

CONTACT STRUCTURES ON 5-MANIFOLDS

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ABSTRACT. Using recent work on high dimensional Lutz twists and families of Weinstein structures we show that any almost contact structure on a 5-manifold is homotopic to a contact structure.

1. INTRODUCTION

Though contact structures have been studied for quite some time we still do not know which odd dimensional manifolds support such structures. Recall that (oriented) contact structures only exist on odd dimensional manifolds M^{2n+1} and are described as $\xi = \ker \alpha$ where α is a 1-form for which $\alpha \wedge (d\alpha)^n$ is a volume form on M . (We will always take M oriented and assume this volume form gives the preferred orientation.) Noting that $d\alpha$ defines a symplectic form on ξ we see that a contact structure gives a reduction of the structure group of TM to $U(n) \times \mathbf{1}$ (more specifically choosing a complex structure J on ξ that is compatible with $d\alpha$ gives the reduction and one may easily check this reduction is independent of the choice of J or α). Any reduction of the structure group of TM to $U(n) \times \mathbf{1}$, or equivalently a choice of hyperplane field η in TM with a complex structure J on η , is called an *almost contact structure* and the fundamental existence question in contact geometry concerns whether or not almost contact structures always come from contact structures.

For open manifolds Gromov [15] showed that all almost contact structures are homotopic to contact structures, but on closed manifolds much less is known. Previously we only had a complete answer in dimensions 1 and 3. The 1 dimensional result is trivial. In dimension 3 almost contact structures are equivalent to plane fields and so all oriented 3-manifolds M admit almost contact structures. Martinet [20] showed that all closed oriented 3-manifolds admit a contact structure and Lutz [19] showed that every plane field is homotopic to a contact structure. The main result of this paper is to extend this existence result to 5-manifolds.

Theorem 1.1. *On any closed oriented 5-manifold any almost contact structure (η, J) is homotopic, through almost contact structures, to a contact structure ξ .*

There have been many partial results towards this theorem. The first breakthrough was due to Geiges [9] who showed that on simply connected 5-manifolds any almost contact structure was homotopic to a contact structure. This result was reproven using open book decompositions by van Koert in [23]. In the papers [12] and [13] Geiges and Thomas extended the existence results to some 5-manifolds with finite fundamental group. In [11] Geiges and Stipsicz were able to prove existence on some other 5-manifolds. The most recent breakthrough is due to Casals, Pancholi, and Presas [1] who proved that all possible first Chern classes of almost contact structures can be realized by contact structures on 5-manifolds. In particular, they established the above theorem for manifolds without 2 torsion and gave a completely general existence result (that is any almost contact 5-manifold admits some contact structure). (Upon receiving a draft of this paper Casals, Pancholi, and Presas have informed the author that their arguments in [1] can be extended to obtain the main theorem above.)

The proof of Theorem 1.1 involves the use of an open book decompositions of a manifold M . In Section 3 we show that every almost contact structure on a closed oriented 5-manifold M can be “supported” by an open book decomposition. The open book decomposition allows us to write M as a neighborhood $N = Y \times D^2$ of an embedded 3-manifold and its complement C that can be expressed as a mapping torus of a 4-manifold X . We further break C into two pieces C' and C'' where $C' = D^4 \times S^1$ and C'' is the mapping torus of $\overline{X - D^4}$. We then use a result of the author and Pancholi [7] to put a contact structure on C' that induces an overtwisted contact structure on

the boundary of $D^4 \times \{\theta\}$ for each $\theta \in S^1$. We then use work of Cieliebak and Eliashberg [4] on Weinstein manifolds to extend this contact structure over C'' . The final extension over N relies on Eliashberg's understanding of overtwisted contact structures on 3-manifolds in [5].

We note that this proof is relatively elementary in that the two most sophisticated tools that it uses are the stable s -cobordism theorem (when quoting Quinn's result below on the existence of open book decompositions [21]) and a symplectic analog of the h -cobordism theorem due to Cieliebak and Eliashberg [4].

We also note that most of the steps in the proof have obvious analogs in higher dimensions except the last step concerning the extension of the contact structure over the neighborhood N of the binding Y , which relies heavily on Eliashberg's understanding of overtwisted contact structures in dimension 3. So one might hope that a similar approach would yield a general existence result in higher dimensions. This is essentially a program laid out by Giroux independently from the work here, though the author does thank Giroux for comments that simplified the arguments in Section 3.

Remark 1.2. In the proof of Theorem 1.1 we make the somewhat surprising observation that any 5-manifold M has a fixed open book decomposition that supports (in a weak sense, see Section 3.3) any almost contact structure on M and more generally any hyperplane field on M can, in some sense, be thought of as a perturbation of the pages of the open book. For a more precise formulation of this fact see Theorem 3.1. This observation is also true in higher dimensions but clearly far from true for open book decompositions of 3-manifolds. The intuitive explanation for this counter-intuitive fact is that the pages of open books in higher dimensional manifolds have more interesting topology than those of 3-dimensional open books.

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2. PRELIMINARY RESULTS AND RECOLLECTIONS

In this section we first recall the notion of an open book decomposition for a 5-manifold and discuss a special open book decomposition on S^5 . We then discuss a generalization of Lutz twists on 5-manifolds and finish by recalling Eliashberg's result on the space of overtwisted contact structures.

2.1. Open book decompositions of 5-manifolds. An *open book decomposition* of a manifold M is a pair (Y, π) where Y is a co-dimension 2 submanifold of M that has a product neighborhood $N = Y \times D^2$ in M and $\pi: (M - Y) \rightarrow S^1$ is a fibration such that $\pi|_{N-Y}$ is the projection onto the θ -coordinate of D^2 where we give D^2 polar coordinates (r, θ) . We call Y the *binding* of the open book decomposition and $\overline{\pi^{-1}(\theta)}$ a *page* of the open book.

There is another useful view of open book decompositions. If we are given a pair (X, ϕ) where X is a compact manifold with boundary and $\phi: X \rightarrow X$ is a diffeomorphism that agrees with the identity map near ∂X , then we can construct a manifold as follows. Let T_ϕ be the mapping torus of ϕ , that is $T_\phi = X \times [0, 1] / \sim$ where $(x, 1) \sim (\phi(x), 0)$ for all $x \in X$. Notice that $\partial T_\phi = Y \times S^1$ where $Y = \partial X$. We now glue $Y \times D^2$ to T_ϕ so that the product structures on the boundary agree. This gives a manifold $M_{(X, \phi)}$. We call (X, ϕ) an *open book decomposition* for a manifold M if M is diffeomorphic to $M_{(X, \phi)}$. Notice that the image of $Y \times \{(0, 0)\}$ in M is the binding of an open book decomposition as above and any open book decomposition as above can be constructed from a pair (X, ϕ) . We will use these notions interchangeably but note that when using the second definition of an open book we must always keep in mind a fixed diffeomorphism between M and $M_{(X, \phi)}$.

We have the following simple observation that we will use later.

Lemma 2.1. *Given open book decompositions (X_i, ϕ_i) of M_i , $i = 1, 2$, define X to be the boundary sum $X_1 \natural X_2$ and ϕ to be the diffeomorphism of X that restricts to ϕ_i on X_i . Then $M_{(X, \phi)} \cong M_{(X_1, \phi_1)} \# M_{(X_2, \phi_2)}$.*

2.2. An open book decomposition for S^5 . In this section we discuss a special open book decomposition of S^5 that we will need to “stabilize” open books of other 5-manifolds to have nice properties that will be needed later.

Theorem 2.2. *There is an open book $(Y_{\text{stab}}, \pi_{\text{stab}})$ for S^5 with page X_{stab} the manifold $(S^2 \times S^2) \# (S^2 \times S^2)$ with an open disk removed. Let ϕ_{stab} be the monodromy of this open book. There is a Morse function $f_{\text{stab}}: X_{\text{stab}} \rightarrow \mathbb{R}$ for the page such that f_{stab} and $f_{\text{stab}} \circ \phi_{\text{stab}}$ can be connected by a family of Morse functions having only 0 and 2-handles.*

Proof. In [22], Saeki showed there was an open book for S^5 with the given page. We sketch the argument here for the convenience of the reader. Given diffeomorphisms ϕ of X_{stab} that is the identity on the boundary we see that the identity map I on the homology chains of X_{stab} and ϕ_* induce the same map on the homology chains in ∂X_{stab} . Thus $I - \phi_*$ induce a map from $H_2(X_{\text{stab}}, \partial X_{\text{stab}}) \rightarrow H_2(X_{\text{stab}})$. Since X_{stab} can be build with just 0 and 2-handles one may easily check that the open book (X, ϕ) gives a homotopy sphere if $I - \phi_*$ is an isomorphism. Using the obvious product basis for $H_2(X_{\text{stab}})$ the matrix

$$\Phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

defines an automorphism of $H_2(X_{\text{stab}})$ that preserves the intersection pairing. As $H_1(X_{\text{stab}}) = 0$ this determines a map $I - \Phi: H_2(X_{\text{stab}}, \partial X_{\text{stab}}) \rightarrow H_2(X_{\text{stab}})$ that is easily seen to be an isomorphism. A result of Wall [24] implies that there is a diffeomorphism ϕ_{stab} of X_{stab} that induces the Φ . Thus $(X_{\text{stab}}, \phi_{\text{stab}})$ give an open book decomposition for S^5 which we denote by $(Y_{\text{stab}}, \pi_{\text{stab}})$.

One may easily show, cf. [17], that the diffeomorphism given in Wall’s theorem can be induced by a sequence of 2-handle slides. More specifically, there is a family of handle decompositions H_t , $t \in [0, 1]$, of manifold $X_t = X_{\text{stab}}$ that starts and ends with the same handle decomposition of $X_{\text{stab}} = X_0 = X_1$. In addition each H_t only consists of 0 and 2-handles and the family describes isotopies of the attaching regions and handles slides. We can use the handle decomposition of X_t to describe a diffeomorphism $\phi_t: X_0 \rightarrow X_t$ and $\phi_1: X_{\text{stab}} \rightarrow X_{\text{stab}}$ is a diffeomorphism that induces the map Φ on $H_2(X_{\text{stab}})$, that is ϕ_1 can be take to be ϕ_{stab} . Now if f_{stab} is a Morse function on $X_0 = X_{\text{stab}}$ that corresponds to H_0 then set $f_t = f_{\text{stab}} \circ \phi_t$. Thus we have a sequence of Morse functions $f_t: X_{\text{stab}} \rightarrow \mathbb{R}$ that correspond to 2-handle slides and $f_1 = f_{\text{stab}} \circ \phi_1$. \square

2.3. A notion of high dimensional Lutz twists. Recall the standard contact structure ξ_{std} on S^3 is obtained as the set of complex tangencies to the unit S^3 in \mathbb{C}^2 . There is an overtwisted contact structure ξ_{ot} on \mathbb{C}^2 in the same homotopy class of plane field as ξ_{std} . This follows from Eliashberg’s classification of overtwisted contact structures [5], or can easily be constructed by performing a full Lutz twist, see Section 2.4 below, on a transverse unknot in ξ_{std} .

