A. Cobordisms

our main goal will be to study cobordisms

a cobordism W^{n+1} is an oriented $(n+1)$ dimensional manifold
and a splitting of its boundary

$$\partial W = (-\partial_- W) \cup (\partial_+ W)$$

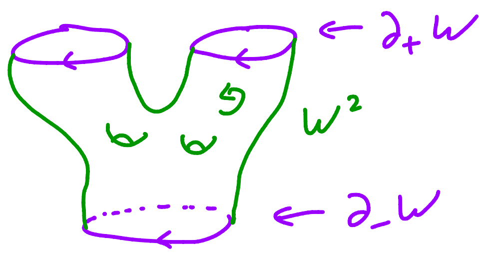
reverse orientation

into disjoint components

lower boundary

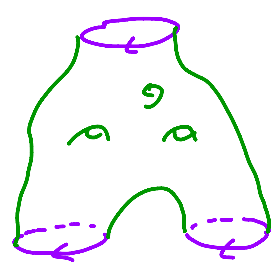
upper boundary

we say W is a cobordism from $\partial_- W$ to $\partial_+ W$



denote: $\partial_- W \xrightarrow{W} \partial_+ W$

Remarks: 1) a cobordism with $\partial = \emptyset$, is just a manifold
2) a cobordism with $\partial_- = \emptyset$, is just a manifold w/ boundary
3) can consider cobordisms w/o orientation too, then
ignor the $-$ before $\partial_- W$
4) notice W is also a cobordism from $-\partial_+ W$ to $-\partial_- W$



and $-W$ is a cobordism from $-\partial_- W$ to $-\partial_+ W$

we will also consider generalizations to cobordisms of manifolds
with boundary

a cobordism of manifolds with boundary is an oriented $(n+1)$ manifold
W^{n+1} whose boundary is written

technically a mfd with corners

$$\partial W = (-\partial_- W) \cup (-\partial_v W) \cup (\partial_+ W)$$

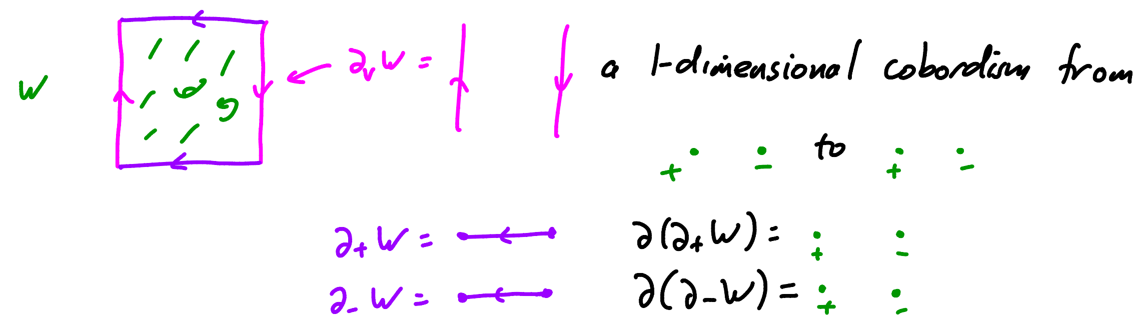
where 1) $\partial_- W \cap \partial_+ W = \emptyset$

2) $\partial_v W$ is a cobordism from $\partial_-(\partial_v W)$ to $\partial_+(\partial_v W)$

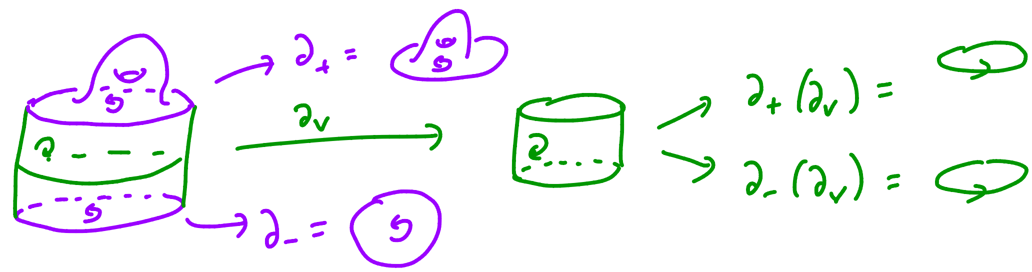
3) $\partial(\partial_+ W) = \partial_+(\partial_v W)$ and $\partial(\partial_- W) = \partial_-(\partial_v W)$

examples:

1)



2)



Remark: if every component of $\partial_+ W$ and $\partial_- W$ have boundary and $\partial_v W$ is a product (i.e. $\partial_v W = [1,0] \times \partial(\partial_- W)$) then W is sometimes called a sutured manifold

