Tight contact structures and Anosov flows

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Abstract

In this note, we apply classic and recent results on the classification of tight contact structures to the problem of existence of Anosov flows on three-manifolds. The ingredients we use are the results of Mitsumatsu on Anosov flows, the homotopy invariant of plane fields as described by Gompf and others, and certain recent classification results of Honda. This yields a novel proof of the nonexistence of Anosov flows on S^3 using only contact topology (and in particular without use of Novikov's Theorem on foliations).

The theory of contact structures, though magnificently old, has of late become central to several key questions in the study of three-manifolds [Eli92a], Seiberg-Witten invariants [LM97, KM97], symplectic geometry [Etn98], knot theory [Rud95], and hydrodynamics [EG98, EG99]. In this note, we reinterpret a result of Mitsumatsu [Mit95] to present an application to a fundamental problem in dynamical systems theory: which manifolds support an Anosov flow.

1 An introduction to contact structures

For the sake of concreteness and applicability, we will restrict all definitions and discussions to the case of contact structures on three-manifolds, noting that several features hold on arbitrary odd-dimensional manifolds. For introductory treatments, see [MS95, Aeb94, Eli92a].

A CONTACT FORM on an oriented three-manifold M is a one-form α on M such that $\alpha \wedge d\alpha$ defines an oriented volume form on M. A CONTACT STRUCTURE is a plane field which is the kernel of a (locally defined) contact form:

$$\xi := \ker(\alpha) = \{v \in T_pM : \alpha(v) = 0, p \in M\}. \tag{1}$$

The orientation induced by $\alpha \wedge d\alpha$ is independent of the defining one-form; hence, ξ has a natural orientation which can agree (a POSITIVE structure) or disagree (a NEGATIVE strucutre) with that of M. According to the Frobenius integrability condition, a contact structure is thus a maximally nonintegrable plane field. In particular, a contact structure is locally twisted at every point and may be thought of as an "anti-foliation." It is usually sufficient to consider contact structures which are the kernel of a globally defined contact one-form: these are COORIENTED contact structures.

Unlike foliations, contact structures are structurally stable, in the sense that not only is a perturbation of a contact form α still a contact form, but also such a perturbation has kernel isotopic to that of α. In fact, a standard application of the Moser method in this context implies that every contact structure is locally contactomorphic to (or, diffeomorphic via a map which carries the contact structure to the kernel of $dz + x\,dy$ on $\mathbb{R}^3$ (see, *e.g.*, [MS95]). Note the similarity with codimension-one foliations, which are locally equivalent to the kernel of dz on $\mathbb{R}^3$.

Example 1.1 The standard positive contact structure on the unit $S^3 \subset \mathbb{R}^4$ is given by the kernel of the one-form

$$\alpha_0 := \frac{1}{2}\left(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3\right). \tag{2}$$

The contact structure $\xi_0^+ := \ker(\alpha_0)$ is the plane field orthogonal to the fibres of the Hopf fibration (orthogonal with respect to the metric on the unit three-sphere induced by the standard metric on $\mathbb{R}^4$). This contact structure induces the positive orientation on S^3 (*i.e.*, $\alpha_0 \wedge d\alpha_0 > 0$). A negative contact structure on S^3 may be obtained by applying an orientation-reversing diffeomorphism.

As in foliation theory, the global features of a contact structure are closely related to those of the manifold in which it sits. The classification of contact structures follows along lines similar to the Reeb-component versus taut perspective in (codimension-one) foliation theory [ET98].

Definition 1.2 Given a three-manifold M with contact structure ξ, let $F \subset M$ be an embedded surface. Then the CHARACTERISTIC FOLIATION on F, F_ξ, is the foliation on F generated by the (singular) line field

$$\mathcal{F} = \{T_pF \cap \xi_p : p \in F\}.$$

A contact structure ξ is OVERTWISTED if there exists an embedded disc $D \subset M$ such that the characteristic foliation D_ξ has a limit cycle. A contact structure which is not overtwisted is called TIGHT.

A priori, Definition 1.2 appears arbitrary. However, if one builds an analogy with foliation theory, this definition becomes more natural [ET98]. Consider a Reeb component in a codimension-one foliation of a three-manifold, as illustrated in Figure 1 (see, *e.g.*, [God91] for definitions). The characteristic foliation induced by the Reeb component on a meridional disc is a foliation by circles with one singularity. The intersection of any single $\mathbb{R}^2$-leaf with the meridional disc is a sequence of concentric circles which "limit" onto the boundary torus, which forms a sort of limit cycle. An overtwisted contact structure is the nonintegrable analogue of this object.