Proposition 2.3. *There is a contact structure ξ on $D^4 \times S^1$ such that ξ induces the overtwisted contact structure ξ_{ot} on $\partial D^4 \times \{\theta\}$ for each $\theta \in S^1$. More precisely, if α_{ot} is a 1-form for which $\xi_{\text{ot}} = \ker \alpha_{\text{ot}}$ then in a neighborhood $S^3 \times (1/2, 1] \times S^1$ of the boundary of $D^4 \times S^1$ the contact structure is given by*

$$\xi = \ker(Kd\theta + t\alpha_{\text{ot}}),$$

where t is the coordinate on $(1/2, 1]$, θ is the coordinate on S^1 , and K is any positive constant. Moreover, ξ is homotopic through almost contact structures to the almost contact structure coming from the tangents to $D^4 \times \{\theta\}$, for each $\theta \in S^1$, where we think of D^4 as the unit disk in \mathbb{C}^2 .

This theorem follows easily from a results of Pancholi and the author [7]. To state the result we first establish some notation. Consider $T^2 \times \mathbb{R}$ with coordinates (ϑ, φ, r) and the contact structure

$$\xi_{T^2 \times \mathbb{R}} = \ker(\cos r d\vartheta + \sin r d\varphi).$$

We will think of $\xi_{[a,b]}$ as the contact structure obtained from $\xi_{T^2 \times \mathbb{R}}$ by restricting it to $T^2 \times [a, b]$. We also denote $T^2 \times \{r\}$ by T_r .

Notice that $T^2 \times [0, \pi/2]$ has boundary $T_0 \cup T_{\pi/2}$ and the characteristic foliation induced by $\xi_{[0, \pi/2]}$ is given by a linear horizontal floatation on T_0 and linear vertical foliation on $T_{\pi/2}$. So if we collapse the leave of the foliation on the boundary of $T^2 \times [0, \pi/2]$, or more precisely perform a contact cut [18], then we obtain S^3 and $\xi_{[0, \pi/2]}$ induces the standard contact structure on S^3 . Similarly performing a contact cut on the boundary of $T^2 \times [0, 5\pi/2]$ will also produce S^3 but this time with the overtwisted contact structure ξ_{ot} . (In particular it is easy to see that this contact structure is obtained from the one above by a full Lutz twist.)

We now define a symplectic structure on a neighborhood of the boundary of $Y \times [0, 1]$, where $Y = T^2 \times [0, 1]$. Using an identification of $[0, 1]$ with $[0, \pi/2]$ we construct a diffeomorphism $\phi_0: Y \times \{0\} \rightarrow T^2 \times [0, \pi/2]$. Then on $(Y \times [0, 1/4]) \cup (T^2 \times ([0, 1/4] \cup [3/4, 1]) \times [0, 1])$ we let λ be $t\phi_0^*\alpha$, where t is the coordinate on $[0, 1]$. Similarly an identification of $[0, 1]$ with $[0, 5\pi/2]$ can be used to build a diffeomorphism $\phi_1: Y \times \{1\} \rightarrow T^2 \times [0, 5\pi/2]$. Then on $Y \times [3/4, 1]$ we can define λ to be $t\phi_1^*\alpha$. If ϕ_1 is chosen correctly, λ will be a well defined 1-form on a closed neighborhood of the boundary of $Y \times [0, 1]$ such that $d\lambda$ is a symplectic form with $Y \times \{0\}$ concave, $Y \times \{1\}$ convex and $(\partial Y) \times [0, 1]$ flat. It is well known that λ cannot be extended over all of $Y \times [0, 1]$ but if there was an extension we could construct a contact form on $Y \times [0, 1] \times S^1$. Pancholi and the author [7] proved that this contact structure does exist.

Lemma 2.4 (Etnyre and Pancholi, 2011 [7]). *With Y and λ as defined above, there is a contact structure ξ on $Y \times [0, 1] \times S^1$ so that near the boundary $\xi = \ker(Kd\theta + \lambda)$, where θ is the coordinate on S^1 and K is any positive constant. Moreover, the contact structure ξ is homotopic through almost contact structures to $\ker(Kd\theta + (t\phi_0^*\alpha))$.*

With this lemma in hand we return to the proof of our main theorem of this section.

Proof of Proposition 2.3. Simultaneously performing a contact cut on $\partial Y \times \{t\}$ for each t gives a contact structure on $S^3 \times [0, 1] \times S^1$. We can glue $D^4 \times S^1$ with the contact structure $Kd\theta + \lambda$, where λ is the standard Liouville form on D^4 , to the lower boundary of $S^3 \times [0, 1] \times S^1$. This results in a contact structure on $D^4 \times S^1$ with the desired properties. \square

2.4. Plane fields and contact structures on 3-manifolds. Let Y be a closed oriented 3-manifold. A contact structure ξ on Y is called *overtwisted* if there is a disk D embedded in Y that is tangent to ξ along the boundary of D and at one point on the interior of D and is transverse to ξ elsewhere. The disk D is called an *overtwisted disk*.

Recall that given any contact structure ξ on Y we can always alter it to be overtwisted by a Lutz twist. To define a Lutz twist let K be a knot in Y that is transverse to ξ . It is easy to show, for example see [10], that K has a standard neighborhood S and the boundary of S has a neighborhood contactomorphic to $(T^2 \times [a, b], \xi_{[a,b]})$ for some $a < b$, where we are using the notation from the previous section. Notice that one can replace the contact structure on this neighborhood with $(T^2 \times [a, b], \xi_{[a, b+2\pi]})$ to obtain a new contact structure on Y . We say that this contact structure is obtained from ξ by a full Lutz twist along K . It is well known, see for example [10] for a nice exposition of this, that the contact structure obtained from ξ by a full Lutz twist is homotopic as a plane field to ξ ; moreover, this homotopy can be done fixing the plane fields along K (which is called the core of the Lutz tube). It is also simple to see that if full Lutz twists are performed parametrically on a family of contact structures then the resulting family is homotopic to the original family (also fixing the planes along the cores of the Lutz tubes).

We fix a base point p in Y and a plane $P \subset TY$. Then denote by $\mathcal{P}_p(Y)$ the space of plane fields on Y that agree with P at p . Let D be a disk in Y that is tangent to P at p (and p is on the interior of D). We let $\mathcal{C}_{ot}(Y)$ be the space of contact structures on Y for which D is an overtwisted disk. An amazing insight of Eliashberg relates these spaces.

Theorem 2.5 (Eliashberg, 1989 [5]). *The natural inclusion map*

$$i: \mathcal{C}_{ot}(Y) \rightarrow \mathcal{P}_p(Y).$$

is a homotopy equivalence.

3. ALMOST CONTACT STRUCTURES

In this section we study hyperplane fields and almost contact structures on 5-manifolds.

3.1. Hyperplane fields on 5-manifolds. Let M be an oriented 5-manifold. To study the homotopy classes of hyperplane fields on M we consider the bundle associated to the tangent bundle TM with fiber the Grassmann manifold of oriented 4-planes in \mathbb{R}^5 :

$$\begin{array}{ccc} G^+(4, 5) & \longrightarrow & E \\ & & \downarrow \\ & & M. \end{array}$$

Hyperplane fields on M correspond to sections of this bundle. So we are interested in studying the homotopy classes of sections of this bundle. Throughout this section we will make no distinction between a hyperplane field and a section of this bundle. Note that $G^+(4, 5)$ is diffeomorphic to S^4 . (One may easily see this by noting that oriented hyperplanes are in one-to-one correspondence with unit vectors in \mathbb{R}^5 , where the correspondence comes from an inner product on \mathbb{R}^5 .)

Studying the obstruction theory of this bundle we first observe that $\pi_n(G^+(4, 5))$ agrees with $[S^n, G^+(4, 5)]$, the set of homotopy classes of maps of S^n to $G^+(4, 5)$ (that is $G^+(4, 5)$ is n -simple). In addition because TM is oriented it is well known that the action of $\pi_1(M)$ on $\pi_n(G^+(4, 5))$ coming from the bundle E is trivial. Thus the obstruction to homotoping two sections of E over the n -skeleton of M lies in $H^n(M; \pi_n(G^+(4, 5)))$. Knowing that

$$\pi_n(G^+(4, 5)) = \begin{cases} 0 & n < 4 \\ \mathbb{Z} & n = 4 \\ \mathbb{Z}/2\mathbb{Z} & n = 5 \end{cases}$$

it is clear that any section s_1 can be homotoped to s_2 on the 3-skeleton of M and the obstruction to homotoping them together on the 4-skeleton is an element

$$d_4(s_1, s_2) \in H^4(M; \mathbb{Z})$$

and is a primary obstruction. Moreover if s_1 and s_2 agree on the 4-skeleton the, non-primary, obstruction to homotoping s_1 to s_2 over the 5-skeleton is an element

$$d_5(s_1, s_2) \in H^5(M; \mathbb{Z}/2\mathbb{Z}).$$

Given a plane field ξ on a 3-manifold Y and an open book decomposition (Y, π) of a 5-manifold M we can construct a hyperplane field as follows. Let $N = Y \times D^2$ be a neighborhood of Y in M , so that $\pi|_{(N-Y)}$ is just the projection to the θ -coordinate of D^2 . On N we take the hyperplane field

$$\ker \{f(r)d\theta + g(r)\alpha\}$$

where α is a 1-form on Y that defines ξ , (r, θ) are polar coordinates on the unit disk D^2 , f is a non-decreasing function equal to r^2 near 0 and constantly 1 near 1, and g is a non-increasing function equal to 1 near 0 and 0 near 1 for which f and g are never simultaneously zero. This hyperplane field can be extended across $(M - N)$ by $\ker d\pi$. Denote this hyperplane field $H(\xi)$. It is clear that H gives a well defined map from the set of plane fields on Y to the set of hyperplane fields on M . Our main observation is that under certain hypothesis this map is onto.