(some times allow certain closed components
e.g. Gabai's sutured 3-mfds are allowed to have T^2 components in $\partial_v W$)

as a first step to understanding cobordisms we consider Morse functions

B. Morse functions

let $f: W \to \mathbb{R}$ be a smooth function

a critical point of f is a point $p \in W$ such that $df_p : T_p W \to T_{f(p)} \mathbb{R}$ is not surjective

in this case that means $df_p = 0$

image of $z: M \to T^*M$, $p \mapsto 0$

notice $df: W \to T^*W$ is a section of the cotangent bundle and p is a critical point $\Leftrightarrow$ df intersects the zero section Z at p

we say a critical point is non-degenerate if df is transverse to Z at p ($f \pitchfork Z$ at p)

exercise:

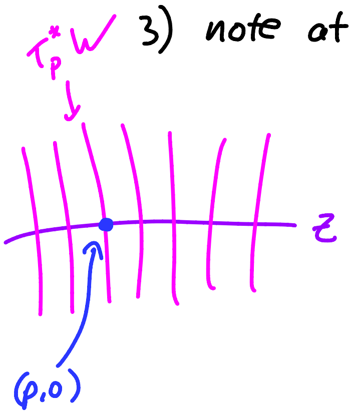
1) Show non-degenerate critical points are isolated. (think what transversality means)

2) let p be a critical point of f. Show p is non-degenerate

$\iff$

$\exists$ local coordinates $\phi : \underset{\mathbb{R}^n}{U} \to \underset{W}{V}$ (can assume $\phi(0) = p$)

such that the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is non-singular



3) note at a point $(p,0) \in Z \subset T^*W$ we have

$$T_{(p,0)}(T^*W) = T_{(p,0)} Z \oplus \underset{T_p^*M}{\underset{\shortparallel}{\ker (d\pi)_{(p,0)}}}$$ where $\pi : T^*W \to W$ is projection

so we have a map $C_p : T_{(p,0)}(T^*W) \to T_p^*W$ only well-defined along zero section!!

if p is a critical point of f then define

$$(d^2f)_p : T_pW \times T_pW \to \mathbb{R}$$
$$(v, w) \longmapsto \underbrace{C_p(d_p(df(v)))}\,\underbrace{(w)}_{\in T_pM}$$

$df : W \to T^*W$
$d_p(df) : T_pW \to T_{(p,0)} T^*W$
$C_p(d_p(df(v))) \in T_p^*M$

if you know what connections are show how to define d^2f_p using one and that it is independent of connection at a critical point p.

show $(d^2f)_p$ is a symmetric bilinear form and

p is a non-degenerate critical point

$\iff$

$(d^2f)_p$ is non-degenerate

$(d^2f)_p$ is called the <u>Hessian</u> also written $\mathrm{Hess}_p f$

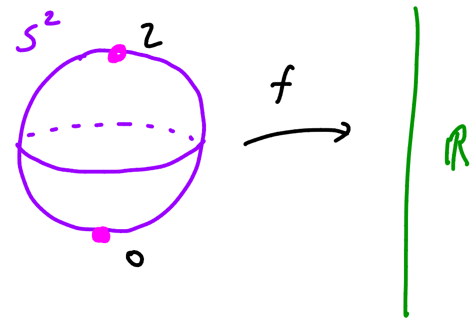
since it is symmetric and bilinear, we may choose a basis in which $(d^2f)_p$ is represented by a diagonal matrix with entries on diagonal ± 1

we define the <u>index of p</u> to be the number of -1's

<u>examples:</u>

1) $W = S^2$ unit sphere in $\mathbb{R}^3$

$f: S^2 \to \mathbb{R} : (x,y,z) \mapsto z$



we have the coordinate chart

$$\phi_u : D^2 \to S^2 : (x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$$

and

$$\phi_\ell : D^2 \to S^2 : (x,y) \mapsto (x, y, -\sqrt{1-x^2-y^2})$$

note: $f \circ \phi_u (x,y) = \sqrt{1-x^2-y^2}$

so $d(f\circ\phi_u) = \frac{1}{(1-x^2-y^2)^{1/2}} \begin{pmatrix} -x \\ -y \end{pmatrix}$ critical point at $(0,0)$

so $(0,0,1)$ crit. pt. of f

$$\left(\frac{\partial^2 f\circ\phi_u}{\partial x_i \partial x_j}\right) = \frac{1}{(1-x^2-y^2)^{3/2}} \begin{pmatrix} x^2-1 & -xy \\ -xy & y^2-1 \end{pmatrix}$$

at $(0,0)$ get $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ so index 2

similarly $(0,0,-1)$ critical point of index 0

<u>exercise:</u> show no critical points along equator

2) Recall $\mathbb{R}P^2 = \mathbb{R}^3 - \{(0,0,0)\}/\sim$ where $(x,y,z) \sim (tx,ty,tz)$ for $t \neq 0$