The classification of overtwisted structures up to contact isotopy coincides with the classification of plane fields up to homotopy [Eli89] and hence reduces to a problem in algebraic topology. The classification of tight structures, on the other hand, is far from complete: for example, it is unknown whether every three-manifold admits a tight contact structure. Like taut foliations, tight contact structures exhibit several "rigid" features which make them relatively rare. The following theorems of Bennequin and Eliashberg are foundational:

Theorem 1.3 *The contact structure ξ_0^+ of Example 1.1 is tight [Ben83] and is the unique tight contact structure on S^3 up to orientation and contact isotopy [Eli92a].*

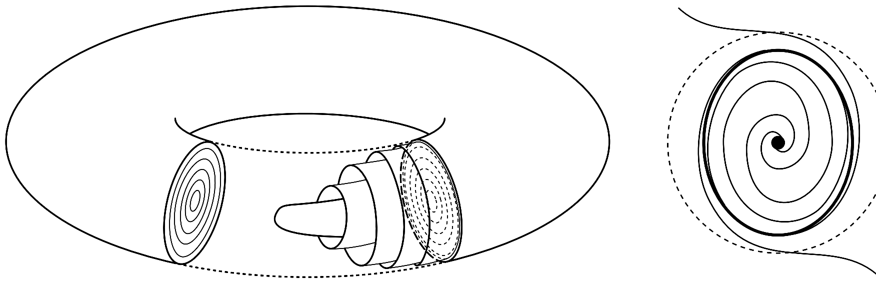

FIGURE 1: A Reeb component in a foliation on a three-manifold (left) can be perturbed into an overtwisted contact structure (right).

2 Anosov flows

Recall that an invariant set Λ of a flow ϕ^t on a Riemannian manifold M is HYPERBOLIC if the tangent bundle $TM|_\Lambda$ has a continuous ϕ^t-invariant splitting into $E^c \oplus E^s \oplus E^u$, where E^c is tangent to the flow direction, and $D\phi^t$ uniformly contracts and expands along E^s and E^u respectively: *i.e.*,

$$\begin{array}{ll} \|D\phi^t(\mathbf{v}^s)\| \leq Ce^{-\lambda t}\|\mathbf{v}^s\| & \text{for } \mathbf{v}^s \in E^s \\ \|D\phi^{-t}(\mathbf{v}^u)\| \leq Ce^{-\lambda t}\|\mathbf{v}^u\| & \text{for } \mathbf{v}^u \in E^u \end{array} ,\, t > 0, \tag{3}$$

for some $C \geq 1$ and $\lambda > 0$. A flow ϕ^t which is hyperbolic on all of M is called an ANOSOV FLOW. Anosov flows are some of the most important types of flows, from dynamical, topological, and geometric perspectives. Fundamental examples of Anosov flows include geodesic flows on surfaces of constant negative curvature, as well as suspensions of hyperbolic toral automorphisms. It is an open question which three-manifolds support an Anosov flow: obstructions have in the past come primarily from foliation theory, since the plane fields $E^c \oplus E^s$ and $E^c \oplus E^u$ are integrable and tangent to taut (and often minimal) foliations [Pla72]. Group-theoretic obstructions exist [PT72], but even these rely to some extent on the geometry of the stable and unstable foliations. We will restrict attention to volume-preserving Anosov flows: in dimension three, certain "anomalous" Anosov flows exist which do not preserve volume (and have other unusual properties) [FW80].

The following beautiful construction was discovered by Mitsumatsu [Mit95] (see also [ET98]).

Theorem 2.1 (Mitsumatsu [Mit95]) *Let X be a vector field generating an Anosov flow on M^3. Then X lies in the transverse intersection of a pair of oppositely oriented tight contact structures.*

There is a pair of transverse integrable plane fields containing X given by $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively: these form the (weak-) stable and unstable foliations. For each $p \in M$, define ξ^s_p to be the subspace of TM_p obtained by rotating $E^c_p \oplus E^s_p$ about the E^c_p

subspace by a fixed angle, say $\pi/4$. One may define ξ^u likewise by rotating $E^c \oplus E^u$ about the E^c direction by $\pi/4$. Under the action of the flow of X, the transverse directions are always rotated away from the stable section E^s and towards the unstable section E^u; hence, $\mathcal{L}_X\xi^s > 0$ and $\mathcal{L}_X\xi^u < 0$. Since a plane field η is integrable if and only if $\mathcal{L}_X\eta = 0$ for every X tangent to η, it follows that ξ^s and ξ^u define contact structures which are furthermore of opposite orientation (follow the directions of twisting). Mitsumatsu then shows that these structures are tight by appealing to a theorem of Eliashberg and Gromov [Eli92b, Gro85] that symplectically semi-fillable structures are tight.