Theorem 3.1. *Let M be any closed oriented 5-manifold and (Y, π) any open book decomposition of M . Then we have the following:*

- (1) *The map H is well defined up to homotopy. That is if ξ and ξ' are to homotopic plane fields on Y then $H(\xi)$ is homotopic to $H(\xi')$.*
- (2) *If the pages of (Y, π) are handlebodies with handles of index 0,1 and 2, then any homotopy class of hyperplane field is in the image of the map H defined above. More specifically, if η is a hyperplane field on M then there is some plane field ξ on Y such that $H(\xi)$ is homotopic to η .*

Remark 3.2. The obvious generalization of this theorem to higher dimensional manifolds is also true and might be of use in trying to construct contact structures homotopic to almost contact structures on higher dimensional manifolds.

Proof. The first statement is clear as we can apply the same construction used in defining H to the homotopy between ξ and ξ' .

The second statement requires a bit more work. We begin by reinterpreting the obstruction cohomology class d_4 . Let η and η' be two hyperplane fields on M . Thinking of η and η' as sections of the bundle $G^+(4, 5) \rightarrow E \rightarrow M$ we homotope them so that they are transverse. Let

$$\gamma(\eta, \eta') = \{x \in M \mid \eta(x) = \eta'(x)\}$$

be the locus where η and η' agree.

Lemma 3.3. *With the notation above, the Poincaré dual of $d_4(\eta, \eta')$ is $[\gamma(\eta, -\eta')] \in H_1(M; \mathbb{Z})$.*

Proof. We begin by recalling how to compute $d_4(\eta, \eta')$. We can assume that η and η' agree on the 3-skeleton of M . Now given a 4-cell D of M the sections η and η' each give a map from $D \rightarrow G^+(4, 5) \cong S^4$ that agree along the boundary. We can thus use η and η' to define a map from $S^4 \rightarrow S^4$ by using η on the upper hemisphere and η' on the lower hemisphere. The degree of this map is the value of $d_4(\eta, \eta')$ on the 4-cell D .

We may isotope η and η' so that the 1-manifold $\gamma(\eta, -\eta')$ is disjoint from the 3-skeleton of M and intersect the 4-skeleton transversely. It is clear that we may compute the degree of the above mentioned map on D as the intersection number of D and $\gamma(\eta, -\eta')$. (You can pick a trivialization of TM over D so that η , say, is constant, then $\gamma(\eta, -\eta') \cap D$ is just the preimage of some point on S^4 .) \square

Lemma 3.4. *Let M be a closed oriented 5-manifold with open book decomposition (Y, π) whose page is a handlebody with handle of index 0, 1 and 2. Fix a plane field ξ on a Y . As ξ' ranges over all homotopy classes of plane field on Y the class*

$$d_4(H(\xi), H(\xi')) \in H^4(M, \mathbb{Z})$$

ranges over all of $H^4(M; \mathbb{Z})$.

In particular, every hyperplane field η on M is homotopic, over the 4-skeleton of M , to $H(\xi')$ for some plane field ξ' on Y .

Proof. Using Lemma 3.3 we prove the first statement by showing that $[\gamma(H(\xi), -H(\xi'))]$ ranges over all of $H_1(M; \mathbb{Z})$ as ξ' ranges over all homotopy classes of plane field on Y . To this end we first observe that the inclusion map

$$i: Y \rightarrow M$$

induces a surjection

$$i_*: H_1(Y; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}).$$

To see this let X be a page of the open book (Y, π) . A neighborhood V_0 of X can be identified with $X \times [0, 1]$ and $V_1 = M \setminus V_0$ is also homeomorphic to $X \times [0, 1]$. Notice that V_0 and V_1 are handlebodies with handles of index less than 3. So M is built from V_0 by adding handles of index greater than 2 (that is the handles in V_1 turned upside down). Thus the inclusion of V_0 into M induces an isomorphism on the first homology $H_1(V_0; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$. Of course the inclusion of X into V_0 is a homotopy equivalence so $H_1(X; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$. Finally notice that X can be built from $Y = \partial X$ by attaching 4-dimensional handles of index 2, 3 and 4 (the handles in X turned upside down). Thus the inclusion map gives a surjection of $H_1(Y, \mathbb{Z})$ onto $H_1(X; \mathbb{Z})$. The claim follows.

We now define a map \tilde{H} that is a slight variant of H . Recall in the definition of H we used the neighborhood $N = Y \times D^2$ of Y in M , where D^2 was the unit disk. We also had functions $f(r)$ and $g(r)$ where $f(r) = r^2$ near 0 and is constantly 1 near 1 and non-decreasing, and $g(r) = 1$ near 0 and is 0 near 1 and non-increasing. We now specify that g should be zero on only the interval $[1/2, 1]$. Now choose $\tilde{g}(r)$ to be a non-increasing function that is 1 near 0 and 0 near 1, but is 0 only on $[3/4, 1]$. For the moment let $\tilde{f} = f$, but we will perturb it later. Define \tilde{H} in the same manner as

H except use \tilde{f} and \tilde{g} in place of f and g . Notice that $H(\xi)$ and $\tilde{H}(\xi)$ are homotopic as hyperplane fields. Moreover, on $(Y \times \{(r, \theta) | r \in [1/2, 3/4]\}) \subset M$, $H(\xi)$ and $\tilde{H}(\xi')$ give disjoint section. Thus $\tilde{H}(\xi)$ can be perturbed, relative to the region $(Y \times \{(r, \theta) | r \in [1/2, 3/4]\}) \subset M$, so that it is transverse to $H(\xi)$ over the complement of $Y \times \{(r, \theta) | r < 1/2\}$. From now on we will denote by $\tilde{H}(\xi)$ this perturbed hyperplane field. Notice that $H(\xi')$ for any other plane field ξ' agrees with $H(\xi)$ on the complement of $Y \times \{(r, \theta) | r < 1/2\}$. Thus $H(\xi')$ is transverse to $\tilde{H}(\xi)$ on the complement of $Y \times \{(r, \theta) | r < 1/2\}$.

We claim with the appropriate choice of \tilde{f} and \tilde{g} , $H(\xi')$ will be transverse to $\tilde{H}(\xi)$ on all of M if ξ and ξ' are transverse on Y . To achieve this we choose \tilde{f} to be zero only on $[0, 1/2]$ (it is still non-decreasing and 1 near 1). It is now a simple exercise to see that $\tilde{H}(\xi)$ and $H(\xi')$ are transverse on $Y \times \{(r, \theta) | r < 1/2\}$. (Though the exercise is somewhat easier if one uses the dual picture to think of the hyperplane fields as unit vector fields.)

Obstruction theory as discussed above shows ξ and ξ' have a difference class $d_2(\xi, \xi')$ that obstructs homotoping them on the 2-skeleton of Y and it is Poincaré dual to the 1-dimensional homology class given by the locus where they agree, which we denote by γ' . From construction it is clear that $\tilde{H}(\xi)$ and $H(\xi')$ on $Y \times \{(r, \theta) | r < 1/2\}$ only agree along γ' . So the Poincaré dual of $d_4(\tilde{H}(\xi), H(\xi'))$ is given by the homology class of $\gamma' \cup \gamma''$ where γ'' is a 1-manifold in the complement of $Y \times \{(r, \theta) | r < 1/2\}$. Notice that γ'' is independent of ξ' . Moreover we know that by the appropriate choice of ξ' we can realize any homology class in $H_1(Y, \mathbb{Z})$ as the Poincaré dual of the difference class between ξ' and ξ . Thus we see that any element of $H_1(M; \mathbb{Z})$ can be realized as the Poincaré dual of the difference class $d_4(\tilde{H}(\xi), H(\xi')) = d_4(H(\xi), H(\xi'))$.

For the last statement recall that the difference class d_4 satisfies

$$d_4(\eta_1, \eta_3) = d_4(\eta_1, \eta_2) + d_4(\eta_2, \eta_3).$$

Given a hyperplane field η the first part of the proof guarantees a plane field ξ' such that

$$d_4(H(\xi), H(\xi')) = -d_4(\eta, H(\xi)).$$

So we see that $d_4(\eta, H(\xi')) = d_4(\eta, H(\xi)) + d_4(H(\xi), H(\xi')) = 0$. Thus $H(\xi')$ is homotopic to η over the 4-skeleton of M . \square

Returning to the proof of Theorem 3.1 we note that given any hyperplane field η on M we can find some plane field ξ on Y such that $H(\xi)$ is homotopic to η on the 4-skeleton of M . We are left to see that ξ can be chosen so that the homotopy can be extended over the 5-skeleton. Recall from above that if two hyperplane fields agree on the 4-skeleton then the obstruction to homotoping them on the 5-skeleton is a class in $H^5(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

We claim that if ξ and ξ' are homotopic over the 2-skeleton of Y and their obstruction to homotopy over the 3-skeleton is odd (this obstruction is an element of $H^3(Y; \mathbb{Z}) = \mathbb{Z}$ and is well defined modulo the divisibility of the Euler class of ξ), then $H(\xi)$ and $H(\xi')$ are homotopic over the 4-skeleton of M but $d_5(H(\xi), H(\xi')) \neq 0$. Thus if η is homotopic to $H(\xi)$ over the 4-skeleton then it will be homotopic to either $H(\xi)$ or $H(\xi')$ over all of M .

To verify the claim we begin by choosing representatives of ξ and ξ' that agree except on the 3-handle of Y . Thus we know that $H(\xi)$ and $H(\xi')$ agree on the complement of $N = Y \times D^2$. To construct M from the complement of N we think of a handle decomposition of N , coming from Y , turned upside down. Thus M is constructed from $\overline{M - N}$ by attaching a 2-handle, some 3 and 4-handles and a 5-handle. The 5-handle comes from the 3-handle of Y and thus we see that $H(\xi)$ and $H(\xi')$ agree on all of M except the 5-handle D^5 .

From the set up we know D^5 is the product of the 3-dimensional 3-handle D^3 with D^2 . The plane fields ξ and ξ' give maps from D^3 to $G^+(2, 3) \cong S^2$ that agree on the boundary of D^3 . The obstruction to finding a homotopy of ξ to ξ' is an element of $H^3(Y, \mathbb{Z})$ that evaluates on D^3 to the ‘‘Hopf invariant’’ of the map $S^3 \rightarrow G^+(2, 3) \cong S^2$ given by ξ on the upper hemisphere of S^3 and ξ' on the lower hemisphere. Denote this map $h: S^3 \rightarrow S^2$.

Similarly the hyperplane fields $H(\xi)$ and $H(\xi')$ each give a map from $D^5 \cong D^3 \times D^2$ to $G^+(4, 5) \cong S^4$ and they agree on ∂D^4 . The obstruction to finding a homotopy of the hyperplane field $H(\xi)$ to

$H(\xi')$ is the homotopy class of the map $H: S^5 \rightarrow S^4$ defined by $H(\xi)$ on the upper hemisphere and by $H(\xi')$ on the lower hemisphere.

