ON SYMPLECTIC COBORDISMS

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ABSTRACT. In this note we make several observations concerning symplectic cobordisms. Among other things we show that every contact 3-manifold has infinitely many concave symplectic fillings and that all overtwisted contact 3-manifolds are "symplectic cobordism equivalent".

1. INTRODUCTION

In this note we make several observations concerning (directed) symplectic cobordisms, Stein cobordisms, and *concave* symplectic fillings for contact 3-manifolds. Symplectic and Stein cobordisms have recently come to the foreground of symplectic and contact geometry, largely due to the introduction of a symplectic field theory (SFT) by Eliashberg, Hofer and Givental [12]. The goal of SFT is to associate an algebraic structure to a given symplectic cobordism. Though clearly a central notion in symplectic and contact geometry, there is surprisingly little concerning symplectic cobordisms in the literature.

We will assume our 3-manifolds are closed and oriented, and our contact structures are oriented and positive. A contact 3-manifold (M_1, ξ_1) is *symplectically cobordant* to another contact manifold (M_2, ξ_2), if there exists a symplectic 4-manifold (X, ω) with $\partial X = M_2 - M_1$ and a vector field v defined on a neighborhood of $(M_1 \cup M_2) \subset X$ for which $\mathcal{L}_v \omega = \omega$, $v \pitchfork (M_1 \cup M_2)$, the normal orientation of $M_1 \cup M_2$ agrees with v and the 1-form $\alpha = \iota_v \omega$ is a contact from for ξ_i when restricted to $M_i, i = 1, 2$. If there is, moreover, an almost complex structure J on X and a strictly plurisubhamonic function $\phi : X \to \mathbb{R}$ such that $\omega = -dJ^* d\phi$ and $M_i, i = 1, 2$, are non-critical level sets of ϕ, then we say (M_1, ξ_1) is *strictly complex cobordant* to (M_2, ξ_2). Such cobordisms have been studied in [9, 13] and can be thought of as the cobordism analog of a Stein manifold. Hence we shall abuse terminology and refer to strictly complex cobordisms as "Stein cobordisms". We say (M_1, ξ_1) is the *concave end* of the cobordism, while (M_2, ξ_2) is the *convex end*. We denote the existence of such a cobordism by $(M_1, \xi_1) \prec (M_2, \xi_2)$ — in the paper we implicitly assume that $\prec$ refers to a Stein cobordism, unless specified otherwise. Note that symplectic (and Stein) cobordism is not an equivalence relation. For example, a *Stein fillable* contact structure

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(M,ξ) (= one satisfying $\emptyset \prec (M,\xi)$) cannot be symplectically cobordant to an overtwisted contact structure, but the opposite is possible. Our first result is:

Theorem 1.1. *Let (M_1,ξ_1) be a contact 3-manifold. Then there exists a Stein fillable contact 3-manifold (M_2,ξ_2) and a Stein cobordism $(M_1,\xi_1) \prec (M_2,\xi_2)$.*

Though this result indicates the overall structure of the "partial order" on contact 3-manifolds induced by cobordisms, there is very little control over the target contact manifold (M_2,ξ_2). On the other hand, when (M_1,ξ_1) is overtwisted, there is complete freedom in choosing (M_2,ξ_2):

Theorem 1.2. *Let (M_1,ξ_1) be an overtwisted contact 3-manifold and (M_2, ξ_2) any contact 3-manifold, tight or overtwisted. Then there exists a Stein cobordism $(M_1,\xi_1) \prec (M_2,\xi_2)$.*

In particular, all overtwisted contact structures are equivalent under symplectic or Stein cobordism!

