Jet Spaces

Jet spaces keep track of derivatives of functions

we say 2 functions $f, g : W \to M$ have k^{th} order contact at $p \in X$ if

1) $f(p) = g(p)$ (call this q)

2) in any coordinate chart
$\phi : \underset{\subseteq \mathbb{R}^n}{U} \longrightarrow \underset{\subseteq W}{V}$ about $p \in W$
$\psi : \underset{\subset \mathbb{R}^m}{U'} \longrightarrow \underset{\subseteq M}{V'}$ about $q \in M$

all partial derivatives of order $\leq k$ of $\psi^{-1} \circ f \circ \phi$ and $\psi^{-1} \circ g \circ \phi$ agree at p

note: $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are k-equivalent at $p \Leftrightarrow k^{th}$ order Taylor Polynomials at p agree

exercise: this is an equivalence rel$^{\underline{n}}$

denote the equiv. relation by $\sim_k$ and the equivalence class of f by $j^k_{p,q}(f)$

set $J^k_{p,q}(W, M) = \{ j^k_{p,q}(f) \mid f : W \to M,\ f(p) = q \}$

and $J^k(W, M) = \bigcup_{\substack{p \in W \\ q \in M}} J^k_{p,q}(W, M)$ this is the jet space of functions from W to M

given $\sigma \in J^k(W, M)$ $\exists$ some p, q s.t. $\sigma \in J^k_{p,q}(W, M)$
we define $\alpha(\sigma) = p$ and call it the source
and $\beta(\sigma) = q$ and call it the target

so we have 2 maps

$$\begin{tikzcd} & J^k(W,M) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ W & & M \end{tikzcd}$$

given $f \in C^\infty(W,M)$ get its k-jet $j^k(f): W \to J^k(W,M)$

$$p \mapsto j^k_{p,f(p)}(f)$$

you should think of $j^k(f)$ as its k^{th} order Taylor polynomial at p

examples: 1) $J^0(W,M) = W \times M$

and $j^0(f): W \to W \times M$ graph!

$$p \mapsto (p, f(p))$$

2) exercise: a) $J^1(W,M) \cong \operatorname{Hom}(TW, TM)$

$$\sigma = j^k_{p,q}(f) \mapsto df_p \in \operatorname{Hom}(T_pW, T_qM)$$

bundle over $W \times M$ with fiber $T_pW \xrightarrow{\ell} T_qM$

b) $J^1(W,\mathbb{R}) \cong T^*W \times \mathbb{R}$

$$j^1(f) = (df, f)$$

Facts: 1) $J^k(W^n, M^m)$ is a smooth manifold of dimension

$$n + m + \dim(B^k_{n,m})$$

space of m-tuples of polynomials of degree $\leq k$ in n variables

e.g. $\dim J^1(W^n, M^m) = n + m + mn$

exercise: work out $\dim(B^k_{n,m})$

2) $\alpha: J^k(W,M) \to W$, $\beta: J^k(W,M) \to M$, $\alpha \times \beta: J^k(W,M) \to W \times M$ are fiber bundles

3) if $h: M \to N$ is smooth then

$$h_*: J^k(W,M) \to J^k(W,N)$$
$$\sigma = j^k_{pq}(f) \mapsto j^k_{p,h(q)}(\phi \circ f)$$

is well-defined and smooth

4) if $g: W \to M$ smooth, then so is $j^k g: X \to J^k(W,M)$

Th$^{\underline{m}}$ (Thom Transversality):

let W and M be smooth manifolds and Σ a submanifold of $J^k(W,M)$. The set of maps $f \in C^\infty(W,M)$ s.t.

$$j^k(f) \pitchfork \Sigma$$

is risidual in $C^\infty(W,M)$

moreover, if f is such that $j^k f \pitchfork W$ on a closed set C in W, then can find approximation $\tilde{f}$ st. $\tilde{f} = f$ on open nbhd of C and $j^k(f) \pitchfork \Sigma$

intersection of countably many open dense sets

$\Rightarrow$ dense here

if W cpt then open & dense

Proof of Morse Theory Th$^{\underline{m}}$:

recall $J^1(W,\mathbb{R}) \cong T^*W \times \mathbb{R}$

let Z = zero section in T^*W and $\Sigma = pr_1^{-1}(Z) \subset J^1(W,\mathbb{R})$

where $pr_1: J^1(W,\mathbb{R}) \to T^*W$

from earlier we know f is Morse $\Leftrightarrow df \pitchfork Z$

this is equivalent to $j^1(f) \pitchfork \Sigma$

now done! by Thom $\boxplus$

fun bonus!