We claim the map $H: S^5 \rightarrow S^4$ factors through a map homotopic to the map obtained from $h: S^3 \rightarrow S^2$ by a “double suspension”. Recall that the suspension of a space X is the quotient of $X \times [0, 1]$ by $X \times \{0\}$ and $X \times \{1\}$. Given a function $f: X \rightarrow X'$ there is clearly an induced suspension map $S(f): S(X) \rightarrow S(X')$. It is well known, and easy to see, that $S^n \cong S(S^{n-1})$. So we are claiming that our map H factors through a map $k: S^5 \rightarrow S^4$ that is homotopic to $S^2(h)$, in other words there is a map $k': S^5 \rightarrow S^5$ such that $H = k \circ k'$. Moreover $k'_*: \pi_5(S^5) \rightarrow \pi_5(S^5)$ is an isomorphism. The Freudenthal suspension theorem implies that the double suspension map from $\pi_3(S^2) \rightarrow \pi_5(S^4)$ is surjective (here we have ignored base points, but due to the simple connectivity of the spaces their inclusion would not alter the statement). Thus, as $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_5(S^4) \cong \mathbb{Z}_2$, we see that $S^2(h)$ is not null homotopic and represents the generator of $\pi_5(S^4)$. Thus k , which is homotopic to $S^2(h)$, induces a surjective map $k_*: \pi_5(S^5) \rightarrow \pi_5(S^4)$. As k'_* gives an isomorphism on $\pi_5(S^5)$ we see H_* is surjective and hence $H: S^5 \rightarrow S^4$ represents the generator of $\pi_5(S^4)$. Thus $d_5(H(\xi), H(\xi'))$ is the mod 2 reduction of 3-dimensional obstruction $d_3(\xi, \xi')$. This finishes the proof.

So we are left to justify the claim about H and $S^2(h)$. We first note that the suspension can be described in terms of joins. Specifically, given a space X its suspension is $S(X) \cong X * S^0$. (Recall the join $X * Y$ of X and Y is obtained from $X \times Y \times [0, 1]$ where each of the sets $X \times \{y\} \times \{0\}$ and $\{x\} \times Y \times \{1\}$ are identified to a point.) As we know that $S^0 * S^0 = S^1$ it is clear that the double suspension can be written $S^2(X) = X * S^1$. Now given a map $f: X \rightarrow Y$ the double suspension map $S^2(f): X * S^1 \rightarrow Y * S^1$ is constructed as follows. We begin by defining the map $X \times S^1 \times [0, 1] \rightarrow Y \times S^1 \times [0, 1]$ by $(x, \theta, t) \mapsto (f(x), \theta, t)$. Composition with the quotient map $Y \times S^1 \times [0, 1] \rightarrow Y * S^1$ we get a map $X \times S^1 \times [0, 1] \rightarrow Y * S^1$. This map clearly descends to a map $X * S^1 \rightarrow Y * S^1$ which we define to be $S^2(f)$.

In our case we start with the map $h: S^3 \rightarrow S^2$ and the map $H: S^5 \rightarrow S^4$. These maps are defined with respect to some trivialization of the tangent space TY over the 3-handle D^3 and of the tangent space TM over D^5 , respectively. Recall $D^5 = D^3 \times D^2$ and over D^5 we have $TM|_{D^5} = TY_{D^3} \oplus TD_{D^2}$. Notice that the unit sphere S^4 in \mathbb{R}^5 is naturally represented as the join of S^2 and S^1 where S^2 is the unit sphere in the first 3 coordinates of \mathbb{R}^5 and the S^1 is the unit sphere in the last 2 coordinates. Thus we think of the S^4 -bundle in $TM|_{D^5}$ as a fiber wise join of the unit sphere bundle in $TY|_{D^3}$ and the unit sphere bundle in TD^2 .

Recall that the S^5 in the domain of H is obtained by gluing two copies of the 5-handle D^5 together. Moreover, $D^5 = D^3 \times D^2$ where D^3 is the 3-handle in Y and D^2 is the fiber in the neighborhood N of Y . We glue the two copies of $D^3 \times D^2$ together in two stages. We first glue along $(\partial D^3) \times D^2$. This yields $S^3 \times D^2$. Writing D_1 and D_2 for the two copies of D^3 glued to form D^3 we see that S^5 is obtained from $S^3 \times D^2$ by gluing $D_1 \times \{\theta\}$ to $D_2 \times \{\theta\}$ for $\theta \in \partial D^2$. In addition we see that the join $S^3 * S^1$ is obtained from $S^3 \times D^2$ (this is $S^3 \times S^1 \times [0, 1] / \sim$ where each $\{p\} \times S^1 \times \{t\}$ is collapsed to a point) by collapsing each $S^3 \times \{\theta\}$ to a point. Thus we clearly see that there is a map $k': S^5 \rightarrow S^5$ obtained by thinking of the domain as two copies of $D^3 \times D^2$ glued together and the image space as $S^3 * S^1$. (Specifically the first S^5 is obtained from $S^3 \times D^2$ by partially collapsing each $S^3 \times \{\theta\}$ while the second is obtained by completely collapsing the spheres.) This is clearly a degree 1 map and hence induces an isomorphism on the homotopy groups of S^5 . Below we will denote S^5 created via the first method by S_1^5 and via the second method as S_2^5 . Clearly $S^2(h): S_2^5 \rightarrow S^4$ and $H: S_1^5 \rightarrow S^4$. We are now left to see that $H = k \circ k'$ where k is homotopic to $S^2(h)$.

Representing S^4 as $S^2 * S^1$ from the splitting of $TM|_{D^5}$ above we can consider “coordinates” on $S^2 * S^1$ to be $(p, \theta, t) \in S^2 \times S^1 \times [0, 1]$ as discussed above. Using coordinates (p, r, θ) on $S^3 \times D^2$, where (r, θ) are polar coordinates on D^2 , we see them map H is given by

$$(p, r, \theta) \mapsto \left(h(p), \theta, \frac{2}{\pi} \tan^{-1} \frac{f(r)}{g(r)} \right).$$

(To see this notice that we can map $S^2 \times S^1 \times [0, 1]$ to the unit sphere in \mathbb{R}^5 by the map $(p, \theta, t) \mapsto (p \cos \frac{\pi}{2}t, \theta \sin \frac{\pi}{2}t)$, where we think of S^2 is the unit sphere in the first 3 coordinates of \mathbb{R}^5 and S^1 as the unit sphere in the last 2 coordinates. Now the map H from $S^3 \times D^2$ to $S^4 \subset \mathbb{R}^5$ is given by $(p, r, \theta) \mapsto (g(r)h(p), f(r)(\cos \theta, \sin \theta))$. So if g and f are chosen so that $g^2 + f^2 = 1$ we get the above representation of H when using the “join coordinates” on S^4 .)

The map $H: S_1^5 \rightarrow S^4$ descends to a map $k: S_2^5 \rightarrow S^4$ so that $H = k \circ k'$. Now choosing a homotopy from $r \mapsto \frac{2}{\pi} \tan^{-1} \frac{f(r)}{g(r)}$ to the identity map $r \mapsto r$ we have an induced homotopy $k: S_2^5 \rightarrow S^4$ to $S^2(h): S_2^5 \rightarrow S^4$. Thus completing the proof. \square

3.2. Almost contact structures. Recall an almost contact structure on an oriented 5-manifold M is a reduction of the structure group of TM from $SO(5)$ to $U(2) \times 1$. This is equivalent to a choice of homotopy class of hyperplane field η on M together with a homotopy class of complex structure J on η . We will always think of an almost contact structure as a pair (η, J) , up to homotopy.

To study the homotopy classes of almost contact structures on M we consider the bundle associated to the tangent bundle TM with fiber $SO(5)/U(2)$

$$\begin{array}{ccc} SO(5)/U(2) & \longrightarrow & E \\ & & \downarrow \\ & & M. \end{array}$$

Homotopy classes of sections of this bundle correspond to homotopy classes of almost contact structures. Using obstruction theory to understand the homotopy classes of sections of this bundle we need to know the homotopy type of the fiber. To this end we recall that $SO(5)/U(2)$ is diffeomorphic to $\mathbb{C}P^3$, see for example [10]. Thus all the relevant homotopy groups are zero except $\pi_2(SO(5)/U(2)) \cong \mathbb{Z}$. From this we see that the first and only obstruction to homotoping one almost contact structure to another is in $H^2(M; \mathbb{Z})$. In particular, we have the following useful observation.

Lemma 3.5. *If two almost contact structures on the same 5-manifold M are homotopic over the 2-skeleton of M then they are homotopic.*

3.3. Almost contact structures and open book decompositions. Recall that in Section 3.1 a map

$$H_{(Y, \pi)}: \{\text{hyperplanes on } Y\} \rightarrow \{\text{hyperplanes on } M\}$$

was assigned to an open book decomposition (Y, π) of a 5-manifold M . We say that an open book (Y, π) supports an almost contact structure (η, J) if there is some hyperplane ξ on the binding Y such that η is (homotopic to) $H_{(Y, \pi)}(\xi)$ and the plane field ξ is a complex sub-bundle of $H_{(Y, \pi)}(\xi)$ on Y (that is ξ on Y is J invariant). Notice that this implies that there is a homotopy of (η, J) so that η is transverse to Y and outside of a small neighborhood of Y is tangent to the pages of (Y, π) and thus defines an almost complex structure on the pages.

Theorem 3.6. *Given an almost contact structure (η, J) on a closed oriented 5-manifold, there is an open book decomposition (Y, π) with pages having a handle decomposition with handles of index less than or equal to two that supports the given almost contact structure.*

Moreover there is an overtwisted contact structure ξ on Y such that $H(\xi) = \eta$ and one can assume that there is a neighborhood $N = Y \times D^2$ of the binding Y such that η is tangent to the pages of the open book outside of N and at each point of N the plane ξ is a J -complex sub-bundle of $H(\xi)$.

To prove this proposition we will need a preliminary lemma.

Lemma 3.7. *If J is a complex structure on the hyperplane field η then there is a C^0 -neighborhood of η in the space of hyperplane fields such that J induces a complex structure for all hyperplane fields in this neighborhood. Moreover, the complex structures are well-defined up to homotopy.*

Proof. Given η choose any line field L on M that is transverse to η . Let \mathcal{O} be a C^0 -neighborhood of η in the space of hyperplane fields that consists of hyperplanes transverse to L . Note that L and η can be used to define a projection $p: TM \rightarrow \eta$. It is clear that at any point $x \in M$ the projection p maps η'_x isomorphically onto η_x for any $\eta' \in \mathcal{O}$. Thus we can use p to pull J back from η to η' .