This result is of interest in that it allows one to construct very explicit examples of tight contact structures on those three-manifolds which admit Anosov flows. We consider the converse problem of using existence and uniqueness theorems for tight contact structures as an obstruction to the existence of an Anosov flow. To proceed, we require a bit of knowledge about the homotopy classification of plane fields on three-manifolds.

3 The three-dimensional invariant

We describe an invariant of plane fields on integral homology three-spheres. This invariant was originally defined by Gompf [Gom98] for any closed three-manifold (cf. [Kup96]), but is simplest to define in the restricted case we consider. Given a coorientable plane field ξ on an oriented homology three-sphere M one can always find an oriented almost-complex 4-manifold X which M bounds (respecting orientations) so that ξ is the field of complex tangencies [Gom98]. Since $H^2(\partial X;\mathbb{Z}) = 0$ we have $c_1(X) \in H^2(X;\mathbb{Z}) \cong H^2(X, \partial X;\mathbb{Z})$. Thus we can think of $c_1^2(X)$ as the integer obtained by pairing $c_1(X) \smile c_1(X)$ with the fundamental class $[X, \partial X] \in H_4(X, \partial X;\mathbb{Z})$. Now define

$$\theta(\xi) := c_1^2(X) - 2\chi(X) - 3\sigma(X), \tag{4}$$

where $\sigma(X)$ is the signature of X and $\chi(X)$ is the Euler characteristic of X. The invariant $\theta(\xi)$ depends only on the homotopy type of ξ and the orientation on M (not on the coorientation of ξ). To see this, fix Y an almost-complex manifold which bounds $(-M, \xi)$ (*i.e.*, M with reversed orientation). Then, consider X_0 and X_1 two almost-complex four-manifolds which bound (M, ξ) with the proper orientation. We can glue X_0 or X_1 to Y along their boundaries to obtain a closed almost-complex manifold W. For such a manifold the Hirzebruch signature theorem (see, *e.g.*, [Kir89]) says that

$$c_1^2(W) = 2\chi(W) + 3\sigma(W). \tag{5}$$

This proves, after noting the additivity of all three terms in Equation 4, that (1) the invariant θ is well-defined; and (2) θ reverses sign upon changing the orientation on M. On homology three-spheres, θ is a complete invariant of plane fields.

Theorem 3.1 *Let ξ_1 and ξ_2 be coorientable plane fields on an oriented homology three-sphere M. Then ξ_1 is homotopic to ξ_2 if and only if $\theta(\xi_1) = \theta(\xi_2)$.*

For a proof of this theorem the reader is referred to [LM97]. This is a special case of a much more general theorem in [Gom98]. This invariant (and the more general version) yields an invariant of homotopy classes of nonsingular vector fields on three-manifolds by associating to any such vector field a transverse plane field. The relationships between the dynamics of a nonsingular vector field X and the information encoded in $\theta(X)$ have been almost completely unexplored[1].

Example 3.2 Let ξ_0^+ denote the standard tight contact structure on S^3 of Example 1.1. One can realize ξ_0^+ as the set of complex tangencies of the unit $S^3 \subset \mathbb{C}^2$ with the standard complex structure, bounding the trivial 4-ball. Hence,

$$\theta(\xi_0^+) = c_1^2 - 2\chi - 3\sigma = 0 - 2(1) - 3(0) = -2.$$

If, however, we consider ξ_0^-, the unique tight contact structure on S^3 which induces the negative orientation, we can realize this as the image of ξ_0^+ under an orientation-reversing diffeomorphism of S^3. Since applying an orientation-reversing diffeomorphism changes the sign of the three-dimensional invariant (with respect to a fixed orientation as per point (2) above), we have $\theta(\xi_0^-) = +2$.

