COMPLEMENTARY LEGS AND SYMPLECTIC RATIONAL BALLS

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ABSTRACT. We completely characterize when a small Seifert fibered space with complementary legs symplectically bounds a rational homology ball in the case $e_0 \leq -1$, and we establish strong obstructions for other values of e_0. Our results highlight a sharp contrast with the smooth category, where many more such Seifert fibered spaces are known to bound smooth rational homology balls. We also complete the classification of contact structures on spherical 3-manifolds with either orientations that admit symplectic rational homology ball fillings.

1. INTRODUCTION

There has been a great deal of work aimed at determining which rational homology 3-spheres bound rational homology 4-balls. In [13], the authors studied this question in the symplectic category for small Seifert fibered spaces. In particular, it was shown that none of the contact structures on a small Seifert fibered space with $e_0 \leq -5$ admit a symplectic rational homology ball filling, even though many such spaces smoothly bound rational homology balls. In [3], Bhupal and Stipsicz characterized which Milnor fillable contact structures on small Seifert fibered spaces with $e_0 \leq -2$ symplectically bounded rational homology balls. They consisted of a family with $e_0 = -4$, two families with $e_0 = -3$, and seven families when $e_0 = -2$. In [13], the authors showed that these were the only contact structures on small Seifert fibered spaces bounding rational homology balls when $e_0 = -4, -3$ and also when $e_0 = -2$ if the space was in the Bhupal-Stipsicz list.

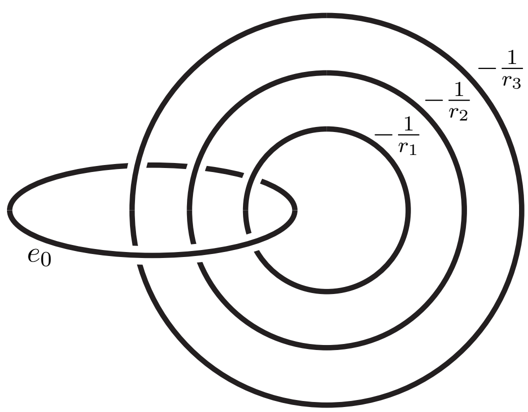

FIGURE 1. A surgery diagram for the small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$, with normalized Seifert invariants.

Here we continue our study by specifically focusing on small Seifert fibered spaces with complementary legs. Let $Y = Y(e_0; r_1, r_2, r_3)$ denote the small Seifert fibered space with normalized Seifert invariants, where $e_0 \in \mathbb{Z}$ and $r_i \in (0,1) \cap \mathbb{Q}$ for $i = 1, 2, 3$, see Figure 1. We say Y has *complementary legs* if two of the r_i add to 1. Without loss of generality, we will assume, for the rest of the paper, that $r_1 + r_3 = 1$. Lecuona [19] has characterized precisely when such a Seifert fibered space smoothly bounds a rational homology ball. This will be the starting point for our analysis in the symplectic category. We will reprove her main result in a way that is more aligned with our symplectic geometric perspective in this paper. To this end, we first recall a result of Lisca. In [21], Lisca showed that a lens space $L(p, q)$ bounds a rational homology ball if and only if $p/q \in \mathcal{R}$, where $\mathcal{R}$ is the collection of rational numbers $p/q > 1$ with $gcd(p, q) = 1$, $p = m^2$, and $q, p - q$, or q^* is of the form

(1) $mh \pm 1$ where $0 < h < m$ with $gcd(h, m) = 1$,
(2) $mh \pm 1$ where $0 < h < m$ with $gcd(h, m) = 2$,
(3) $h(m \pm 1)$ where $h > 1$ divides $2m \mp 1$, or
(4) $h(m \pm 1)$ where $h > 1$ is odd and divides $m \pm 1$,

where $0 < q^* < p$ is the inverse of q mod p. We note that item (2) above did not appear in [21] though the proof there does produce it, see Remark 1.5 in [2]. In Theorem 1.1 below, and throughout this paper, $[a_0, a_1, \ldots, a_n]$ denotes the continued fraction

$$a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}}$$

We can now give the characterization of when a small Seifert fibered space with complementary legs bounds a rational homology ball.