The only choice made in this construction was the choice of L , but clearly as we vary L this only changes the complex structures induced on η' by a homotopy. \square

Proof of Theorem 3.6. Quinn [21] proved that there is an open book decomposition (Y, π) of M whose pages have a handle decomposition with handles only of index less than or equal to two. Let (η, J) be an almost contact structure on M . By Proposition 3.1 there is a hyperplane field ξ on Y such that the induced hyperplane field $H(\xi)$ is homotopic to η . Using Lemma 3.7 we can pull J back along this homotopy so that $H(\xi)$ also has a complex structure J' . Thus $(H(\xi), J')$ is an almost contact structure that is homotopic to (η, J) . To simplify notation we now assume our given almost contact structure is $(H(\xi), J)$.

We are left to see that we can arrange that ξ on Y is J -invariant. We will do this by altering $H(\xi)$ on the neighborhood N of the binding Y to construct a new almost contact structure and then show that this is homotopic to $(H(\xi), J)$ through almost contact structures.

Recall $N = Y \times D^2$ where D^2 is the unit disk in \mathbb{R}^2 . We write D^2 as the union $D_{1/4} \cup A_{1/4, 1/2} \cup A_{1/2, 3/4} \cup A_{3/4, 1}$, where $D_{1/4}$ is the disk of radius $1/4$ and $A_{a,b}$ denotes the annulus $\{(r, \theta) : a \leq r \leq b\}$. We can homotope $(H(\xi), J)$ so that it is independent of the coordinates on D^2 near $Y \times \{(0, 0)\}$. Now one can use a map from $A_{3/4, 1}$ to D that is the identity near the outer boundary of the annulus, a diffeomorphism away from the inner boundary and collapses the inner boundary to the origin, to pull $(H(\xi), J)$ back to an almost contact structure on $Y \times A_{3/4, 1}$, call the resulting almost contact structure (η', J') . Notice that η' on $Y \times \{p\}$ for any point on the inner boundary of $A_{3/4, 1}$ is homotopic to $TY \oplus \mathbb{R}$ (where we can take \mathbb{R} to point in the radial direction if we like). We can use this homotopy to extend η' to a hyperplane field on $A_{1/2, 3/4}$ so that on the inner boundary of this annulus the hyperplane field is $TY \oplus \mathbb{R}$. We can also use the homotopy to extend J' over this annulus so that it is independent of the angular coordinate on the annulus.

To continue to define our almost contact structure we need to discuss complex structures on $TY \oplus \mathbb{R}$. Recall that complex structures on a 4-dimensional bundle correspond to reductions of the structure group from $SO(4)$ to $U(2)$ and thus correspond to sections of a principal $SO(4)/U(2)$ bundle. As $SO(4)/U(2)$ is homotopy equivalent to S^2 and the bundle $TY \oplus \mathbb{R}$ is trivial, we see that complex structures on this bundle correspond to maps $Y \rightarrow S^2$. We can concretely see this by noting that a unit vector in the \mathbb{R} direction of $TY \oplus \mathbb{R}$ will be mapped by the complex structure to a unit vector in TY . The span of these two vectors is a complex line in $TY \oplus \mathbb{R}$ and there is a unique, up to homotopy, complex structure on the orthogonal complement of this complex line. Thus a unit vector field on Y , or dually an oriented plane field, determines the complex structure on $TY \oplus \mathbb{R}$. Clearly there is some plane field, say ξ' , on Y that corresponds to J' on $TY \oplus \mathbb{R}$ (thought of as a hyperplane field along the inner boundary of $Y \times A_{1/2, 3/4}$). We can assume that ξ' is an overtwisted contact structure.

We can now define η' on $D_{1/4}$ using the form $f(r) d\theta + g(r) \alpha'$, where α' is a contact form for ξ' and f and g are functions analogous to the ones used in the definition of $H(\xi)$ in Section 3.1. With the appropriate choice of f and g this is a contact form on the interior of $D_{1/4}$ and the complex structure discussed in the previous paragraph thought of as defined on η' restricted to $Y \times \partial D_{1/4}$ extends to a complex structure compatible the contact structure η' on the interior of $D_{1/4}$. In particular at any point in $Y \times D_{1/4}$ the plane field $\xi \subset TM$ can be assumed to be a J' -complex line in η' . Finally by our choice of J' and ξ' above we can extend η' over $Y \times A_{1/4, 1/2}$ to be $TY \oplus \mathbb{R}$ and extend J' by the homotopy from the paragraph above.

We finally note that η' can be homotoped on $Y \times (A_{1/2, 3/4} \cup A_{3/4, 1})$ so that it is $TY \times \mathbb{R}$ at every point. Thus η' is homotopic to $H(\xi')$ and the complex structure J' has the properties stated in the theorem. We are left to see that $(H(\xi'), J')$ is homotopic to $(H(\xi), J)$. To this end notice that if U is a neighborhood of a page of the open book (Y, π) then M has a handle decomposition with U being the union of 0, 1, and 2-handles and the other handles being index 3 and above. Since $(H(\xi'), J')$ and $(H(\xi), J)$ agree along the cores of the 0, 1 and 2-handles we see from Lemma 3.5 that they are homotopic on all of M . \square

4. COBORDISMS

This section consists of two subsections. In the first we recall the notions of Weinstein cobordisms and Cieliebak and Eliashberg's "Weinstein flexibility results". The following subsection examines Morse functions on the pages of open book decompositions of 5-manifolds.

4.1. Weinstein cobordisms and flexibility. A *cobordism* W is an compact n -manifold with boundary $\partial W = -\partial_- W \cup \partial_+ W$. We say that W is a cobordism from $\partial_- W$ to $\partial_+ W$. A *Morse cobordism* is a pair (W, f) where W is a cobordism and $f: W \rightarrow \mathbb{R}$ is a Morse function having $\partial_\pm W$ as regular level sets.

Recall that a 1-form λ on a manifold W is called a *Liouville form* if $d\lambda$ is a symplectic form. Given such a 1-form there is a unique vector field v such that $\iota_v d\lambda = \lambda$. This vector field is called the *Liouville vector field* associated to λ . Notice that if $\omega = d\lambda$ is a symplectic structure on W then λ can be recovered from v and ω by $\lambda = \iota_v \omega$. We call (ω, v) a Liouville structure on W if $d\iota_v \omega = \omega$.

A *Weinstein cobordism* is a tuple (W, ω, v, f) where W is a compact cobordism, (ω, v) is a Liouville structure on W so that v points out of W along $\partial_+ W$ and into W along $\partial_- W$, and f is a Morse function on W that is constant on each boundary component and for which v is a gradient-like vector field. Recall v is gradient-like for f if there is some constant $\delta > 0$ and some metric so that

$$df(v) \geq \delta(|v|^2 + |df|^2).$$

It is well known, see for example [4], that the Weinstein structure gives W a handle decomposition with handles of index less than or equal to half the dimension of W . We also note that $\iota_v \omega|_{\partial_\pm W}$ is a contact form on $\partial_\pm W$ and thus Weinstein cobordisms are cobordisms between contact manifolds.

Cieliebak and Eliashberg define a *flexible* Weinstein cobordism on a 4-manifold W to be one with only 0 and 1-handles or one whose lower boundary is an overtwisted contact 3-manifold and all the 2-handles are attached along Legendrian knots that have overtwisted disks in their complement.

Theorem 4.1 (Cieliebak and Eliashberg 2012, [4]). *Let (W, f) be a 4-dimensional Morse cobordism such that f has no critical points of index larger than 2. Let η be a non-degenerate 2-form on W and w a vector field near $\partial_- W$ such that (η, w, f) is a Weinstein cobordism structure on some neighborhood N of $\partial_- W$. Suppose that the contact structure induced by $\iota_w \eta$ on $\partial_- W$ is overtwisted. Then there is a flexible Weinstein cobordism structure (ω, v, f) on W such that*

- (1) $(\omega, v) = (\eta, w)$ on some neighborhood N' of $\partial_- W$, and
- (2) the non-degenerate 2-forms ω and η are homotopic on W relative to N' .

We now discuss a slight refinement of the above theorem that follows directly from the proof of that theorem, [3]. We first establish some notation. A *relative cobordism* W is a compact n -manifold with boundary $\partial W = \partial_- W \cup \partial_+ W \cup \partial_v W$ where $\partial_v W \cong -[0, 1] \times \partial(\partial_- W) \cong [0, 1] \times \partial(\partial_+ W)$. We say W is a relative cobordism from the manifold with boundary $\partial_- W$ to the manifold with boundary $\partial_+ W$. We say the *vertical boundary* of W is $\partial_v W$. (There is of course an analogous notion of relative Morse cobordism.)

A *relative Weinstein cobordism* structure on a relative cobordism W is a triple (ω, v, f) as in the ordinary definition of Weinstein cobordism and in addition, f must have no critical points near $\partial_v W$ and v must be tangent to $\partial_v W$. As in the case of Weinstein cobordisms a relative Weinstein cobordism is flexible if it only has 0 and 1-handles or if on its lower boundary the induced contact structure is overtwisted and all the 2-handles are attached along Legendrian knots that have overtwisted disks in their complement.

Theorem 4.2 (Cieliebak and Eliashberg 2012, [3, 4]). *Let (W, f) be a 4-dimensional relative Morse cobordism such that f has no critical points of index larger than 2. Let η be a non-degenerate 2-form on W and w a vector field defined near $\partial_- W \cup \partial_v W$ such that (η, w, ϕ) is a Weinstein cobordism structure on some neighborhood N of $\partial_- W$ and on some neighborhood N_v of $\partial_v W$. Suppose that the contact structure induced by $\iota_w \eta$ on $\partial_- W$ is overtwisted. Then there is a flexible Weinstein cobordism structure (ω, v, f) on W such that*

- (1) $(\omega, v) = (\eta, w)$ on some neighborhood N' of $\partial_- W$ and N'_v of $\partial_v W$ (N' can be any subset of N on which (η, w) gives a Weinstein cobordism structure and similarly for N'_v and N_v), and
- (2) the non-degenerate 2-forms ω and η are homotopic on W relative to $N' \cup N'_v$.

A Weinstein homotopy on a cobordism W is a smooth family of Weinstein structures $(\omega_t, v_t, f_t), t \in [0, 1]$, where we allow the functions f_t to have birth-death type degenerations (that is, they form a generic family of functions), and (ω_t, v_t) is a smooth family of Liouville structures on W . We now state a slight strengthening of Cieliebak and Eliashberg's Weinstein flexibility theorem from [4]. Just as for Theorem 4.2 above, the proof of this is the same as for the non-relative version in [4].

