb{Z}_{x_1 x_3} & 4 \end{cases}$

and $H^*(EU(2)) = \mathbb{Z}$

so the Leray-Serre spectral sequence has E_∞ term

$$\begin{array}{cccc} \vdots & & & \iddots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \mathbb{Z} & 0 & 0 & \cdots \end{array}$$

and $E_2^{s,t} = H^s(BO(2); H^t(U(2)))$ write H^s for $H^s(BU(2))$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & & 0 & & 0 \\
H^0_{x_1 x_3} \arrow[rrd, "d_2"] & 0 & H^2_{x_1 x_3} \arrow[rrd, "d_2"] & 0 & H^4_{x_1 x_2} \\
H^0_{x_3} \arrow[rrd, "d_2"] & 0 & H^2_{x_3} & 0 & H^4_{x_3} \\
0 & 0 & 0 & 0 & 0 \\
H^0_{x_1} \arrow[rrd, "d_2"] & 0 & H^2_{x_1} & 0 & H^4_{x_1} \\
H^0 & 0 & H^2 & 0 & H^4
\end{tikzcd}$$

must be 0 or lives to ∞

same once we know $H^1 = 0$ so $E_2^{1,n} = 0$

$d_2 : E_2^{0,1} \to E_2^{2,0}$ must be an isomorphism
or $E^{0,1}$ lives to ∞ or a
quotient of $E^{2,0}$ does

so $H^2 \cong \mathbb{Z}$ gen by $c_1 = dx_1$

thus $c_1 \cdot x_1$ generates $E_2^{2,1} \cong \mathbb{Z}$
and $d_2 c_1 \cdot x_1$ can't be 0 or $E^{2,1}$ lives to ∞
so $d_2 c_1 \cdot x_1 = c_1 \cdot c_1$ is a generator of H^4

$d_2 x_3 = 0$ since target group 0

$d_2\, x_1 x_3 = c_1 \cdot x_3$ is the generator
of $E_2^{2,3} = H^2_{x_3}$

$d_2\, x_1 x_3 c_1 = c_1 \cdot x_3 \cdot c_1$ is a generator
of $E_2^{4,3}$

so $E_3^{s,t}$ is

$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
0	0	0	0	$H^4_{x_1 x_3}$	0
$H^0_{x_3}$	0	0	0	$H^4_{x_3} / x_3 \cdot c_1^2$	0
0	0	0	0	0	0
0	0	0	0	$H^4_{x_1}$	0
H^0	0	0	0	$H^4 / c_1 c_1$	0

must be 0
or live to ∞

now can go back and update $E_2^{s,t}$

$$\begin{tikzcd}[row sep=small, column sep=small]
0 & \vdots & 0 & \vdots & 0 & \vdots & & \vdots & \\
H^0_{x_1x_3} \arrow[drr, "d_2"] & 0 & H^2_{x_1,x_3} \arrow[drr, "d_2"] & 0 & H^4_{x_1x_3} & 0 & H^6_{x_1x_3} & 0 & H^8_{x_1x_3} \\
H^0_{x_3} \arrow[drr, "d_2"] & 0 & H^2_{x_3} & 0 & H^4_{x_3} & 0 & H^6_{x_3} & 0 & H^8_{x_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H^0_{x_1} \arrow[drr, "d_2"] & 0 & H^2_{x_1} & 0 & H^4_{x_1} \arrow[drr, "d_2"] & 0 & H^6_{x_1} \arrow[drr, "d_2"] & 0 & H^8_{x_1} \\
H^0 & 0 & H^2 & 0 & H^4 & 0 & H^6 & 0 & H^8
\end{tikzcd}$$

must be 0 since only thing that could kill it dies at E_3

$d_2\, x_1c_1\cdot c_1 = c_1\cdot c_1\cdot c_1$ must be non zero since only thing that could kill it dies in E_3

similarly $d_2\, x_1c_1c_1c_1 = c_1^4$ must be nonzero

so now E_3 is

$$\begin{tikzcd}[row sep=small, column sep=small]
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & 0 & \vdots & \\
0 & 0 & 0 & 0 & \ker d_2 & 0 & \ker d_2 & 0 & \ker d_2 \\
H^0_{x_3} \arrow[dddrrrr, "d_4"] & 0 & 0 & 0 & H^4_{x_3}/x_3\cdot c_1^2 & 0 & H^6_{x_3}/x_3c_1^3 & 0 & H^8_{x_3}/x_3c_1^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ker d_2 & 0 & \ker d_2 & 0 & ? \\
H^0 & 0 & 0 & 0 & H^4/c_1c_1 & 0 & H^6/c_1^3 & 0 & H^8/c_1^4
\end{tikzcd}$$