It is interesting to compare the previous two theorems with recent work of Epstein-Henkin [13] and de Oliveira [5] which deal with cobordisms between CR-structures. (Here "CR-structure" will mean "strictly pseudoconvex CR-structure".) On any 3-manifold M, there is a 1-1 correspondence between CR-structures and pairs (ξ, J) consisting of a contact structure ξ and an almost complex structure J on ξ. We say a CR-structure (ξ, J) on M is *fillable*, if there is a compact, connected, complex manifold X with $\partial X = M$, so that the complex tangencies to M are ξ and the induced complex structure on ξ is J. In [13] it was shown that if a CR-manifold (M_1,ξ_1,J_1) is Stein cobordant to a fillable CR-manifold (M_2,ξ_2,J_2), then (M_1,ξ_1,J_1) is also fillable. Here we assume Stein cobordisms of CR-manifolds respect complex structures. Thus, if $(M_1,\xi_1,J_1) \prec (M_2,\xi_2,J_2)$ is a Stein cobordism but (M_1,ξ_1) is not Stein fillable, then (M_2,ξ_2,J_2) cannot be a fillable CR-structure, even if (M_2,ξ_2) is a Stein fillable contact structure. De Oliveira [5] gave some interesting examples of complex (but not Stein) cobordisms from non-fillable CR-structures to fillable ones, thus showing the necessity of having a Stein cobordism in the Epstein-Henkin result.

Our last result is:

Theorem 1.3. *Any contact 3-manifold has infinitely many concave symplectic fillings which are mutually non-isomorphic and are not related to each other by a sequence of blow-ups and blow-downs.*

A *convex (resp. concave) symplectic filling* of (M,ξ) is a symplectic cobordism (X,ω) from $\emptyset$ to (M,ξ) (resp. from (M,ξ) to $\emptyset$). The phrase "symplectic filling", without modifiers, is usually reserved for "convex symplectic filling". Having a (convex) filling is quite restrictive for a contact 3-manifold — for instance, it implies the contact structure is tight. (Note, however, that there are many tight contact structures without such fillings due to Eliashberg [11], Ding-Geiges [6], and Etnyre-Honda [14].) We show that, on the contrary, concave fillings are not restrictive at all. Though this was believed for a long time, and specific isolated contact manifolds with infinitely many such fillings are easy to come by, the degree to which concave fillings are not restrictive is perhaps a little surprising.

We assume the reader is more or less familiar with contact geometry and hence we do not include any background material here. We refer the reader to [2] for the basics of contact geometry, [8] for Lutz twisting, and [1, 12, 9] for the notions of Stein and symplectic cobordisms.

2. Legendrian surgeries

In this section we give a description of *Legendrian surgery*, both on the 3-manifold level and as a source of Stein filling on the 4-manifold level. There is some related material in [21] for Legendrian surgeries.

Let (M, ξ) be a contact manifold and $L \subset M$ a closed Legendrian curve. Let $N(L)$ be a *standard tubular neighborhood* of the Legendrian curve L, with convex boundary and two parallel dividing curves. Choose a framing for L (and a concomitant identification $\partial N(L) \simeq \mathbb{R}^2/\mathbb{Z}^2$) so that the meridian has slope 0 and the dividing curves have slope ∞. With respect to this choice of framing, a Legendrian surgery is a -1 surgery, where a copy of $N(L)$ is glued to $M \setminus N(L)$ so that the new meridian has slope -1. Here, even though the boundary characteristic foliations may not exactly match up a priori, we use Giroux's Flexibility Theorem [15, 20] and the fact that they have the same dividing set to make the characteristic foliations agree. This gives us a new manifold (M', ξ').

The following proposition describes Legendrian surgery on the 4-manifold level.

Proposition 2.1. *Let (M', ξ') be a contact manifold obtained by Legendrian surgery along L in (M, ξ), in a 3-dimensional manner. Then there exists a Stein cobordism from (M, ξ) to (M', ξ'), obtained by attaching a 2-handle along $N(L)$.*

Proof. We apply Lemma 2.2 below to obtain a Stein cobordism $X = M \times [0, 1]$. Then Legendrian surgery corresponds to attaching a 2-handle along $N(L) \subset M \times \{1\}$ in a Stein (resp. symplectic) manner, which yields a Stein (resp. symplectic) cobordism from (M, ξ) to (M', ξ'). (See Eliashberg [9].) □

Lemma 2.2. *Let (M, ξ) be a contact structure. Then there exists a thickening of M to $X = M \times [0, 1]$ and a Stein cobordism from (M, ξ) to itself.*

A proof of this fact appears in [7].

