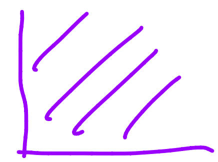
Manifolds with corners

a <u>manifold with corners</u> is a topological space W that is Hausdorff, 2^{nd} countable, and has coordinate charts in $\mathbb{R}^n$, $\mathbb{R}^n_+$, or $\mathbb{R}^n_{++}$

$\{(x_1 \dots x_n) \mid x_n \geq 0\}$ $\qquad$ $\{(x_1 \dots x_n) \mid x_n, x_{n-1} \geq 0\}$

topologically W is a manifold with boundary and $\partial W =$ pts corresp to $x_n = 0$ in $\mathbb{R}^n_+$ and $x_n x_{n-1} = 0$ in $\mathbb{R}^n_{++}$

now let $\mathcal{A}$ be an atlas of coordinate charts on W from $\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_{++}$ that are smoothly compatible

this gives W a smooth structure

now W is <u>not</u> a smooth manifold with boundary!

let $\lrcorner W = \{p \in W : p$ corresp to $x_n = x_{n-1} = 0$ in some chart in $\mathbb{R}^n_{++}\}$

$\partial W = \{p \in W : p$ corresp $x_n = 0$ in $\mathbb{R}^n_+$ or $x_n = x_{n-1} = 0$ but not $x_n = x_{n-1} = 0$ in $\mathbb{R}^n_{++}\}$

<u>exercise</u>: 1) $\lrcorner W$ is an $(n-2)$-dim'l manifold

2) $\overline{\partial W} = \partial W \cup \lrcorner W$

3) if $\lrcorner W$ is 2 sided in $\overline{\partial W}$ the each component of $\overline{\partial W} \setminus \lrcorner W$ has boundary a subset of $\lrcorner W$

4) if W_1, W_2 are manifolds w/∂, then $W_1 \times W_2$ is a manifold with corners

lemma:

1) If C is a component of ∂W ($\partial W = \emptyset$), then there is an embedding $[-\varepsilon, 0] \times C \to W$ whose image is a nbhd of C and $\{0\} \times C$ maps to C by the identity. Moreover any 2 such embeddings differ by ambient isotopy

2) if C is a component of ∂W and C is 2-sided, then $\exists$ an embedding $h : [-\varepsilon, 0] \times [-\varepsilon, 0] \times C \to W$ whose image is a nbhd of C, $h(0,0,p) = p$, and $h(\{0\} \times [-\varepsilon, 0] \times C)$ is a collar of one component of C is $\overline{\partial W} \setminus \partial W$ and $h([-\varepsilon, 0] \times \{0\} \times C)$ is a collar in the other. Moreover any 2 such embeddings differ by ambient isotopy

maybe prove later

lemma (rounding corners):

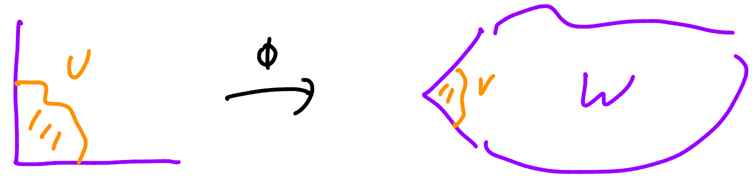
if W is a manifold with corners, there is a manifold with boundary M and a homeomorphism $h : W \to M$ that is a diffeomorphism off of ∂W

Moreover M is unique up to diffeomorphism

Proof: $M = W$ as a topological mfd

for $p \in W \setminus \partial W$ we take coord. charts of M from W

for $p \in \partial W$ we have a coord chart



$z \mapsto z^2$

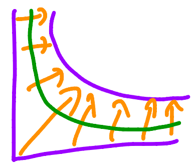
let $S : \mathbb{R}^n_+ \to \mathbb{R}^n_{++} : (x_1, \ldots x_{n-2}, x_{n-1}, x_n) \longmapsto (x_1, \ldots x_{n-2}, \underbrace{x_{n-1}^2 - x_n^2, 2x_{n-1}x_n}_{z \mapsto z^2})$

and let $\phi \circ S$ be a coord chart for M

this gives an atlas on M & $\therefore$ a smooth str. $\blacksquare$

exercise:

take a vectorfield v along $\partial W \cup \partial W$ pointing into W
let M be a co-dim 1 submfd of W in collar nbhd defined by v and $\pitchfork$ to v everywhere
Then the "interior" to M of W is diffeomorphic to W with rounded corners from lemma



exercise: Show B^{n+m} is $B^n \times B^m$ with corners rounded

we can reverse "rounding corners"

lemma:

If M is a submanifold of ∂W that is codim 1 and separating then $\exists$ a manifold with corners N and a homeomorphism $\phi : W \to N$ that is a diffeo off of M and takes M to $\angle N$

just reverse proof of above lemma

exercise: show that the two processes above are inverses

Cutting and Gluing

Given 2 manifolds W_1 and W_2 with boundary let $M_i \subset \partial W_i$
and $f: M_1 \to M_2$ be a diffeomorphism

by lemma above $\exists$ collar nbhds
$$\phi_i : [-\varepsilon, 0] \times M_i \to W_i$$

let $W = W_1 \cup W_2 / p \sim f(p)$ with the quotient topology
let $q: W_1 \cup W_2 \to W$ be quotient map

define $\phi: [-\varepsilon, \varepsilon] \times M_1 \longrightarrow W : (t,p) \mapsto \begin{cases} q(\phi_1(t,p)) & t \in [-\varepsilon, 0] \\ q(\phi_2(-t, f(p))) & t \in [0, \varepsilon] \end{cases}$

easy to check this is a topological embedding

define charts on $\operatorname{int} W_i \subset W$ from W_i (and q)
and on $p \in M_1 \subset W$ from ϕ

easy to check these charts are compatible so W has
a smooth str

we say W is the result of gluing W_1 and W_2 via f
and denote it $W_1 \cup_f W_2$

lemma:
the smooth str on $W_1 \cup_f W_2$ is well-defined upto diffeo
and the natural inclusions
$$W_i \to W_1 \cup_f W_2$$
are smooth embeddings

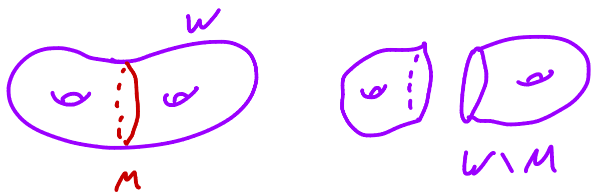
the proof is a nice exercise since collar nbhds
well-defined upto ambient isotopy

exercise:

1) If $M_1 \subset \overline{\partial W_1}$ is a component of $\partial W_1 \setminus \angle W_1$ then we can still glue to get manifold (with less corners)

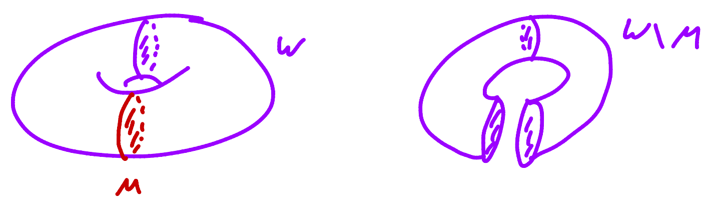
2) the result of gluing W and $\partial W \times [0,1]$ by
$$\begin{aligned} \partial W &\to \partial W \times \{0\} \\ p &\mapsto (p,0) \end{aligned}$$
is diffeomorphic to W

now if $M \subset W$ is a codim 1 submanifold let N be a tubular nbhd of M in W the manifold
$$W \setminus M = \overline{W - N}$$
is called the result of cutting W along M



exercise: 1) Show cutting and gluing are inverse opperations

2) Show how to cut along a submfd w/∂ to get a manifold with corners



Important examples.

Connected sum: let W_1, W_2 be 2 n-manifolds

$f_i : D^n \hookrightarrow W_i$ embeddings st. f_1 preserves and f_2 reverses orientation if W_i oriented

let $W_1 \# W_2 = \overline{W_1 - D_1} \cup_{f_1 \circ f_2^{-1}} \overline{W_2 - D_2}$

one can show $W_1 \# W_2$ is well-defined if W_i connected since orientation pres. embeddings $D^n \to W^n$ are unique upto isotopy (since normal bundle of a point)

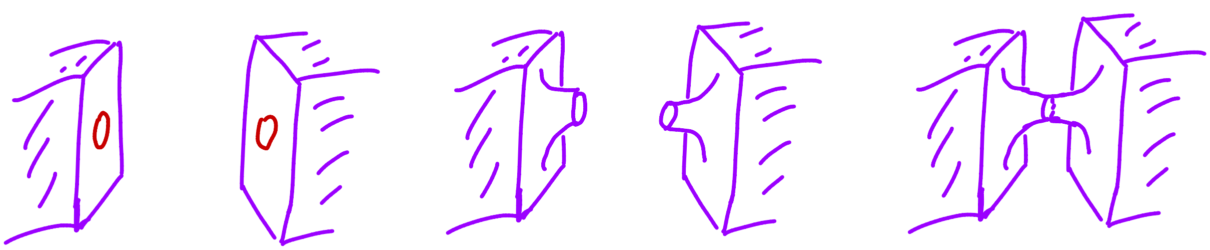
boundary connected sum

if W_1, W_2 are n-manifolds with boundary and

$f_i : D^{n-1} \to \partial W_i$ are embeddings

then we can introduce corners along $f_i(\partial D^{n-1})$

and set $W_1 \natural W_2 = W_1 \cup_{f_1 \circ f_2^{-1}} W_2$



exercise: 1) if ∂W_i connected, then $W_1 \natural W_2$ well-def.

2) $W^n \# S^n \cong W$

3) $W^n \natural D^n \cong W$

4) $\partial(W_1 \natural W_2) = \partial W_1 \# \partial W_2$