Theorem 1.1 (Lecuona 2019, [19]). *Let $Y = Y(e_0; r_1, r_2, r_3)$ be a small Seifert fibered space with complementary legs (i.e., $r_1 + r_3 = 1$) whose surgery diagram is depicted in Figure 1. Perform $(-e_0 - 1)$ Rolfsen twists on the $(-1/r_2)$-framed surgery curve to obtain a new surgery diagram of Y such that the new framing on the horizontal curve is -1. Denote the new framing on the $(-1/r_2)$-framed surgery curve by $-1/r_2'$ and note that*

$$-1/r_2' = n + \frac{1}{[a_1^2, \ldots, a_{n_2}^2]}$$

for some uniquely determined integers $n, a_1^2, \ldots, a_{n_2}^2$, with $a_i^2 \geq 2$ for $1 \leq i \leq n_2$. Then Y smoothly bounds a rational homology ball if and only if $[a_1^2, \ldots, a_{n_2}^2] \in \mathcal{R}$.

1.1. **Symplectic rational homology balls.** We now consider when a small Seifert fibered space with complementary legs can bound a symplectic rational homology ball.

Theorem 1.2. *A small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$ with complementary legs and $e_0 \leq -2$ does not admit a rational homology ball symplectic filling.*

Remark 1.3. We would like to emphasize that Theorem 1.2 confirms our Conjecture 1.5 in [13], for any small Seifert fibered space $Y(-2; r_1, r_2, r_3)$ with complementary legs.

Remark 1.4. We would like to point out that when $e_0 \leq -3$, Theorem 1.2 also follows directly from [13, Theorem 1.1] since one can easily verify that the small Seifert fibered spaces described by the plumbing diagrams in $\mathcal{QHB}$ (depicted in [13, Figure 3]) do not have complementary legs. Any small Seifert fibered space $Y(-2; r_1, r_2, r_3)$ with complementary legs is an L-space and therefore, [13, Corollary 1.6] implies that $Y(-2; r_1, r_2, r_3)$ with a pair of complementary legs admits *at most four* contact structures that might admit rational homology ball symplectic fillings. Hence, the results in our earlier work [13] almost prove Theorem 1.2 here, but fall short when $e_0 = -2$.

Remark 1.5. We note that from Theorem 1.1 there are many small Seifert fibered spaces with complementary legs and any e_0 that bound smooth rational homology balls. So we see a stark contrast between the smooth and symplectic categories.

We now turn to the case $e_0 \geq -1$. In Lemma 4.5 of [13], several symplectic rational homology ball fillings of small Seifert fibered spaces $Y(-1; r_1, r_2, r_3)$ were constructed.[1] Here, we prove that these are the only such fillings of contact structures on $Y(-1; r_1, r_2, r_3)$ with complementary legs.

Theorem 1.6. *A small Seifert fibered space with complementary legs and $e_0 = -1$, carries a contact structure which is symplectically fillable by a rational homology ball if and only if it is of the form $Y(-1; r, m^2/(nm^2 - mh + 1), 1 - r)$ for some integer $n > 1$, and some relatively prime integers $0 < h < m$, or $n \geq 1$, $m = 1$ and $h = 0$. Moreover, there are precisely $2n$ distinct tight contact structures with this property on such a small Seifert fibered space. Furthermore, for $2(n-1)$ of these contact structures, the symplectic rational homology ball filling is unique.*

According to [15] we know that any tight contact structure on $Y = Y(e_0; r_1, r_2, r_3)$ with $e_0 \geq 0$ is obtained from contact surgery on the link $L_1 \cup L_2 \cup L_3$ in $S^1 \times S^2$ shown in Figure 7. We call each of the L_i a leg of the Seifert fibration. Recall that the contact surgery on each of the L_i is determined by a sequence of signs related to the continued fraction expansion of $-1/r_i$. We say that the contact surgery on leg L_i is *consistent* if all the signs are the same. If Y has complementary legs, then we say that the contact structure coming from contact surgery is *balanced* if each of the complementary legs is consistent, but they have opposite signs.