3. Open book decompositions

Recall an *open book decomposition* of a 3-manifold M consists of a link K, called the *binding*, and a fibration $f\colon (M \setminus K) \to S^1$ such that each fiber F in the fibration is a Seifert surface for K. The manifold $M \setminus K$ is obtained by taking $F \times [0, 1]$ with coordinates (x, t) and identifying $(x, 0) \sim$

$(\phi(x), 1)$ via the monodromy map $\phi : F \xrightarrow{\sim} F$. Following Thurston and Winkelnkemper [26], we construct a contact structure on M from an open book decomposition: Let λ be a primitive for an area form on F and let $\lambda_t = t \cdot \lambda + (1-t) \cdot \phi^* \lambda$, $t \in [0,1]$. The 1-form $\alpha = dt + \lambda_t$ is a contact 1-form on $F \times [0,1]$ which glues to give a contact structure on $M \setminus K$. One easily checks that α extends over K. If (M, ξ) is obtained in this manner, then we say that the open book decomposition of M is *adapted to* ξ. We now have the following recent result of Giroux [16]:

Theorem 3.1. *Any contact structure ξ on a closed 3-manifold M admits an open book decomposition of M which is adapted to ξ.*

The following lemma (and more importantly its converse) is due to the efforts of many people, beginning with the work of Loi and Piergallini [23] (also see [25] for an earlier effort), and recently culminating in the work of Giroux [16] (see also [3, 24]).

Lemma 3.2. *If the monodromy $\phi : F \to F$ for an open book can be expressed as a product of positive Dehn twists, then the adapted contact structure is Stein fillable.*

Proof. If a manifold M_n has an open book decomposition with fiber F, an m-times punctured genus g surface, and monodromy $\phi = id$, then the manifold is the connected sum of $n = 2g + m - 1$ copies of $S^1 \times S^2$. (To see this, note that M_n with the binding removed is $F \times S^1$ and the co-core of each 1-handle in F is an annulus. Now, when the binding is replaced, these annuli become essential 2-spheres.) This open book decomposition can be seen as the boundary of a (positive) Lefschetz fibration on a 4-manifold X that M_n bounds. From this one may easily conclude that the contact structure ξ_n, adapted to the open book decomposition, is Stein filled by X (*cf.* [3, 23]).

Assume ϕ consists of a single positive Dehn twist along a closed curve $\gamma \subset F$. Then the manifold M is obtained from M_n by a Dehn surgery along γ with surgery coefficient one less than the framing induced on γ by the fiber. But we can also make γ a Legendrian curve in F so that the framings given by the contact structure and the fibers agree. (In other words, the twisting number of γ relative to F is zero.) This is made possible by applying (a variant of) the Legendrian Realization Principle (for details see [20]). Although ∂F is not Legendrian, for the purposes of the Legendrian Realization Principle we may assume that ∂F is the dividing set of the convex surface F and realize any closed curve $\gamma \subset \text{int } F$ as a Legendrian curve, provided γ is *non-isolating*, *i.e.*, every component of $F \setminus \gamma$ nontrivially intersects ∂F. Thus (M, ξ) is obtained from (M_n, ξ_n) by a Legendrian surgery and hence is Stein fillable, provided γ is non-isolating. The only way our γ could be isolating is if it were separating but then we use the argument in Lemma 1 of [23] and write a positive Dehn twist about the separating curve γ as a product of positive Dehn twists about non-separating curves. Thus we are left with the case where ϕ is the product of $k > 1$ positive

Dehn twists about non-separating curves and we just perform k Legendrian surgeries on different leaves. □

We are now ready to prove Theorem 1.1. It should be pointed out that the strategy of proof is similar to the proof strategy in [6], where it is proved that "most" universally tight contact contact structures on torus bundles over the circle are not (strongly) symplectically fillable.