4 A tight obstruction to Anosov flows

Lemma 4.1 *Let X be a vector field contained in the transversally orientable plane field η on an oriented three-manifold M. Then the three-dimensional invariants of X and η agree.*

Proof: Choose Z a vector field transverse to η, and let ζ denote the plane field spanned by Z and X. Since Z and X are nowhere collinear, we may homotope Z to X within ζ. $\diamond$

The classification of tight contact structures on S^3 (Theorem 1.3) thus yields a simple proof of the nonexistence of Anosov flows on S^3:

Theorem 4.2 *There are no Anosov flows on S^3.*

Proof: Assume X is an Anosov flow on S^3. Then X lies in the transverse intersection of a pair of oppositely oriented tight contact structures ξ^+ and ξ^- which are homotopic as they contain a common vector field. Theorem 1.3 implies that ξ^+ and ξ^- are contact isotopic to ξ_0^+ and ξ_0^- respectively. Therefore, with respect to the positive orientation on S^3, the calculation of Example 3.2 yields the contradiction $-2 = \theta(\xi^+) = \theta(\xi^-) = 2$. $\diamond$

This result, though well-known and easily proved via Novikov's theorem on foliations, provides an alternate motivation for classifying tight contact structures on three-manifolds, as well as extends the range of applications of contact topology to include the field of dynamical systems. Note in addition, that smoothness issues concerning the foliations associated to Anosov flows (which are quite delicate — the foliations are only Hölder continuous in general) are not an issue when working with the associated (smooth) contact structures.

[1]The invariant θ is a dynamical invariant in the sense that topologically conjugate vector fields have equal θ-values.

5 Homotopy uniqueness

Every three-manifold possesses a countable infinity of homotopy classes of cooriented contact structures (or, equivalently, nonsingular vector fields). It is an elusive conjecture that the tight structures comprise at most finitely many homotopy classes on any fixed three-manifold.[2]

One may consider a dynamically motivated analogue to this question: *How many homotopy classes of Anosov vector fields exist on a given manifold?* It is unknown to us whether this question has been considered before. We offer the following first step via a recent classification of certain tight contact structures due to Honda [Hon99b].

Theorem 5.1 *Given M a hyperbolic torus bundle over S^1, there is exactly one homotopy class of Anosov fields on M.*

Proof: The recent classification of Honda [Hon99b] for tight contact structures on T^2-bundles over S^1 implies that there are many different tight contact structures on a hyperbolic torus bundle over S^1. However, Honda shows that there is a unique homotopy class of universally tight structures — that is, those structures for which no cover yields an overtwisted structure. Given an Anosov vector field, any cover also yields an Anosov vector field, which has the corresponding pair of transverse tight structures by Mitsumatsu[3]. Thus, the contact structures associated to an Anosov field are universally tight, and the result follows by Honda's classification. $\diamond$

The unique homotopy class for a hyperbolic torus bundle is, roughly speaking, the class which is "parallel" to the S^1 direction corresponding to the natural Anosov flow obtained by the suspension of the hyperbolic monodromy.

6 Miscellany

There are several ways in which problems concerning the dynamics of flows on three-manifolds can be assisted by understanding the classification of tight contact structures. As this latter subject is in its infancy and growing rapidly [Etn97, Hon99a, Hon99b], we are optimistic that the following problems may have contact-topological solutions:

Hyperbolic manifolds

> It is an open problem whether one can find an Anosov flow on every closed oriented hyperbolic three-manifold (with homology three-spheres being of particular interest). It is thus important to try and classify universally tight contact structures on hyperbolic manifolds. Of course, as with taut foliations, this is not a simple task.

[2]Kronheimer and Mrowka [KM97] proved that there are finitely many classes of fillable contact structures, and it is an open problem to find a tight contact structure which is not fillable.

[3]For a noncompact cover one must appeal to stronger results of Gromov on pseudoholomorphic curves to show that fillability induces tightness.

Conformally Anosov flows

These flows, defined independently by Mitsumatsu [Mit95] and Eliashberg and Thurston [ET98], are flows which have the same dynamics on the projectivized normal bundle to the flow as an Anosov flow does. Such flows are more general than Anosov flows (*e.g.*, they can arise on T^3, whereas Anosov flows cannot); however, they still appear as (and are indeed equivalent to) the intersection of a pair of transverse oppositely oriented contact structures. It is an open problem to classify which manifolds admit conformally Anosov flows.

Legendrian flows

Besides the Anosov fields considered thus far, several other important flows in dynamical systems are Legendrian, or tangent to a contact structure. It is an interesting question which nonsingular vector fields must be Legendrian, and in particular how the tight/overtwisted dichotomy manifests itself. A simple corollary of a recent theorem of Honda [Hon98] yields the first set of examples of nonsingular vector fields on S^3 which are not Legendrian. As these examples are all Morse-Smale, it follows that they cannot preserve any volume form. It remains an open problem to find an obstruction for nonsingular volume-preserving vector fields on the three-sphere.[4]

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