In Lemma 4.5 of [13], several symplectic rational homology ball fillings of Y with $e_0 \geq 0$ were constructed. Our final result below shows that, for Seifert fibered spaces with complementary legs, these fillings account for *almost* all fillings that could exist.

Theorem 1.7. *Let Y be a small Seifert fibered space with complementary legs and $e_0 \geq 0$, equipped with a balanced contact structure ξ. Then (Y, ξ) bounds a symplectic rational homology ball if and*

[1] The Seifert invariants in [13] appear different than the ones given in our theorem here, but substituting h for $m - h$ and n for $n + 1$ equates the results there and in this paper. The discrepancy comes from the different surgery descriptions utilized in the two papers.

only if Y is of the form $Y(e_0; r, s, 1-r)$, where s is given by

$$-1/s = \frac{-m^2 + mh - 1}{m^2 + (e_0+1)(-m^2 + mh - 1)} < -1,$$

for some $0 < h < m$ relatively prime (note that once $e_0 \geq 0$ is fixed, only certain values of h and m will satisfy the desired inequality). Moreover, on such a Y, there is a unique balanced contact structure (up to the orientation on the plane field) that bounds a symplectic rational homology ball.

If ξ is not a balanced contact structure, then Y might bound a symplectic rational homology ball if it is given by the surgery diagram in Figure 2 where $[a_1^2, \ldots, a_{n_2}^2] \in \mathcal{R}$. Given such a Y there is at most one, respectively two, contact structures (up to orientation of the plane field) that can bound a symplectic rational homology ball if $e_0 = 0$, respectively $e_0 > 0$.

Remark 1.8. We conjecture that none of the contact structures in the second paragraph of the theorem admit symplectic rational homology ball fillings. If so, then the theorems above completely characterize when a Seifert fibered space with complementary legs bounds a symplectic rational homology ball. The result below gives some evidence towards this conjecture, and also answers positively the prediction made in [13, Remark 1.21]. (See also Section 1.2).

Proposition 1.9. *A small Seifert fibered space $Y = Y(e_0; \frac{1}{2}, s, \frac{1}{2})$ with $e_0 \geq 0$ equipped with a contact structure ξ admits a rational homology ball symplectic filling if and only if Y is of the form as in the first part of Theorem 1.7 and ξ is balanced.*

Remark 1.10. We point out a subtle difference between characterizing smooth rational homology ball fillings vs symplectic rational homology ball fillings. It is a standard fact that the lens space $L(p,q)$ arises as the double cover of S^3 branched over the two-bridge link $K(p,q)$. In [21, Corollary 1.3], Lisca not only characterized which lens spaces bound smooth rational homology balls but also proved that $L(p,q)$, assuming p is odd, bounds a rational homology ball exactly when $K(p,q)$ is a smoothly slice knot. In particular, a rational homology ball that $L(p,q)$ bounds can be constructed by taking the double branched cover of B^4 branched over the slice disk. This result was later extended by Lecuona [19] to cover Seifert fibered spaces with complementary legs that arise as the double cover of S^3 branched along a Montesinos link that can be put in the form $K(p,q) \sqcup U$ where U is the unknot. When determining symplectic rational homology ball fillings for lens spaces or Seifert fibered spaces with complementary legs, a natural guess would be that perhaps the two-bridge knots involved are symplectically or Lagrangian slice, which would require the knot to be at least quasi-positive. It is well-known, by combining [4, 18, 28] that this is never the case. (See also [26], for two new proofs of this fact).

The strategy of proof for Theorem 1.2 involves the θ-invariant calculations. The θ-invariant is an invariant of homotopy classes of plane fields, defined in [17]. It is well-known, see for example [13], that if a contact 3-manifold (M, ξ) symplectically bounds a rational homology ball, then $\theta(\xi) = -2$. In Sections 4 and 5 we show by explicit calculations that any contact structure on $Y(e_0; r_1, r_2, r_3)$ with complementary legs and $e_0 \leq -2$ has θ-invariant larger than -2. We prove Theorem 1.6 (see Theorem 3.3) and 1.7 (see Theorem 3.4) in Section 3, by constructing Stein cobordisms from lens spaces to certain small

Seifert fibered spaces and using facts from [13]. Section 2 discusses small Siefert fibered spaces with complementary legs and relates prior work in the area to conventions needed here.