Theorem 4.3 (Cieliebak and Eliashberg 2012, [3, 4]). *Let (ω_0, v_0, f_0) and (ω_1, v_1, f_1) be two flexible Weinstein structures on a 4-dimensional relative cobordism W from an overtwisted contact structure on $\partial_- W$ to some contact structure on $\partial_+ W$. Let*

- (1) $f_t, t \in [0, 1]$, be a Morse homotopy without critical points of index greater than two, and
- (2) $\beta_t, t \in [0, 1]$, be a homotopy of non-degenerate 2-forms connecting ω_0 and ω_1 such that there are neighborhoods N of $\partial_- W$ and N_v of $\partial_v W$ and vector fields w_t defined on $N \cup N_v$ connecting $v_0|_{N \cup N_v}$ to $v_1|_{N \cup N_v}$ such that (β_t, w_t, f_t) has the structure of a relative Weinstein cobordism when restricted to N and when restricted to N_v .

Then there are sub-neighborhoods N' of $\partial_- W$ in N and N'_v of $\partial_v W$ in N_v and there is a homotopy $(\omega_t, v_t, f_t), t \in [0, 1]$, of Weinstein structures agreeing with (β_t, w_t, f_t) on $N' \cup N'_v$, such that the paths of 2-forms β_t and ω_t , for $t \in [0, 1]$, are homotopic relative to $N' \cup N'_v$ and the endpoints.

4.2. Open books and stabilization. Let (Y, π) be an open book decomposition of a closed oriented 5-manifold M . Let $N = Y \times D^2$ be a closed neighborhood of Y in M . We can think of π as a fibration of $M \setminus N$ (and π extends over $N - Y$ by projection to the θ -coordinate of D^2).

We denote the pages (that is fibers of $\pi: (M \setminus N) \rightarrow S^1$) of the open book by $X_\theta = \pi^{-1}(\theta)$. Fixing some $\theta_0 \in S^1$, notice that

$$(M \setminus N) \setminus X_{\theta_0} \cong X \times [0, 1]$$

where $X = X_{\theta_0}$. Moreover there is some diffeomorphism $\phi: X \rightarrow X$ that is the identity near ∂X so that the mapping torus

$$T_\phi = X \times [0, 1]/(x, 1) \sim (\phi(x), 0)$$

is diffeomorphic to $M \setminus N$.

Let f is a Morse function for X_{θ_0} . We denote f on $X \times \{0\}$ by f_0 and think of $\phi^* f$ is a Morse function on $X \times \{1\}$, which we denote by f_1 .

Theorem 4.4. *Given an open book (Y, π) for a closed oriented 5-manifold M and a Morse function $f: X \rightarrow \mathbb{R}$ on a page of the open book with critical points of index less than or equal to 2, there is another open book (Y', π') with monodromy ϕ' and Morse function $f': X' \rightarrow \mathbb{R}$ on its pages so that, in the notation above, when $M \setminus N'$ is cut open along the page X' to obtain $X' \times [0, 1]$ the Morse function $f_0 = f'$ on $X' \times \{0\}$ and $f_1 = \phi'^* f'$ on $X' \times \{1\}$ can be extended to a family of functions $f_t: (X' \times \{t\}) \rightarrow \mathbb{R}, t \in [0, 1]$, satisfying the following:*

- (1) f_t is either a Morse function or has a birth/death degenerate critical point,
- (2) no f_t has critical points if index larger than 2,
- (3) each f_t has a unique index 0 critical point,
- (4) the index 0 critical is constant in t and it can be connected to the boundary by a gradient like flow line that is also constant in t and ϕ' fixes a neighborhood of the index 0 critical point and flow line.

Proof. Let $f_0 = f$ and $f_1 = f \circ \phi$. Let $f_t, t \in [0, 1]$, be a generic family of functions connecting f_0 to f_1 . It is well known, see [2], that such a family has the following structure. For all but finitely many $\{t_1, \dots, t_k\} \subset (0, 1)$ the functions f are Morse functions whose critical points all have distinct values. At each t_i there are either precisely two critical points with the same value or all critical points have distinct values, but one of the critical points is a birth or death point.

The *Cerf graphic* for the family $\{f_t\}$ consists of the points $(t, u) \in [0, 1] \times \mathbb{R}$ where u is a critical value of f_t . It is now a standard argument using manipulations of the Cerf graphic, see [2, 16], that we can alter f_t relative to f_0 and f_1 so that there are no births or deaths of index 0 or 4 critical

points. Thus f_t has a unique index 0 critical point for all t . One can isotope ϕ and then the functions $\{f_t\}$, relative to f_0 and f_1 , so that the critical point and a gradient-like flow line is fixed for all t ; moreover, one can further isotope ϕ and then the functions relative to f_0 and f_1 so that they all agree on a neighborhood of the index 0 critical point and flow line.

We are left to show how to remove the index 3 critical points. We will give a modification of Fenn and Rouke's extension [8] of Kirby's arguments in [16] to the non-simply connected setting. We will be considering the family X_t of cobordisms upside down. That is we use the functions $\{-f_t\}$ so in order to eliminate the index 3 critical points of the $\{f_t\}$ we will eliminate the index 1 critical points of the $\{-f_t\}$.

We begin by identifying a product neighborhood $N = (\partial X) \times [a, b]$ of ∂X so that $\partial X = (\partial X) \times \{a\}$ and isotoping ϕ so that it is the identity map on N . We can also assume that $f|_N: (\partial X) \times [a, b] \rightarrow [a, b]$ is just the projection map (and thus $-f_0$ and $-f_1$ will also be the projection map when restricted to N). We will denote the copy of N in X_i by N_i , $i = 0, 1$.

Following Kirby we can modify $\{-f_t\}$ so that all the index 1 critical points occur as shown on the left of Figure 1. Looking at the bottom most line of index 1 critical points we can isotope $\{-f_t\}$

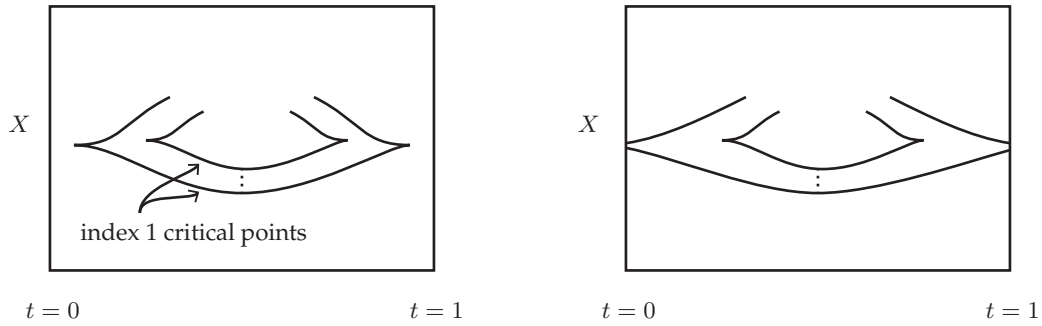


FIGURE 1. On the left, the birth and death of index 1 critical points of f_t . On the right we have moved the birth death points for the lowest index 1 critical point across the left and right boundary of the Cerf diagram.

so that the birth and death points are pushed past the edge of the Cerf graphic as shown on the right of Figure 1. We can assume that the canceling 1/2-handle pair in X_0 and X_1 is contained a small 4-ball $B_0 \subset N_0$ and $B_1 \subset N_1$, respectively, where $B_0 = B_1$.

More precisely, we can assume that an index 1-handle, h_1^1 and canceling index 2-handle, h_2^2 are attached to a 3-ball \tilde{B}_i in $(\partial X) \times \{c\} \subset N_i$, for some $c \in (a, b)$ and $i = 0, 1$; and moreover $\tilde{B}_0 = \tilde{B}_1$. The attaching sphere for h_0^2 is a circle A_0 that is the union of two arcs a_1^0 and a_2^0 with $a_1^0 \subset \tilde{B}_0$ and a_2^0 contained in h_0^1 . Let γ_0 be an arc in \tilde{B}_0 that connects the two components of the attaching region of h_0^1 and is parallel to a_1 . There is an arc γ_0^1 in h_0^1 that is parallel to the core of h_0^1 and completes γ_0 into a circle C_0 . Notice that the 2-handle h_0^2 shows that the circle C_0 is null-homotopic and in particular bounds an embedded disk D_0 .

Moving through the cobordisms $(X_t, -f_t)$ the attaching region of the 1-handle changes by an isotopy in $(\partial X) \times \{c\}$ (since it is the bottom most 1-handle it does not slide over the other 1-handles). This can be extended to an ambient isotopy of $(\partial X) \times \{c\}$ and thus we may push γ_0 to a curve γ_1 in $(\partial X) \times \{c\} \subset N_1$. We can further extend this ambient isotopy to X so that D_0 can be pushed to the disk D_1 in X_1 . In particular we get simple closed curves C_t in X_t . Identifying X with X_0 we let X' be the result of surgering C_0 in X . This will replace the 1-handle (that is index 1 critical point) with a 2-handle (that is index 2 critical point). We can use the circles C_t to surger all the X_t simultaneously to get X'_t that fit together to form $X' \times [0, 1]$. Moreover the functions $-f'_t$ induced from the $-f_t$ and this surgery have one less critical point of index 1. We claim that

- (1) $X' = X \# (S^2 \times S^2)$
- (2) we may extend ϕ to a diffeomorphism ϕ' of X' and
- (3) we may still assume that $f'_1 = f'_0 \circ \phi'$.

Once we establish these three claims we will see how to alter the pair (X', ϕ') to an open book decomposition (X'', ϕ'') of M with the desired properties.

We first observe that since h_0^1, h_0^2 and γ_0 are attached in B_0 and so the surgery on C_0 with the appropriate framing in X_0 clearly results in $X_0 \# (S^2 \times S^2)$. Since we can surger all the C_t in X_t (using the same framing) at the same time and preserve the product structure (we could interpret this as surgering the annulus that is formed as the union of the C_t in $X \times [0, 1]$) we see that the surgered family of cobordisms $X \times [0, 1]$ becomes $X' \times [0, 1]$ where $X' = X \# (S^2 \times S^2)$ and thus establish Claim (1). We note that we can extend the functions f_t restricted to the complement of the surgered regions to the surgery regions to obtain new functions f'_t .

We now establish Claims (2) and (3) by analyzing the curve C_1 . Notice that C_1 runs over the 1-handle h_1^1 and no other 1-handles (again since we are considering the bottom most 1-handle it does not slide over the other 1-handles). In particular $C_1 = \gamma_1 \cup \gamma'_1$, where γ'_1 is the analog of γ'_0 that runs over the 1-handle h_1^1 in X_1 .