Proof of Theorem 1.1. If (M_1, ξ_1) is Stein fillable, then we are done by Lemma 2.2. Therefore, let (M_1, ξ_1) be a contact structure which is not Stein fillable. By Theorem 3.1, there exists an open book decomposition for M_1 which is adapted to ξ_1. Let K be the binding, $f\colon (M_1 \setminus K) \to S^1$ the fibering of the complement, F the fiber, and ϕ the monodromy map. Since (M_1, ξ_1) is not Stein fillable, any product decomposition of ϕ into Dehn twists must contain some negative Dehn twists. We view each Dehn twist as being done on a separate fiber. On a fiber just after one on which a negative Dehn twist was done along γ, we can take a parallel copy of γ and perform a positive Dehn twist, which is tantamount to a Legendrian surgery. If a compensatory positive Dehn twisted is added whenever there is a negative Dehn twist, then we will have a new monodromy map ϕ' with only positive Dehn twists. Of course ϕ' will define a different manifold M_2 and a different contact structure ξ_2. However, since the difference in between the monodromy for M_1 and for M_2 is just several positive Dehn twists, we can get from (M_1, ξ_1) to (M_2, ξ_2) by a sequence of Legendrian surgeries. Thus we have a Stein cobordism from (M_1, ξ_1) to (M_2, ξ_2). □

4. Overtwisted Contact Structures

In this section we prove Theorem 1.2. The proof will be broken down into two propositions.

Proposition 4.1. *Any overtwisted contact manifold is Stein cobordant to any overtwisted contact manifold.*

Proof. Let $(M_i, \xi_i), i = 1, 2$ be two overtwisted contact manifolds. It is a well-known fact in 3-manifold topology that we can find a link L in M_1 such that a certain integer Dehn surgery on L will yield M_2. Thus we can construct a topological cobordism X from M_1 to M_2 by attaching 2-handles with the appropriate framing to $M_1 \times [0,1]$. Moreover, one can adapt the proof of Lemma 4.4 in [19] to show that we may assume that X has an almost complex structure with complex tangencies ξ_i on M_i. We now apply the following theorem of Eliashberg (Theorem 1.3.4 in [9]):

Theorem 4.2 (Eliashberg)**.** *Let (X, J) be a compact, almost complex (real) 4-manifold with boundary $\partial X = M_2 - M_1$. Assume M_1 is J-concave, J is integrable near M_1, and the corresponding contact structure (M_1, ξ_1) is* *overtwisted. If the cobordism (X, J) from M_1 to M_2 consists of only 2-handle attachments, then there exists a deformation of J (rel M_1) to an integrable complex structure $\widetilde{J}$ on X for which M_2 is $\widetilde{J}$-convex.*

Using this theorem, we obtain a Stein structure on X for which the complex tangencies on M_1 are ξ_1 and on M_2 are some contact structure ξ' homotopic to ξ_2 as a 2-plane field. Now, we are done if ξ' is overtwisted, since overtwisted contact structures are classified by their 2-plane field homotopy type [8]. But we can easily ensure that the contact structure on M_2 is overtwisted by adding some extra Lutz twists to (M_1, ξ_1) that are disjoint from the regions where the 2-handles are attached. □

Proposition 4.3. *Given a tight contact manifold (M, ξ), there exists an overtwisted contact structure ξ' on M in the same homotopy class as ξ and which satisfies $(M, \xi') \prec (M, \xi)$.*

Proof. Given (M, ξ), take a Legendrian curve $L \subset M$ and its standard neighborhood $N(L)$. Choose a framing as in Section 2 so that the slope of the dividing set of $\partial N(L)$ is ∞. Now, identify slopes $s \in \mathbb{R} \cup \{\infty\}$ with their respective "angles", $[\theta_s] \in \mathbb{R}/\pi\mathbb{Z}$. In order to distinguish the different amounts of "wrapping around", we will choose a lift $\theta_s \in \mathbb{R}$ instead. There exists an exhaustion of $N(L)$ by concentric T^2, where the angles of the dividing curves on the tori monotonically increase over the interval $[\frac{\pi}{2}, \pi)$ as the T^2 move towards the core.