This paper can be thought of as a continuation of [13]. As such, we refer to Section 2 of that paper for the background on standard facts from contact geometry, smooth and contact surgeries, and homotopy invariants of plane fields.

1.2. **Spherical 3-manifolds.** We now turn to the classification of spherical 3-manifolds (with either orientations) that admit rational homology ball symplectic fillings. We refer the reader to our earlier work [13], for basic definitions of spherical 3-manifolds that will be used below. It was shown by Choe and Park [6] that a spherical 3-manifold Y bounds a smooth rational homology ball if and only if Y or $-Y$ is homeomorphic to one of the following manifolds:

(1) $L(p,q)$ such that $p/q \in \mathcal{R}$,
(2) $D(p,q)$ such that $(p-q)/q' \in \mathcal{R}$,
(3) T_3, T_{27} and I_{49},

where p and q are relatively prime integers such that $0 < q < p$, and $0 < q' < p-q$ is the reduction of q modulo $p-q$. So it is clear that when searching for spherical 3-manifolds bounding symplectic rational homology balls, we only need to consider these 3-manifolds.

Suppose that ξ is a contact structure on the spherical 3-manifold Y, oriented as the link of the corresponding quotient surface singularity. In [13], we showed that if (Y,ξ) admits a symplectic rational homology ball filling, then Y is orientation-preserving diffeomorphic to a lens space $L(m^2, mh-1)$ for some coprime integers $0 < h < m$, and ξ is contactomorphic to ξ_{can}. We also observed however, that this result does not *necessarily* hold true for a spherical 3-manifold equipped with the orientation opposite to the canonical one it carries when viewed as the singularity link. We showed that $-T_3 = Y(-1; 2/3, 1/2, 1/3)$ admits two non-isotopic tight contact structures, both of which admits a symplectic rational homology ball filling (we note that this follows from Theorem 1.6 too), but neither $-T_{27} = Y(3; 2/3, 1/2, 1/3)$ nor $-I_{49} = Y(0; 4/5, 1/2, 1/3)$ admits a symplectic rational homology ball filling.

Here we complete the classification of all spherical 3-manifolds with either orientations, which admit rational homology ball symplectic fillings, by resolving the remaining case of "oppositely oriented" dihedral-type spherical 3-manifolds.

Theorem 1.11. *The only spherical* 3*-manifolds that admit symplectic rational homology fillings are*

(1) $L(m^2, mh-1)$ *for relatively prime* $0 < h < m$*, with the contact structure* $\pm\xi_{can}$*, or*
(2) $-D((n+1)m^2 - mh + 1, nm^2 - mh + 1)$ *for relatively prime* $0 < h < m$ *or* $m = 1$ *and* $h = 0$*, and* $n \geq 1$ *with one of* $2n$ *possible contact structures, or*
(3) $-T_3 = Y(-1; 2/3, 1/2, 1/3)$ *with one of two non-isotopic contact structures.*

1.3. **Final remarks and questions.** Our work in [13] and in this paper strongly indicates a positive answer to the following general conjecture.

Conjecture 1.12. *A Seifert fibered space $Y = Y(e_0; r_1, r_2, \ldots, r_n)$, which is also a rational homology sphere, carries a contact structure ξ such that (Y, ξ) admits rational homology ball symplectic filling if and only if $n \leq 4$ and either*

(1) $e_0 \leq -2$, Y is the link of a complex surface singularity, whose resolution graph belongs to a finite list of families provided by Bhupal and Stipsicz [3, Figure 1 and Figure 2]*, and ξ is the canonical contact structure, or*
(2) $e_0 = -1$, Y is a small Seifert fibered space with complementary legs as in Theorem 1.6, and ξ is one of the balanced contact structures described in the theorem, or
(3) $e_0 \geq 0$, Y is a small Seifert fibered space with complementary legs as in the first part of Theorem 1.7, and ξ is the unique balanced contact structure described in the theorem.