$d_3 = 0$ in region shown so $E_4 = E_3$

$d_4 : H^{0,3}_4 \to H^{4,0}_4$ must be an isomorphism or something lives to ∞

set $c_2 = d_4 x_3$

and $H^4/\langle c_1^2\rangle \cong \mathbb{Z}\langle c_2\rangle$

so $H^4 \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by c_1^2 and c_2

but now back to E_2 again and notice

$$
\begin{aligned}
d_2 : E_2^{4,1} &\to E_2^{6,0} \\
c_1^2 x_1 &\mapsto c_1^3 \\
c_2 x_1 &\mapsto c_2 c_1
\end{aligned}
$$

must be an isomorphism or a subgroup of $E^{4,1}$ lives to ∞

$\therefore H^6 = E_2^{6,0} = \mathbb{Z} \oplus \mathbb{Z}$ generated by $c_1^3,\ c_1 c_2$

now E_3 is

$$
\begin{tikzcd}
 & & & & \vdots & & \vdots & & \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & 0 & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & \ker d_2 & 0 & \ker d_2 \\
H^0_{x_3} & 0 & 0 & 0 & H^4_{x_3}/x_3 c_1^2 \arrow[ddddrrrr, "d_4"] & 0 & H^6_{x_3}/x_3 c_1^3 & 0 & H^8_{x_3}/x_3 c_1^4, x_3 c_2 c_1^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? \\
H^0 & 0 & 0 & 0 & H^4/c_1 c_1 & 0 & 0 & 0 & H^8/c_1^4, c_2 c_1^2
\end{tikzcd}
$$

again $E_3 = E_4$

$d_4 : E_4^{4,3} \to E_4^{8,0}$ must be an isomorphism

so $H^8 = E_4^{8,0} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

generated by $c_1^4, c_1^2 c_2, c_2^2$

let's inductively assume that for $k \leq 4n$

$$H^k(BU(2)) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \text{ odd} \\ \text{freely gen. by } c_1^{2\ell}, c_1^{2\ell-2}c_2, \dots, c^{\ell} & k = 4\ell \\ \text{"} \quad \text{"} \quad c_1^{2\ell+1}, c^{2\ell}c_2, \dots, c_1 c^{\ell} & k = 4\ell+2 \end{cases}$$

consider E_2

$$\begin{tikzcd}[row sep=small, column sep=small]
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H^{4n-2}_{x_1x_3} & 0 & H^{4n}_{x_1x_3} & 0 & H^{4n+2}_{x_1x_3} & 0 & H^{4n+4}_{x_1x_3} \\
0 & H^{4n-2}_{x_3} & 0 & H^{4n}_{x_3} & 0 & H^{4n+2}_{x_3} & 0 & H^{4n+4}_{x_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H^{4n-2}_{x_1} & 0 & H^{4n}_{x_1} \arrow[drr, "d_2"] & 0 & H^{4n+2}_{x_1} \arrow[drr, "d_2"] & 0 & H^{4n+4}_{x_1} \\
0 & H^{4n-2} & 0 & H^{4n} & 0 & H^{4n+2} & 0 & H^{4n+4}
\end{tikzcd}$$

d_2 must be an isomorphism since if not the only thing that could kill ker d_2 is d_4, but as above we know the domain of d_4 dies in E_3 term

also if d_2 not onto then $E_3^{4n,0}$ will be nontrivial and live to ∞

so H^{4n+2} is generated by d_2 of $x_1 c_1^{2n}, x_1 c_1^{2n-2} c_2, \dots, x_1 c_2^{n}$

i.e. by $c_1^{2n+2}, c_1^{2n}c_2, \dots, c_1c_2^{2n}$

d_2 is injective or a subgroup of $E_2^{4n+2,1}$ lives to infinity since only d_4 could kill it but, its domain dies in E_3

as above we will see E_3 is

$$\begin{array}{ccccc}
\langle c_2^n x_3\rangle & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H^{4n+4}/c_1^{2n+2}, c_1^{2n}c_2, \dots, c_1^2c_2^{4n}
\end{array}$$

and $E_3 = E_4$ now d_4 must be an isomorphism

so H^{4n+4} is generated by $c_1^{2n+2}, c_1^{2n}c_2, \dots, c_1^2c_2^n, c_n^{n+1}$

exercise: try computing $H^*(BU(3))$

can you come up with a proof that

$$H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$$

using spectral sequences?

(there are other proofs of this)