Thm (Whitney immersion thm):

$\boxed{\begin{array}{l} W, M \text{ smooth manifolds with } \dim M \geq 2 \cdot \dim W \\ \text{then immersions are residual in } C^\infty(W,M) \end{array}}$

rank $df_p = \dim M$ $\forall p$ — open dense if W compact

to set this up let $\dim W = n < \dim M = m$

given $\sigma \in J^1(W,M)$ let f represent σ $(\sigma \in J^1_{pq}(W,M))$

to corank $\sigma = n - \operatorname{rank} df_p$

set $S_r = \{\sigma \in J^1(W,M) \mid \operatorname{corank} \sigma = r\}$ $\quad r = 0, \dots, n$

note: $f: M \to W$ an immersion $\Leftrightarrow$ $f \cap \left(\bigcup_{r=1}^{n} S_r\right) = \emptyset$

exercise: S_r is a submfd of codim $= (m-n+r)r$

Steps: 1) consider vector spaces V, U of dim n, m, resp.

let $L^r(V,U) = \{\text{linear } L: V \to U \text{ of corank} = r\}$

show $L^r(V,U)$ a submfd of $\operatorname{Hom}(V,U)$ of codim $= (m-n+r)r$

Hint: If $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ an $m \times n$ matrix with A a $k \times k$ invertable matrix

then $\operatorname{rank} S = k \Leftrightarrow D - CA^{-1}B = 0$

Step 2: show S_r a bundle over $W \times M$ with fiber $L^r(\mathbb{R}^m, \mathbb{R}^n)$

Proof: for $r \geq 1$ $\quad$ codim $S_r = (n-m+r)r \geq (2n - n + 1) = n+1$

so if $f: W \to M$ s.t. $j^1(f) \pitchfork S_r$ then $j^1(f) \cap S_r = \emptyset$

$\therefore$ Thom $\Rightarrow$ result $\blacksquare$

sometimes need to look at jets of a function at many points of W

let $W^{(s)} = \{(p_1, \dots p_s) \in \overbrace{W \times \dots \times W}^{W^s} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq W^s$

consider $(J^r(W,M)) \xrightarrow{\alpha^s} W^s$ source map

and set $J^r_s(W,M) = (\alpha^s)^{-1}(W^{(s)})$

this is s-tuples of r-jets w/distinct sources

given $f: W \to M$ we get the s-multi r-jet

$$j^r_s(f): W^{(s)} \to J^r_s(W,M)$$
$$(p_1, \dots, p_s) \mapsto (j^r_{p_1}(f), \dots, j^r_{p_s}(f))$$

Thm (Thom Multijet Transversality):

let W and M be smooth manifolds and Σ a submanifold of $J^k_s(W,M)$. The set of maps $f \in C^\infty(W,M)$ s.t.

$$j^k_s(f) \pitchfork \Sigma$$

is residual in $C^\infty(W,M)$

moreover, if f is such that $j^k_s f \pitchfork W$ on a closed set C in W, then can find approximation $\tilde{f}$ s.t. $\tilde{f} = f$ on open nbhd of C and $j^k_s(\tilde{f}) \pitchfork \Sigma$

Cor:

The set of Morse functions $f: W \to \mathbb{R}$ all of whose critical values are distinct is dense in $C^\infty(M, \mathbb{R})$

Proof: $\dim J^1_2(W, \mathbb{R}) = 4n+1$, $\dim W^{(2)} = 2n$ $(\dim W = n)$

let $\Delta_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid x = y\} \subset \mathbb{R}^2$

let Z = zero section in T^*W and $\Sigma = pr_1^{-1}(Z) \subset J^1(W,\mathbb{R})$

where $pr_1 : J^1(W,\mathbb{R}) \to T^*W$

(recall f is Morse $\Leftrightarrow f \pitchfork \Sigma$)

and $S = \beta^2(\Delta) \cap (\Sigma \times \Sigma) \subset J_2^1(W,\mathbb{R})$

exercise: In local coords on W, check S is a submanifold of codim $= 2n+1$

Thom $\Rightarrow$ dense set of $f \in C^\infty(M,\mathbb{R})$ have $j_2^1(f) \pitchfork S$

domain $j_2^1(f)$ is $W^{(2)}$ of dim $2n$

$\therefore j_2^1(f) \cap S = \emptyset$

given this if p,q are critical points of f

then $(j_p^1(f), j_q^1(f)) \in Z \times Z$

since $j_2^1(f) \cap S = \emptyset$ this $\Rightarrow f(p) \neq f(q)$ ▦

Th$^{\underline{m}}$ (Whitney Embedding Th$^{\underline{m}}$):

> let W, M be mfds with $\dim M = m \geq 2\underbrace{\dim W}_{n}+1$
> Then the set of 1-1 immersions of $W \to M$ is dense in $C^\infty(W,M)$
> If W is compact then so is the set of embeddings

Proof: $f : W \to M$ is 1-1 iff $j_2^0(f) \cap (\beta^2)^{-1}(\Delta_M) = \emptyset$

$\beta^2 : J_2^0(W,M) \to M^{(2)}$ is surjective so $(\beta^2)^{-1}(\Delta_M)$ a submfd

of codim $=$ codim $(\Delta_M \subset M \times M) = \dim M > 2\dim W$

domain $j_2^0(f)$ has dim $= 2 \dim W$

$\therefore$ if $j_2^0(f) \pitchfork (\beta^2)^{-1}(\Delta_M)$, then $j_2^0(f) \cap (\beta^2)^{-1}(\Delta_M) = \emptyset$

Thom $\Rightarrow$ dense set of f with this property $\therefore$ such f are 1-1

can in addition take f an immersion from previous thm (really need immersions & 1-1 maps residual, not just dense, but they are!) $\boxplus$