We end with an interesting remark about orientations. As a consequence of our results in Theorem 1.2 and 1.6, we obtain many examples of a small Seifert fibered space with no rational homology ball symplectic filling with either orientation, as well as many that have rational homology ball fillings with one orientation. We are not aware of a 3-manifold, other than S^3, that has a rational homology ball symplectic filling with both orientations. So we ask:

Question 1.13. *Suppose Y is a closed, connected and oriented 3-manifold. Is it true that if Y and $-Y$ both admit rational homology ball symplectic fillings, then $Y = S^3$?*

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2. Small Seifert fibered spaces with complementary legs

Recall the Seifert fibered space $Y(e_0; r_1, r_2, r_3)$ has complementary legs if there are distinct i and j such that $r_i + r_j = 1$. We will order the singular fibers such that $r_1 + r_3 = 1$. In this section, we will consider the smooth topology and the contact topology of such Seifert fibered spaces.

2.1. **Smooth Seifert fibered spaces with complementary legs.**

We begin by describing convenient surgery diagrams for small Seifert fibered spaces with complementary legs.

Lemma 2.1. *Any small Seifert fibered space with complementary legs is obtained from $S^1 \times S^2$ by Dehn surgery on a regular fiber of some Seifert fibration of $S^1 \times S^2$ and has a surgery diagram given in Figure 2. Here $r/s, p/q > 1$, $r/s = [a_1^1, \ldots, a_{n_1}^1]$, $p/q = [a_1^2, \ldots, a_{n_2}^2]$, and $r/(r-s) = [a_1^3, \ldots, a_{n_3}^3]$, all the $a_i^j \geq 2$, and $n \in \mathbb{Z}$. Moreover, any such Seifert fibered space is a rational homology sphere.*

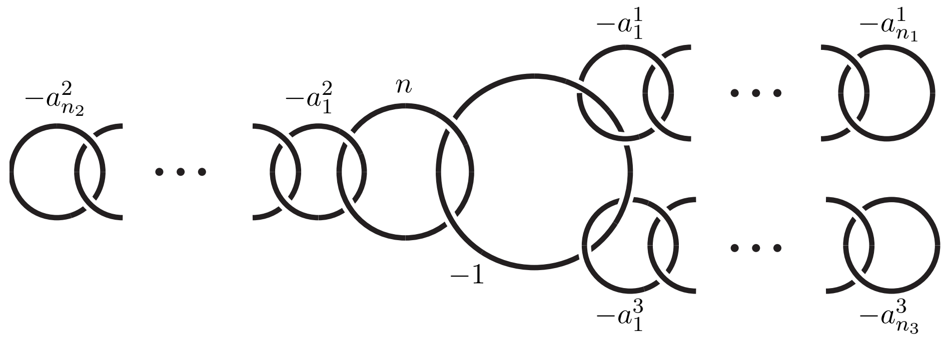

FIGURE 2. Surgery diagram for a small Seifert fibered space with complementary legs. Here $r/s > 1$, $r/s = [a^1_1, \ldots, a^1_{n_1}]$, $p/q = [a^2_1, \ldots, a^2_{n_2}]$, and $r/(r-s) = [a^3_1, \ldots, a^3_{n_3}]$.

Remark 2.2. We note that the Seifert fibered space in Figure 2 is not in normalized form if n is not less than -1. Using Rolfsen twists, one may easily show

$$\begin{array}{ll} e_0 \geq 0 & \text{if } n = -1 \\ e_0 = -1 & \text{if } n \leq -2 \\ e_0 = -2 & \text{if } n \geq 1 \\ e_0 \leq -3 & \text{if } n = 0. \end{array}$$

In fact, if $n - q/p$ is in the interval $(-1/k, -1/(k+1))$ for some integer $k \geq 1$ (which requires that $n = -1$), then $e_0 = k - 1$ and if $-q/p$ is in $(1/(k+1), 1/k)$ for some integer $k \geq 1$ (which requires that $n = 0$), then $e_0 = -k - 2$.