$\tilde{f}(x,y,z) = \frac{x^2+2y^2}{x^2+y^2+z^2}$ induces a function f on $\mathbb{R}P^2$

<u>exercise:</u> Show f has 3 critical points, they are non-degenerate, and have indicies 0, 1, 2.

a function $f: W \to \mathbb{R}$ is called Morse if all its critical points are non-degenerate (i.e. $df \pitchfork z$) and on interior of W

one may apply Thom's jet transversality theorem to establish
might talk about this later

Thm:

> let W be any manifold
> then the set of Morse functions is dense in $C^\infty(W,\mathbb{R})$
> Moreover, if C is a closed set on which f is already Morse
> then $\exists$ a Morse function $\tilde{f}: W \to \mathbb{R}$ that is ε-close
> to f and $\tilde{f} = f$ on an open set containing C

there are simpler ways to show any manifold has a Morse function but this result makes it easier to find Morse functions that respect aspects of the manifold.

Example 1:

If W is a manifold with compact boundary, then $\exists$ a Morse function

$$f: W \to \mathbb{R}$$

such that $f(W) \subset [0,\infty)$,

0 is a regular value of f, and

$f^{-1}(0) = \partial M$

Proof: It is a standard fact that $\exists$ an embedding

$$e: ([0,\varepsilon) \times \partial W) \to W$$

such that $e(\{0\} \times \partial W) = \partial W$ and

$\mathrm{im}(e)$ is a nbhd of ∂W in W

define $f = \pi \circ e$ on $e([0,\varepsilon/2] \times \partial W)$ where

$\pi: [0,\varepsilon) \times \partial W \to [0,\varepsilon)$ is projection

and extend f by $\varepsilon/2$ over the rest of M

$\exists \hat{f}$ that is smooth and agrees with f on $e([0,\varepsilon/3]\times\partial W)$ and is close enough to f so that $\hat{f}(p) > \varepsilon/4$ $\forall\, p \notin e([0,\varepsilon/3]\times\partial W)$

now $\hat{f}$ Morse on $e([0,\varepsilon/3]\times\partial W)$ (no critical pts)

so we may approximate $\hat{f}$ by a Morse function $\tilde{f}$ that agrees with $\hat{f}$ on a nbhd of $e([0,\varepsilon/3]\times\partial W)$ and is $> \varepsilon/4$ else where

so 0 is a regular value of $\tilde{f}$ and

$\tilde{f}^{-1}(0) = \partial W$

Example 2:

a cobordism W admits a Morse function f with

$f(W) \subset [0,1]$,

$0,1$ regular values, and

$f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$

Proof is similar to above (exercise)

Example 3:

a cobordism W of manifolds with boundary where $\partial_v W \approx [0,1]\times\partial_- W$ admits a Morse function f with

$f(W) \subset [0,1]$,

$0,1$ regular values,

$f^{-1}(0) = \partial_- W$, $f^{-1}(1) = \partial_+ W$, and

$f|_{\partial_v W} : [0,1]\times\partial_- W \to [0,1]$ is projection

Proof similar to above (exercise)

Example 4:

If Σ is a submanifold of W, then $\exists$ a Morse function $f: W \to \mathbb{R}$ s.t. $f|_\Sigma$ is Morse, the critical points of $f|_\Sigma$ are critical points of f, and they have the same index.

Proof: let $\tilde{f}: \Sigma \to \mathbb{R}$ be any Morse function

let $N \xrightarrow{\pi} \Sigma$ be a tubular neighborhood of Σ in W

extend $\tilde{f}$ to $\hat{f}: N \to \mathbb{R}$ by $\hat{f}(p) = \tilde{f}(\pi(p)) + |r(x)|^2$

where $r: N \to [0, \infty)$

$x \mapsto$ distance x to Σ

use some metric on W

now arbitrarily extend $\hat{f}$ to a smooth function on W and then approximate by a Morse function f agreeing with $\hat{f}$ on N

now just compute the critical points in N and their index (exercise)

Remark: One may also use jet transversality to show that for a generic Morse function all the critical points have distinct values

Fact: On a compact manifold W, a function $f: W \to \mathbb{R}$ is "stable" $\Leftrightarrow$ f is a Morse function with critical points having distinct values

a function $f: W \to \mathbb{R}$ is "stable" if "functions near f are "the same" as f (upto diffeomorphism)"

i.e. $\exists$ an open set U around f in $C^\infty(W, \mathbb{R})$

s.t. $\forall g \in U$, $\exists$ diffeomorphisms $\phi: W \to W$ and $\psi: \mathbb{R} \to \mathbb{R}$ such that $g = \psi \circ f \circ \phi$