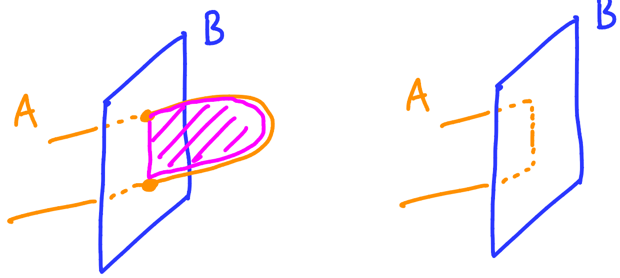
since $n \geq 5$, f can be assumed to be an embedding
(embeddings are dense in $C^\infty(M^n, X^m)$ if $m > 2n$)

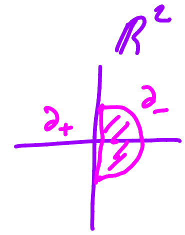
moreover, if codim $A, B > 2$, then int f disjoint from $A \cup B$

if codim $B = 2$, need to assume disk is disjoint from B

now one use D^2 to guide an isotopy of A



let $D^2 = \{(x,y) \mid x^2+y^2 \leq 1,\ x \geq 0\}$



can find nbhd U of $D^2 \times \{0\}$ in $D^2 \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}$

and embedding $\phi : U \to W$ st. $\phi|_{D^2 \times \{0\}} = f$

and $\phi^{-1}(A) = U \cap (\partial_+ D^2 \times \mathbb{R}^{k-1} \times \{0\})$

$\phi^{-1}(B) = U \cap (\partial_- D^2 \times \{0\} \times \mathbb{R}^{n-k-1})$

now one can write an explicit isotopy in U

F. Heegaard Splittings of 3-manifolds

let $H_g = $ (0-handle) $\cup$ g (1-handles) st. H_g is oriented

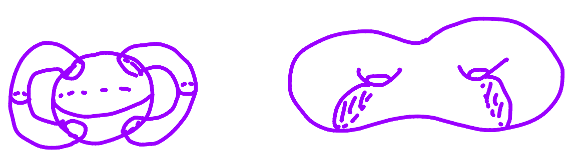
this is called a genus g handlebody

notice that since any embedding of $S^0 \hookrightarrow S^2$ is isotopic to any other

attaching sphere of 1-handle

and $\exists$ unique way to "frame" 1-handle to get oriented mfd

we see H_g only depends on g



exercise: 1) Show an oriented 3-mfd M is a handlebody of genus g
$\iff$
$\exists\, g$ disjoint embedded disks $D_1, \dots, D_n$
s.t. $M \setminus \bigcup D_i \cong B^3$

2) If Σ is an oriented surface w/ $\partial \neq \emptyset$, then $\Sigma \times [0,1]$ is a handlebody.

Given a closed oriented 3-manifold M^3, it has a handle decomposition with one 0-handle and one 3-handle
can assume all the 1-handles come before 2-handles

let $\Sigma \subset M$ be ∂W_1

note: W_1 is a handle body of genus $g = \#$ 1-handles
and Σ genus g sfc

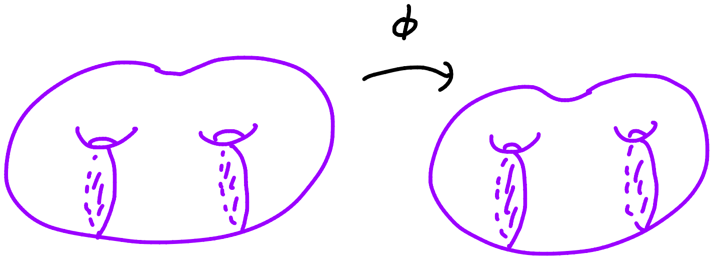
similarly $W' = \overline{M - W_1}$
turned upside down is a handle body of genus
$g' = \#$ 2-handles

since $\partial W_1 = \Sigma = \partial W'$ we see $g = g'$
and M is the union of 2 genus g handle bodies!