Proof. Given a small Seifert fibered space $Y = Y(e_0; r_1, r_2, r_3)$ in normalized form, we see it has complementary legs if $-1/r_1 = -r/s$ and $-1/r_3 = -r/(r-s)$ and both quantities will be less than -1. We can perform a Rolfsen twist on the singular fiber $-r/(r-s)$ so that the singular fiber is now a singular fiber labeled r/s, and the e_0 term has been increased by 1. We may now perform Rolfsen twists on the second singular fiber so that the central

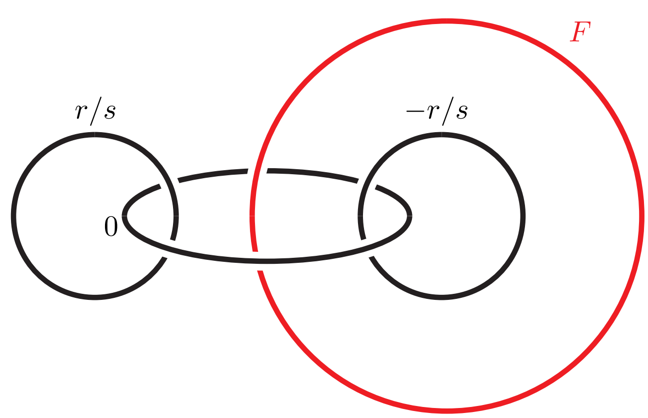

FIGURE 3. A surgery diagram for $S^1 \times S^2$ with a Seifert fibered structure where F is a regular fiber.

curve has surgery coefficient 0. If we remove the second singular fiber from the surgery diagram for Y, then the resulting manifold is $S^1 \times S^2$, and the curve that one would surger to obtain Y is a regular fiber in a Seifert fibered structure on $S^1 \times S^2$. If this is not clear, see [13, Lemma 4.1]. So Y is obtained from Figure 3 by Dehn surgery on F. This establishes the first claim of the lemma. We note for future reference that the framing of F given by the Heegaard torus on which F sits agrees with the 0-framing in the figure.

For the second claim, we note that we can undo the Rolfsen twist on the r/s fiber to get back to a $-r/(r-s)$ fiber and now the central curve has coefficient -1. Moreover, a continued fraction expansion of the surgery coefficient will be of the form

$$n + \frac{1}{[a_1^2, \ldots, a_{n_2}^2]},$$

for some integer n and integers $a_i^2 \geq 2$. This establishes the second claim of the lemma.

Finally, recall that a small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$ in normal form is a rational homology sphere if and only if $e_0 + r_1 + r_2 + r_3 \neq 0$. Note that $e_0 \in \mathbb{Z}$, and $r_i \in (0, 1)$, by definition. The last claim of the lemma follows since we have $r_1 + r_3 = 1$ in our case. $\square$

We characterize when a small Seifert fibered space with complementary legs bounds a rational homology ball, but first recall the notion of a normal sum [16]. Suppose K_i is a knot in Y_i for $i = 0, 1$. The *normal sum of Y_0 and Y_1 along K_0 and K_1* is the result of removing neighborhoods of K_i from Y_i and identifying the resulting boundaries by a diffeomorphism that sends a longitude of K_0 to a longitude of K_1 (thus the result depends on a framing on the K_i) and the meridian of K_0 to the meridian of K_1. One can effect a normal sum by taking the connected sum of Y_0 and Y_1 on a neighborhood of a point on K_0 and K_1. Then in $Y_0 \# Y_1$ we have the connected sum of K_0 and K_1. The normal sum is then the result of Dehn surgery on $K_0 \# K_1$ with framing determined by the framings on the K_i.

Proposition 2.3. *A small Seifert fibered space with complementary legs bounds a rational homology ball if and only if it is obtained by normal summing a lens space $L(p, q)$ that bounds a rational homology ball and $S^1 \times S^2$ along the core of a Heegaard torus in the lens space and the fiber of a Seifert fibration on $S^1 \times S^2$. In particular, it will be given by the surgery diagram in Figure 2, where $p/q = [a_1^2, \ldots, a_{n_2}^2] > 1$ is in $\mathcal{R}$ and $n \in \mathbb{Z}$.*

Before setting the stage for the proof we observe that Theorem 1.1 is now an immediate consequence of Proposition 2.3.