Now, let (M, ξ') be the overtwisted 3-manifold obtained by performing a full Lutz twist along L. This replaces $N(L)$ by the solid torus N, where the angles of the dividing curves of an exhaustion by tori monotonically increase over the interval $[\frac{\pi}{2}, 3\pi)$. We claim that a full Lutz twist $(M, \xi) \overset{L}{\rightsquigarrow} (M, \xi')$ is the inverse process of a sequence of Legendrian surgeries along the same core. To see this, take a Legendrian curve K in (M, ξ') in the same isotopy class as L, whose standard neighborhood $N(K) \subset N$ has an exhausting set of tori which spans the interval $[3\pi - \frac{3\pi}{4}, 3\pi)$. Note this implies that $\mathrm{tb}(K) = 1$ (when measured with respect to the trivialization of N we are using). Thus Legendrian surgery on K corresponds to 0-Dehn surgery. Moreover after Legendrian surgery, the new N "rotates" in the interval $[\frac{\pi}{2}, \frac{5\pi}{2})$. Repeated application (total of 4 times) of Legendrian surgery will get us back to (M, ξ). Note, however, that the intermediate manifolds are not necessarily diffeomorphic to M. We leave it to the reader to check that the four surgeries correspond to Dehn surgery on the link $K_0 \cup K_1 \cup K_2 \cup K_3$, where K_0 is K, each K_i is a meridian to K_{i-1} for $i = 1, 2, 3$, (and not linked with K_j if $|j - i| > 1$) and the surgery coefficients are all 0. □

Combining Propositions 4.1 and 4.3, we immediately get Theorem 1.2.

5. Concave Fillings

In this section we prove Theorem 1.3. Before we set out on the proof, we give a straightforward proof of this theorem for overtwisted contact structures.

Lemma 5.1. *Theorem 1.3 is true for any overtwisted contact structure.*

Proof. Given any overtwisted contact structure (M,ξ), we know by Theorem 1.2 that there is a Stein cobordism (X,ω) from (M,ξ) to (S^3,ξ_{std}). Let (Y,ω') be any closed symplectic 4-manifold. Use Darboux's theorem to excise a small standard ball around a point in Y and obtain a manifold Y' with concave boundary (S^3,ξ_{std}). We then obtain a concave filling of (M,ξ) by gluing (X,ω) to $(Y',\omega'|_{Y'})$. It is clear that there are infinitely many choices for (Y,ω') that will yield infinitely many different concave fillings for (M,ξ). □

Lemma 5.2. *Theorem 1.3 is true for any Stein fillable contact structure.*

Proof. Let (M,ξ) be Stein filled by (X,ω). According to Corollary 3.3 in [22], there is a symplectic embedding of (X,ω) into a compact Kähler minimal surface S of general type. If we take $Y=\overline{S\setminus X}$, then $(Y,\omega|_Y)$ will be a concave symplectic filling of (M,ξ).

A slight modification of the above argument will produce infinitely many concave fillings. Specifically, in a small standard 3-ball $(B^3,\xi_{std})\subset(M,\xi)$, there exist a right-handed Legendrian trefoil knot with $tb=1$ and a linking Legendrian unknot with $tb<0$. If we add 2-handles to X along these Legendrian knots, we obtain a new Stein manifold (X',ω'). Embed X' in a compact Kähler surface S and remove X to obtain a concave symplectic filling (Y',ω') of (M,ξ). In the layer $X'\setminus X$ in Y' there exists a symplectically embedded torus T. To see this note that the manifold N obtained from B^3 by attaching a 2-handle along a right-handed trefoil knot with framing 0 is a "cusp neighborhood", see [17], and thus it supports a symplectic structure containing may symplectic tori. Now our manifold X' is symplectomorphic to $X\cup N$ with a 1-handle attached (this can be done in a symplectic fashion [9]). Let $E(n)$ be the elliptic surface obtained by taking the normal sum [18] of $n\geq 1$ copies of the rational elliptic surface along regular fibers. Then consider the symplectic manifold $Y_n=E(n)\#_T Y'$, obtained by taking the normal sum of Y' along T and $E(n)$ along a regular fiber. These concave fillings of (M,ξ) are not related by blowing up and down, since if they were, then the compact manifolds S_n, obtained from S by normal summing with E_n, would also be so related. However, this is not the case, as $b_2^+(S_n)=b_2^+(S)+2n$ and b_2^+ is unchanged by blowing up and down. □

Theorem 1.3 now follows from Lemma 5.2 and Theorem 1.1.

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