The curve C_1 bounds the disk D_1 in $N_1 = (\partial X_1) \times [a, b]$ and so is null-homotopic. The 2-handle h_1^2 is attached along a curve $A_1 = a_1^1 \cup a_1^2$ where a_1^1 is in \bar{B}_1 and a_1^2 is in the handle h_1^1 . The arcs a_1^1 and γ_1 can be connected in $(\partial X_1) \times \{c\}$ to form a simple closed curve c_1 . The 2-handle h_1^2 can be used to show that C_1 and c_1 are isotopic and thus c_1 is null-homotopic in $(\partial X_1) \times \{c\}$ (since $(\partial X_1) \times \{c\}$ is homotopy equivalent to N_1). We can use this null-homotopy to isotope γ_1 in $(\partial X_1) \times \{c\}$ so that it is parallel to a_1^1 (of course this isotopy may cross a_1^1). The family of functions $-f_t$ for $t \in [0, 1]$ can be extended to a family for $t \in [0, 2]$ that is constant for $t \in [1, 2]$. The isotopy above allows us to extend $C_t, t \in [0, 1]$ to a family of curves C_t in X_t for $t \in [0, 2]$ so that $-f_2 = -f_0 \circ \phi$ and $C_0 = \phi(C_2)$. Thus the C_t form a torus in the mapping torus of ϕ and we can perform surgery on C_t in X_t simultaneously for all t . Reparameterizing the interval $[0, 2]$ establishes Claims (2) and (3).

Given $X'_t, -f'_t$ and ϕ' as above we construct X'' by connect summing X' with $S^2 \times S^2$. More specifically we add two 2-handles to X' to obtain $X'' = X' \# S^2 \times S^2$. Since these 2-handles are attached to $\partial X'$ and ϕ' is the identity map here we can clearly extend ϕ' over X'' . Moreover we can extend the $-f'_t$ over X'' to obtain Morse functions on $X''_t = X''$ that satisfy $f''_1 = f''_0 \circ \phi'$.

Recall that $X'' = X \# Z$ where $Z = S^2 \times S^2 \# S^2 \times S^2$. Using Theorem 2.2 we can perform handle slides on the Z part of X'' , encoded by Morse functions $f''_t, t \in [1, 2]$, to obtain a diffeomorphism $\psi: X'' \rightarrow X''$ that is the identity away from Z and gives the monodromy for S^5 described in the theorem. Thus setting $\phi'' = \phi' \circ \psi$ we have a diffeomorphism of X'' and a family of Morse functions $f''_t, t \in [0, 2]$, such that $f''_2 = f''_0 \circ \phi''$. Moreover it is clear that (X'', ϕ'') describes the open book decomposition obtained from (X, ϕ) by boundary summing with the open book from Theorem 2.2. That is (X'', ϕ'') is an open book for $M \# S^5$ according to Lemma 2.1. Moreover the Morse functions f''_t have one less 3-handle than f_t . Thus continuing the above construction for each 3-handle in f_t we eventually arrive at an open book and Morse functions as described in the theorem. \square

5. ALMOST CONTACT STRUCTURES ARE HOMOTOPIC TO CONTACT STRUCTURES

We are now ready to assemble the pieces discussed above to prove that an almost contact structure on a 5-manifold can be homotoped to a contact structure.

Proof of Theorem 1.1. Let (η, J) be any almost contact structure on the closed oriented 5-manifold M . Let (Y, π) be an open book decomposition for M with pages having no handles of index larger than two. Theorem 4.4 allows us to assume that there is a Morse function f on a page of (Y, π) so that f and $f \circ \phi$, where ϕ is the monodromy of (Y, π) , can be connected by a family of functions as described in the theorem. By Theorem 3.6 there is an overtwisted contact structure ξ on Y such that the hyperplane field $H(\xi)$ is homotopic to η and using the homotopy we can think of η as $H(\xi)$ and J as a complex structure on $H(\xi)$. Thus J induces an almost complex structures J_θ on the pages $X_\theta = \pi^{-1}(\theta), \theta \in S^1$, of (Y, π) .

We break M into several parts. Let $N = Y \times D^2$ be a neighborhood of the binding Y in M , as in Theorem 3.6. Let $C = M \setminus N$ be the complement of the interior of N . We can think of π as a fibration $C \rightarrow S^1$ (that extends over $N - Y$ as projection onto the θ -coordinate of D^2). By a slight abuse of notation we will refer to the fibers of $\pi: C \rightarrow S^1$ as X_θ and J_θ will be the almost complex

structures on X_θ . Letting $X = X_{\theta_0}$ for some fixed $\theta_0 \in S^1$ we can write

$$C = X \times [0, 1] / (x, 1) \sim (\phi(x), 0)$$

for some diffeomorphism $\phi: X \rightarrow X$ that is the identity near the boundary of X . According to Theorem 4.4 we can write X as a union of X' and X'' where X' is a neighborhood of the fixed index 0 critical point of $f: X \rightarrow \mathbb{R}$ and $X'' = X \setminus X'$ is the complement of the interior of X' ; moreover, ϕ is the identity on X' . Thus we may decompose C as

$$C = C' \cup C'',$$

where $C' = D^4 \times S^1$ is the mapping torus of $\phi|_{X'}$ and C'' is the mapping torus of $\phi|_{X''}$.

Glancing at Steps 1 and 2 below will convince the reader that it is relatively easy to create a contact structure on C (using Proposition 2.3 for C' and Theorem 4.3 for C''). However, it is not easy to extend this over N . To achieve this extension we need to be more careful in Steps 1 and 2. For this reason we need to further decompose X'' . More specifically, we will show below that there is a cylinder $W' = D^3 \times [0, 1]$ in X'' that breaks X'' into two relative cobordisms $X'' = W' \cup W$ where W is a relative cobordism from D^3 to $Y \setminus B$ (where B is a ball in Y). Moreover ϕ is the identity on X' and W' , the complex structure J restricted to X' is standard and restricted to W' is compatible with (part of) the symplectization of the standard contact structure on the 3-ball D^3 . Moreover we have a Morse function $f: X'' \rightarrow \mathbb{R}$ that satisfies the properties in Theorem 4.4 and has a gradient like vector field that is non-zero on W' and tangent to $\partial W' \cap \partial W$.

We begin by constructing a model situation. Let B be the ball of radius 1 in \mathbb{C}^2 with the standard complex structure. The complex tangencies to $S^3 = \partial B$ give the standard contact structure ξ_{std} on S^3 . Let D^3 be a small ball in $S^3 = \partial B$ on which the contact structure is standard (*i.e.* a Darboux ball). Gluing $D \times [1, 2]$ to B where $D \times \{1\}$ is identified with $D \subset \partial B$ gives another ball R and using a portion of the symplectization of $(\xi_{std})_B$ we can extend the standard symplectic structure and complex structure on B to all of R . Denote this complex structure J' .

We now carefully construct the decomposition $C' \cup C''$ on C . There is some neighborhood U of ∂X such that ϕ is the identity map on U and hence each page X_θ has a corresponding neighborhood U_θ that can be identified with U . In particular, $U' = \cup_\theta U_\theta$ is a neighborhood of ∂C in C and is diffeomorphic to $U \times S^1$. We can assume that $U \cong Y \times [0, 2]$ and the construction in Theorem 3.6 allows us to assume that under the identifications of U with U_θ the complex structure J_θ is independent of θ on U . Again referring to the proof of Theorem 3.6, $Y \times \{t\}$ in each U_θ is J_θ convex and the J_θ complex tangencies to $Y \times \{t\}$ are ξ (recall that ξ is the overtwisted contact structure on Y such that $\eta = H(\xi)$).

Looking at the proof of Theorem 4.4 we can assume the ball X' is a small neighborhood of a point in U' . Moreover we can take an arc γ from $\partial X'$ to ∂X that is transverse to all the $Y \times \{t\}$ in U . Clearly there is a diffeomorphism from R , constructed above, and X' union a neighborhood of γ so that B maps to X' and $D \times [1, 2]$ maps to a neighborhood of γ . We can also arrange that $D \times \{t\}$ maps to a ball in $Y \times \{t\}$ and by (a parametric version of) Darboux's theorem the J' -complex tangencies to $D \times \{t\}$ in R map to the J -complex tangencies to the ball in $Y \times \{t\}$. We can now homotope, if necessary, the complex structure J so that it agrees with J' on the image of R and the complex tangencies to $Y \times \{t\}$ are not changed for $t \in (1, 2]$ (but we might change the complex tangencies on $[0, 1]$). Since J is θ -invariant in U' we can clearly arrange this on each page X_θ . We now set $W = X' \setminus R$ and $W' = D \times [1, 2]$ to get the desired decomposition of X'' and notice that Theorem 4.4 still gives the desired Morse function.

Step 1: *The contact structure on C' .* Proposition 2.3 gives a contact structure ζ on $C' = D^4 \times S^1$ such that ζ induces the overtwisted contact structure ξ_{ot} on $\partial D^4 \times \{\theta\}$ for each $\theta \in S^1$. The proposition also gives the following specific form for the contact structure near $\partial C'$: if α_{ot} is a 1-form for which $\xi_{ot} = \ker \alpha_{ot}$ then in a neighborhood $S^3 \times (1/2, 1] \times S^1$ of the boundary of $D^4 \times S^1$ the contact structure is given by

$$\zeta = \ker(Kd\theta + t\alpha_{ot}),$$

where t is the coordinate on $(1/2, 1]$, θ is the coordinate on S^1 , and K is any positive constant. Proposition 2.3 allows us to arrange that $\ker(\alpha_{ot})$ is standard on the disk D^3 where W' intersects $\partial X'$.

We note that ξ_{ot} is homotopic to ξ_{std} on S^3 relative to the disk D^3 and thus there is a homotopy of the almost complex structure on $(D^4 = X') \cup X''$, relative to W' , so that $\zeta \cap (\partial D^4 \times \{\theta\})$ is the set of complex tangencies to $\partial D^4 = \partial_- X''$. By Proposition 2.3, ζ is homotopic, through almost contact structures, to $\ker(Kd\theta + \lambda)$, where λ is a primitive for the standard symplectic form on D^4 . And this contact structure is homotopic to the almost contact structure η' coming from the tangencies to B^4 (with the almost complex structure above).

Step 2: Extending the contact structure to C'' . Recall that

$$C'' = X'' \times [0, 1]/(x, 1) \sim (\phi(x), 0).$$

In addition $X'' = W' \cup W$ and we can think of W' as part of the symplectization of the standard contact structure on a 3-ball D^3 , that is $W' = D^3 \times [1, 2]$ and if α is a contact form for the standard contact structure on D^3 then $d(t\alpha)$ is a symplectic form on W' and it is compatible with the almost complex structure J . Thus we see that $W' \times S^1 \subset C''$ already has a contact structure given by $\ker(Kd\theta + \alpha)$. Note also that $(d(t\alpha), v = \partial_t, f|_{W'})$ gives W' the structure of a relative Weinstein cobordism.