The classification of small Seifert fibered space with complementary legs that bound rational homology balls was established by Lecuona in [19, Corollary 3.3]. The notation in that paper is different from the notation used here, and the results there are stated up to possibly reversing orientation, where the above result holds for any orientation. So, we will discuss the classification given in [19, Corollary 3.3] and see how it relates to our statement above. The result there says a Seifert fibered space with complementary legs bounds a rational homology ball if and only if it is homeomorphic, up to orientation, to one described by the surgery diagram in Figure 4.

We first note that the Seifert fibered spaces considered in Figure 4 all have $e_0 \leq -2$. Recall that for a small Seifert fibered space Y we have $e_0(Y) + e_0(-Y) = -3$. Thus, given any such Y, either Y or $-Y$ has $e_0 \leq -2$ and Y bounds a rational homology ball if and only if $-Y$ does. So when trying to determine if Y bounds a rational homology ball, we

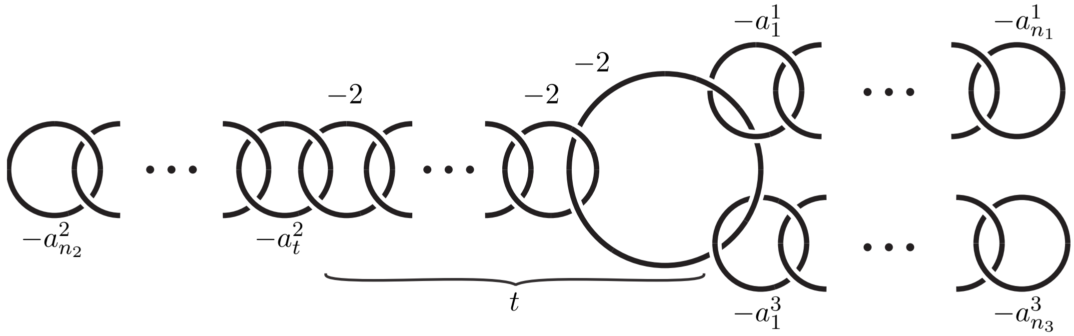

FIGURE 4. Surgery diagram for a small Seifert fibered space with complementary legs from Lecuona's paper [19]. Here $r/s > 1$, $r/s = [a^1_1, \ldots, a^1_{n_1}]$, and $r/(s-r) = [a^3_1, \ldots, a^3_{n_3}]$, where $0 \leq t \leq n_2$, $a^2_t > 2$, and $[a^2_t - 1, a^2_{t+1}, \ldots, a^2_{n_2}]$ is in $\mathcal{R}$.

can always choose the orientation so that $e_0 \leq -2$. Thus, Figure 4 does cover all possible smooth Seifert fibered spaces with complementary legs; however, as we are interested in studying contact structures, it is important to keep track of orientations, and thus we will use Figure 2.

We will now see that Figures 2 and 4 describe the same Seifert fibered spaces with $e_0 \leq -2$. We begin with the case of $e_0 = -2$, and so in Figure 2, we have $n \geq 1$. We will focus on the central unknot and the chain to its left, as the remainder of the diagram is unchanged. The equivalence is shown in Figure 5. The top row shows the chain mentioned above. A blow-up has been performed between the n and -1-framed components to obtain the second row. The last row is obtained by continued blow-ups between the -1-framed knot and what has become of the n-framed knot under the blow-ups. Now one blows down the 1-framed unknot to obtain the left-most chain in Figure 4 with $-a^2_1$ replaced with $-a^2_1 - 1$.

We now show the equivalence between Figures 2 and 4 when $e_0 < -2$. In this case, we will have $n = 0$ in Figure 2. Sliding the $-a^2_1$-framed unknot over the -1-framed unknot in Figure 2 will result in Figure 6. One may now easily see that the 0-famed unknot links the -1-framed uknot as a meridian and does not link any other component in the diagram. Thus, a sequence of handle slides will disentangle these two components from the rest of the diagram, and they can be deleted. This results in Figure 4 with $-a^2_1$ replaced by $-a^2_1 - 1$ (and $t = 0$) as desired.

We note that after the above discussion, Lemma 2.3 follows from [19, Corollary 3.3], but we give a proof here as we will need the ideas in our discussion of symplectic rational homology balls below. We begin with a lemma that is equivalent to Proposition 3.1 in [19].