Σ called a Heegaard surface for M and $M \setminus \Sigma$
a Heegaard splitting

$\exists$ embeddings $H_g \xrightarrow{\psi_1} M$, $H_g \xrightarrow{\psi_2} M$ for cpts of $M \setminus \Sigma$

let $\phi = \psi_2|_{\partial H_g}^{-1} \circ \psi_1^{-1}|_{\partial H_g} : \Sigma_g \to \Sigma_g$
this is a diffeo. and



$$M \cong H_g^1 \cup H_g^2 \Big/_{p \in \partial H_g^1 \sim \phi(g) \in \partial H_g^2}$$

so any 3-mfd obtained by gluing 2 handlebodies together!

really simple manifolds!

so all 3-mfds described by diffeos of surfaces

another way to describe 3D handle decompositions "Kirby pictures" A.K.A. "Heegaard Diagrams"

just as for surfaces two 3-manifolds M_1, M_2 are diffeomorphic $\Leftrightarrow$ $\hat{M}_1, \hat{M}_2$ are diffeomorphic where $\hat{M}_1 = \overline{M_1 - B^3}$

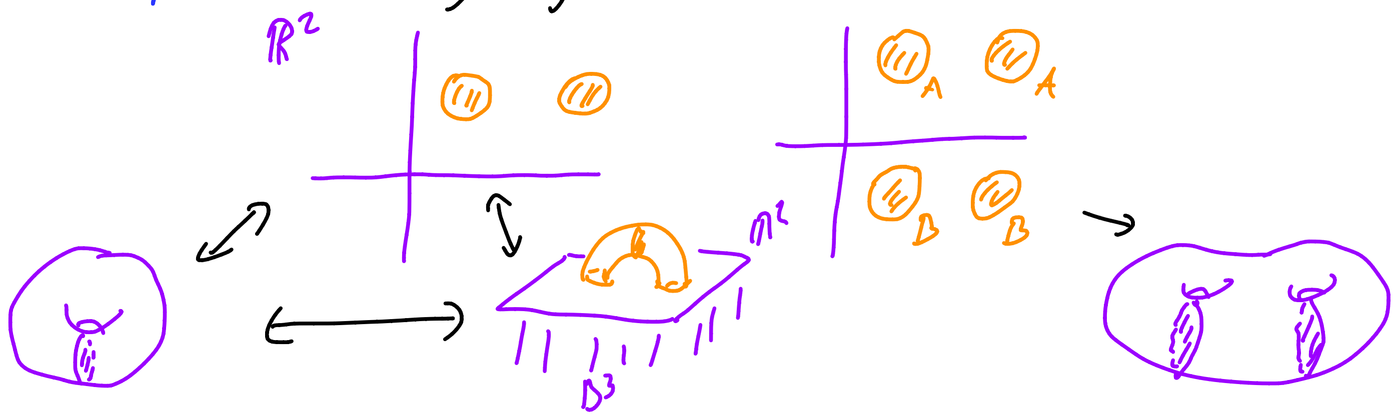
so can understand a 3-mfd M by looking at

$$M^2 = (0\text{-handle}) \cup k(1\text{-handles}) \cup k(2\text{-handles})$$

handles attached to $\partial(0\text{-handle}) = S^2$

and attaching regions of handles can be assumed to be disjoint from a pt in S^2, ie in $\mathbb{R}^2 = S^2 - \{pt\}$

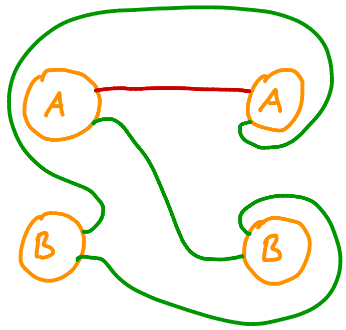
example: attaching region of 1-handle is $S^0 \times D^2$



attaching sphere of 2-handle is S^1 (and attaching region uniquely determined by this $\pi_1(O(1)) = \{e\}$, just thicken S^1)

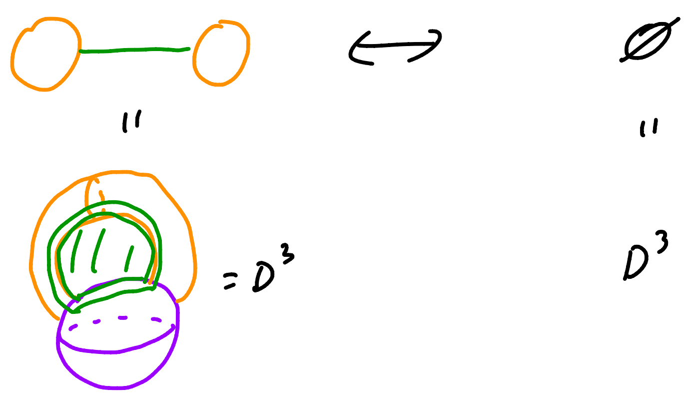
example:

1)