We are left to consider

$$W \times [0, 1]/(x, 1) \sim (\phi(x), 0).$$

We will denote $W \times \{t\}$ by W_t and the almost complex structure on W_t by J_t .

Recall that f and $f \circ \phi$ are connected by a family of functions as in Theorem 4.4. Thinking of f as a Morse function on W_0 and $f \circ \phi$ as a Morse function on W_1 the family of functions from Theorem 4.4 can be thought of as giving functions $f_t: W_t \rightarrow \mathbb{R}$. (We are of course considering the restriction of f to W , but leave this out of the notation for convenience.)

Recall that J_t are all the same near the lower boundary $\partial_- W_t$ so the complex tangencies on $S^3 = \partial_- X_t$ give the standard overtwisted contact structure ξ_{ot} . In addition all the J_t are the same near the vertical boundary $\partial_v W$. By extending W into W' and X' slightly we can assume that there is a neighborhood of $\partial_- W$ that has the structure of a Weinstein cobordism and there is a neighborhood of $\partial_v W$ that has the structure of a Weinstein cobordism.

Recall that for any almost complex structure the space of compatible non-degenerate 2-forms is contractible. Thus we can use the almost complex structures J_t to find a path β_t of non-degenerate 2-forms on W_t such that $\beta_1 = \phi^* \beta_0$. Moreover we can assume that β_i agrees with the 2-forms defining the Weinstein cobordism structures on the neighborhoods of $\partial_- W_t$ and $\partial_v W_t$.

We now claim that there are vector fields v_i on W_i , $i = 0, 1$ such that $(\phi^{-1})_* v_1 = v_0$ and we can assume that (β_0, v_0, f_0) and (β_1, v_1, f_1) are flexible Weinstein structures on W_0 and W_1 , respectively.

To see this we first note that via homotopy we can assume that β_t and f_t are fixed near $t = 0$ and $t = 1$. Now recall that Theorem 4.2 says that there is a homotopy of β_0 to a symplectic form ω and there exists a vector field v such that (ω, v, f_0) is a flexible relative Weinstein cobordism structure on W_0 . We can now extend $W \times [0, 1]$ to, say, $W \times [-\epsilon, 1]$, for some $\epsilon > 0$, and use the homotopy of 2-forms between β_0 and ω to define non-degenerate 2-forms on W_t for $t \in [-\epsilon, 0]$. We can also extend f_t so that it is constant on $[-\epsilon, 0]$. Moreover pulling back the homotopy of 2-forms via ϕ we can also extend our β_t and f_t over $[1, 1 + \epsilon]$. We now have (β_t, f_t) defined for all $t \in [-\epsilon, 1 + \epsilon]$ so that the structures at the endpoints, together with v and $\phi_* v$, define relative Weinstein cobordism structures on W . Moreover all the properties from Theorem 4.4 for the functions f_t are unchanged and $W \times [-\epsilon, 1 + \epsilon]$ can be used to recover C'' . Thus re-parameterizing $[-\epsilon, 1 + \epsilon]$ to $[0, 1]$ establishes the claim. (Notice that the homotopy of the 2-forms can be taken to be fixed in a neighborhood of $\partial_- W \cup \partial_v W$.)

Noting that $f_t, t \in [0, 1]$, is a Morse homotopy without critical points of index greater than 2, we can now apply Cielieback and Eliashberg's theorem, Theorem 4.3 above, to get a path (ω_t, v_t, f_t) , $t \in [0, 1]$, of Weinstein structures such that ω_t is homotopic to β_t relative to the end points and a neighborhood of $\partial_- W$ and $\partial_v W$. Now set $\lambda_t = \iota_{v_t} \omega_t$ and consider the 1-form

$$\alpha = Kdt + \lambda_t$$

on W , where K is some constant. Note

$$\alpha \wedge d\alpha \wedge d\alpha = K dt \wedge d\lambda_t \wedge d\lambda_t + 2 dt \wedge \frac{d\lambda_t}{dt} \wedge \lambda_t \wedge d\lambda_t.$$

So for any K sufficiently large, α defines a contact form on $W \times [0, 1]$ that extends the one on $W' \times [0, 1]$ and clearly descends to a contact form on C'' . Since we have a precise formula for the contact form near $\partial_- X \times S^1 \subset C''$ that agrees with the one near $\partial C' \subset C'$ we can clearly extend the contact form from Step 1 over C'' using α . We denote the associated contact structure ζ . Moreover the contact structure ζ is clearly homotopic through almost contact structures to the tangents to the pages of the open book on C'' (just let the constant K go to infinity).

We wish to use Eliashberg's computation of the homotopy type of overtwisted contact structures, Theorem 2.5. To this end we study the loop of contact structures $\xi_\theta = \ker(\lambda_\theta)$ on Y . First notice that the homotopy of 1-forms from the β_t , compatible with the J_t , to the symplectic forms ω_t , can be covered by a homotopy of almost complex structures $(J_t)_s$, $s \in [0, 1]$. We can arrange that the complex structures $(J_t)_1$ that are compatible with ω_t also have ξ_θ as the set of complex tangencies to $Y = \partial_+ X''$.

The original almost complex structures on X_θ induced the constant loop of plane fields ξ on Y as the complex tangencies to $\partial_+ X'' = Y$. The homotopy of almost complex structures on $(J_t)_s$ induces a homotopy of this loop to the loop $\xi_\theta = \ker(\lambda_t)$. Moreover this loop is constant on the ball $\partial_+ W' \subset Y$. Thus ξ_θ is homotopic to a constant loop of plane fields in the space $\mathcal{P}_p(Y)$ (for any $p \in \partial_+ W'$). (We are using the notation from Section 2.4.) To apply Theorem 2.5 we need to see that we can assume that ξ_θ have a fixed overtwisted disk through p .

Step 3: *Create a fixed overtwisted disk in each fiber of ∂C .* We may identify a neighborhood N'' of ∂C in C with $(1/2, 1] \times Y \times S^1$. Under this identification the contact form α can be written $K d\theta + s\alpha_\theta$, where s is the coordinate on $(1/2, 1]$, θ is the coordinate on S^1 and α_θ is a contact form on Y for each θ . Moreover in $D^3 = \partial_+ W' \subset Y$ the contact forms are independent of θ and induce the standard contact structure on the ball. Let S be a solid torus in D^3 that contains the point p discussed in the previous paragraph and let $T^2 \times [0, 1]$ be a neighborhood of ∂S that does not contain p . We use the contact structure constructed in Lemma 2.4 to remove $(1/2, 1] \times T^2 \times [0, 1] \times S^1$ from C and replace it with the contact structure on $(1/2, 1] \times T^2 \times [0, 1] \times S^1$ described in the lemma. In particular we can assume that S is a "Lutz tube" and, more to the point, there is a disk D in D^3 that is overtwisted in the contact structures ξ'_θ induced on Y thought of as the boundary of X_θ with this new contact structure on C . Each ξ'_θ differs from ξ_θ by a Lutz twist. And by the discussion in Section 2.4 we see that the loop ξ'_θ is homotopic to ξ_θ in $\mathcal{P}_p(Y)$. This completes Step 3. For convenience we rename ξ'_θ to ξ_θ .

Step 4: *Construct a contact structure on N .* From the previous step we know that $\xi_\theta = \ker \alpha_\theta$ is a loop of contact structures on Y with a fixed overtwisted disk that is contractible in $\mathcal{P}_p(Y)$. Thus according to Eliashberg's result, Theorem 2.5, the loop is contractible through contact structures. That is for each $r \in [0, 1]$ there is a loop α_θ^r of contact forms on Y such that $\alpha_\theta^1 = \alpha_\theta$ and α_θ^0 is independent of θ . Moreover, we can assume that α_θ^0 is independent of r near 0 and near 1. Consider the 1-form

$$\beta = f(r) d\theta + g(r) \alpha_\theta^r$$

on $Y \times D^2$. One can compute that

$$\beta \wedge d\beta \wedge d\beta = g(f'g - g'f) dr \wedge d\theta \wedge \alpha_\theta^r \wedge d\alpha_\theta^r - g^2 dr \wedge d\theta \wedge \frac{\partial \alpha_\theta^r}{\partial r} \wedge \left(f d\alpha - g \alpha \wedge \frac{\partial \alpha_\theta^r}{\partial \theta} \right).$$

Where α_θ^0 is independent of r the last term vanishes. Thus as long as g is positive and $(f'g - g'f) > 0$, β will be a contact form. Where α_θ^r does depend on r we will take g to be constant and $f(r) = e^{cr}$. If g is any fixed constant and c is chosen sufficiently large, β will be a contact form in this region too.

Specifically, let $0 < r_1 < r_2 < 1$ be constants such that α_θ^r is independent of r outside $[r_1, r_2]$. Let $r_3 < r_4$ be any numbers in $(r_2, 1)$. We can choose $f(r)$ to be r^2 near 0, e^{cr} on $[r_1, r_2]$, strictly

increasing on $[0, r_4]$ and some large constant K on $[r_4, 1]$. We also choose $g(r)$ to be 2 on $[0, r_3]$, strictly decreasing on $[r_3, 1]$ and equal to $2 - r$ near 1.

If D_a is the disk of radius a then we clearly see that β defines a contact structure on $Y \times D_a$ for any $a > 1$.

Using the coordinates $(1/2, 1] \times Y \times S^1$ for a neighborhood of ∂C in C from above, we can glue $Y \times D_a$ to C by a diffeomorphism

$$\Psi: \{(p, r, \theta) : 1 < r < a\} \rightarrow (1/2, 1] \times Y \times S^1 : (p, r, \theta) \mapsto (2 - r, p, \theta).$$

It is clear that Ψ pulls the contact from from Step 2 back to the one constructed in above. Thus Ψ can be used to glue the contact structures on N and C together, thus extending ζ on C over N .

Step 5: See that ζ is homotopic to (η, J) . Notice in the construction above ζ is clearly homotopic to (η, J) along a page of the open book (Y, π) . Thus, just as in the proof of Theorem 3.6 we see that (η, J) and ζ are homotopic as almost contact structures. (Recall this is simply because almost contact structures on a 5-manifolds are determined up to homotopy by their restriction to the 2-skeleton.) \square

Remark 5.1. We note that in [14] Giroux defines the notion of a contact structure being *supported* by an open book. While we use open books to construct our contact structures, none of the contact structures coming from Theorem 1.1 are supported by the open book used in the proof of the theorem. In particular, it is possible that the given open book does not support any contact structures.

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