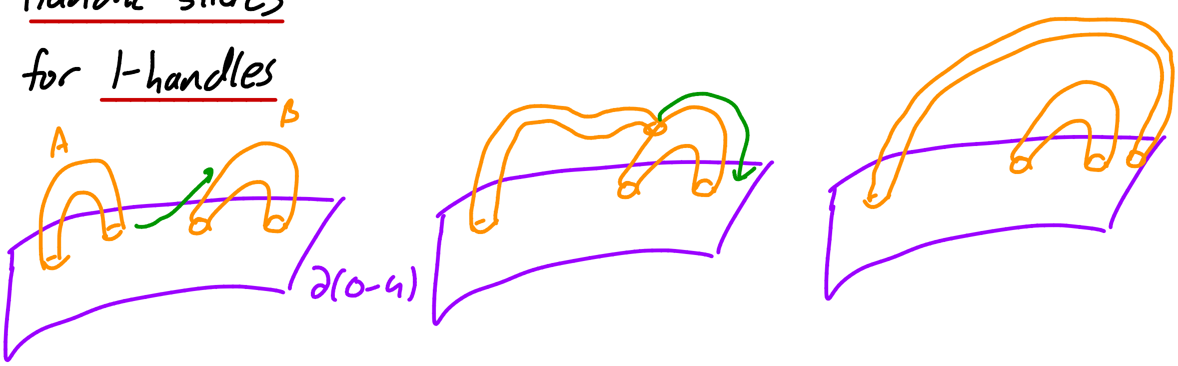
exercise: this is $\mathbb{R}P^3 = S^3/\text{identify antipode}$

just like for surface we have cancelling 1,2-pairs



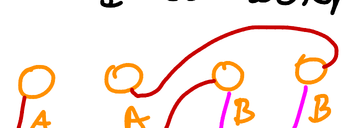
and handle slides

for 1-handles



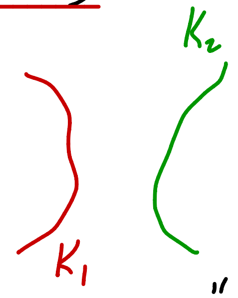
Heegaard diagram

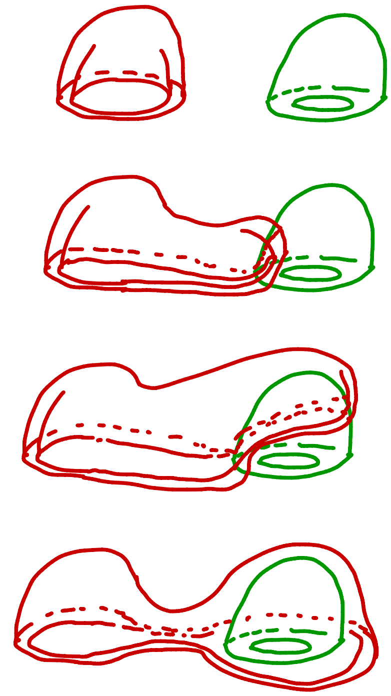
can isotop back to



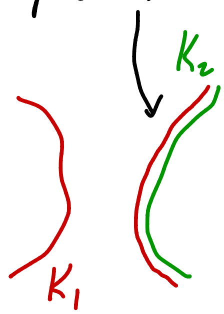
for <u>2-handles</u>:

to push K_1 over K_2

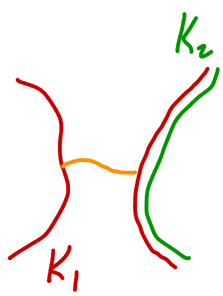




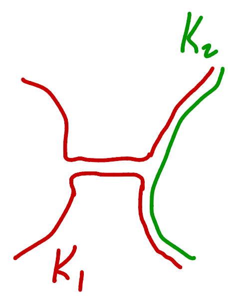
take a "push off" of K_2



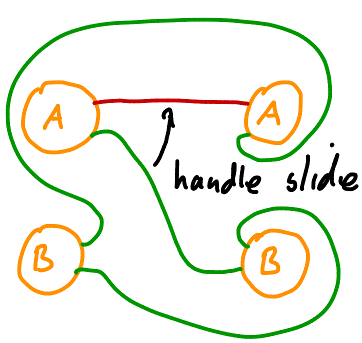
choose arc K_1 to push off



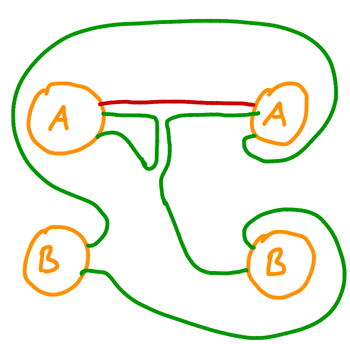
"band" K_1 to push off along arc



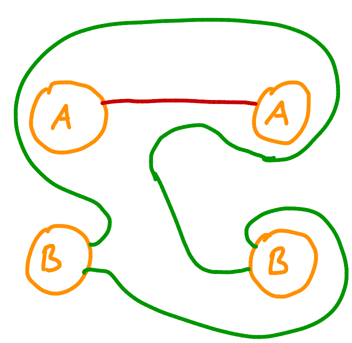
<u>example</u>:



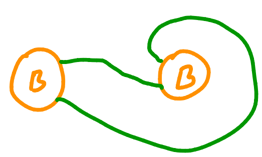
handle slide ⇒



⇒ isotop



⇒ cancel. 1,2-pair



exercises: Draw pictures of $S^1 \times S^2$, $L(p,q)$, $\Sigma_g \times [0,1]$,
$T^3 = S^1 \times S^1 \times S^1$

given a Heegaard surface $\Sigma \subset M^3$

let α be an arc embedded in Σ

let β be the result of isotoping interior of α of Σ

so $\alpha \cup \beta = \partial D^2$ s.t. $D^2 \cap \Sigma = \alpha$

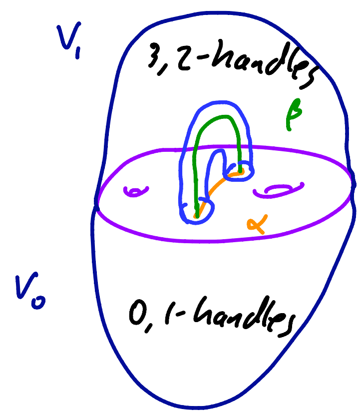
note $\beta \subset$ one component, say V_1, of $M \setminus \Sigma = V_0 \cup V_1$

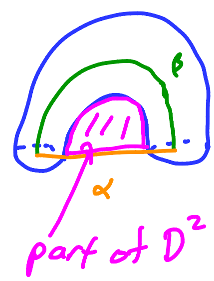
let $N = \text{nbhd } \beta$ in $V_1 = D^2 \times [0,1]$

let $V_1' = \overline{V_1 - N}$, $V_0' = V_0 \cup N$

claim: $\Sigma' = \partial V_0'$ a new Heegaard splitting of M

indeed





so we add a 1-handle to V_0
so V_0' still handle body
then attach nbhd of D^2 to V_0'
(ie. 2-handle) and now
add other 2-handles & 3-handle
ie. $V_1' =$ 3-handle $\cup$ 2-handles
so handlebody

going from Σ to Σ'
is called a stabilization

note on handle decomposition level we just added a cancelling pair of handles

Th$^{\underline{m}}$ (Reidemeister-Singer):

$$\boxed{\text{any 2 Heegaard surfaces for } M \text{ are isotopic after possibly stabilizing each surface}}$$

to prove this (from our perspective) we need something new.

G. Cerf and Smale theory

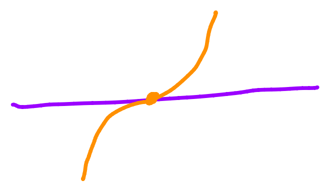
first Cerf theory addresses how to get between 2 Morse functions

recall a critical point p of $f: M \to \mathbb{R}$ is non-degenerate if $df: M \to T^*M$ is $\pitchfork$ zero section at p

if df not $\pitchfork$ at p what's the next closest thing?

"order 1 tangency, in one direction"

eg



$x \mapsto x^3$

we say p is <u>embryonic</u> if $\ker \mathrm{Hess}_p f$ is 1 dimensional and, in local coords, 3^{rd} derivative of f in direction of $\ker \mathrm{Hess}_p f$ is non-zero

we call functions f with non-degen. and embryonic critical points <u>generalized Morse functions</u>

i.e. $\mathrm{im}\, df_p \cap$ zero section = line and in that direction "$\frac{\partial^3 f}{\partial x^3} \neq 0$"

similar to Morse lemma we have

lemma:

> If p is an embryonic critical point of f, then $\exists$ local coords about p in which f takes the form
>
> $$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + x_n^3$$

k is index of p

Similar to Th^m about the existence of Morse functions we have

Th^m (Cerf):

> If $f_0, f_1 : M \to \mathbb{R}$ are Morse functions, then for generic paths $f_t: M \to \mathbb{R}$ connecting f_0, f_1 we can assume $\exists$ a finite numbere of $t_1 < \dots < t_k$ such that