Lemma 2.4. *There is a rational homology cobordism from the lens space $L(p, q)$ with $p/q = [a^2_1, \ldots, a^2_{n_2}] > 1$ to the Seifert fibered space shown in Figure 2.*

Recall a cobordism X form Y_0 to Y_1 is a rational homology cobordism if the relative homology $H_k(X, Y_i; \mathbb{Q}) = 0$ for all k. (If this is true for one i then it is true for the other i as well.)

Proof. Given a 4-manifold X with boundary, we attach a *round* 1*-handle* to X as follows. The handle is $H = (D^1 \times D^2) \times S^1$ and it is attached to ∂X by an embedding $(\partial D^1) \times$

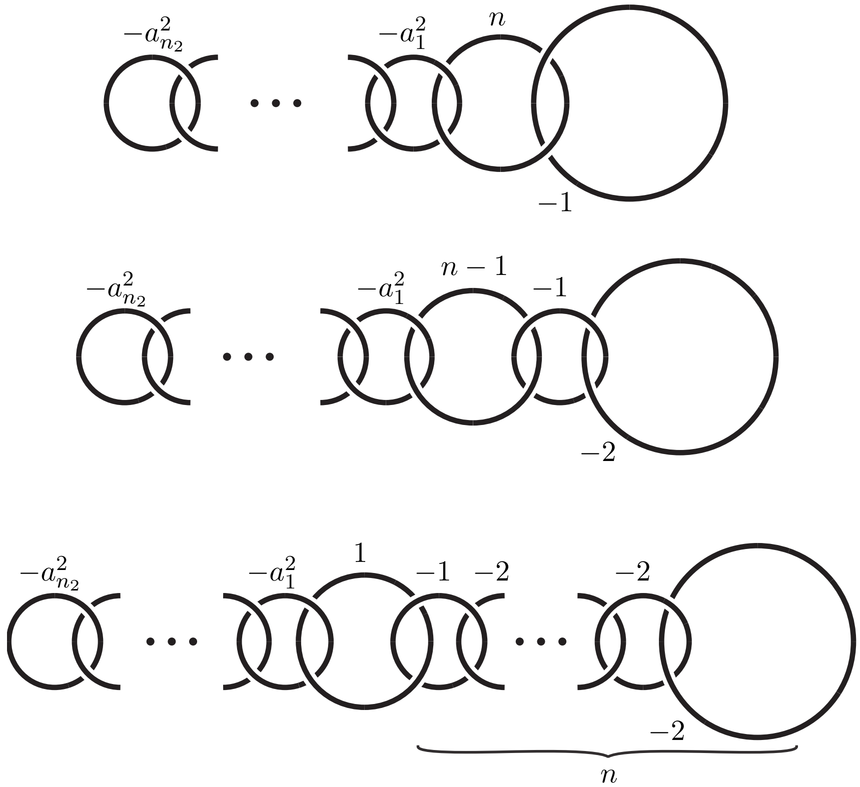

FIGURE 5. Equating Figures 2 and 4 when $e_0 = -2$.

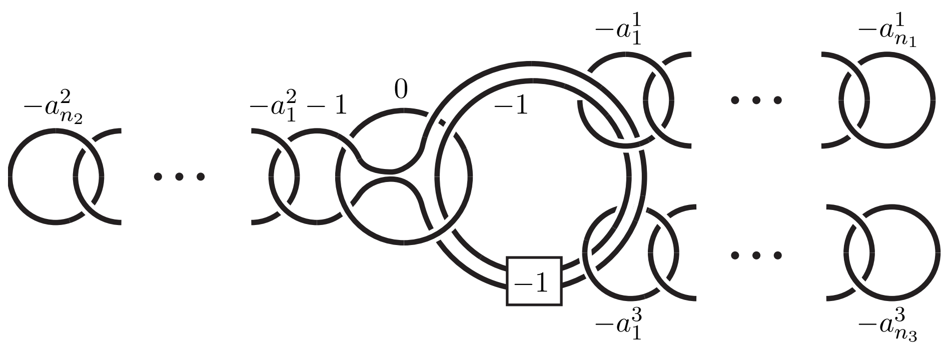

FIGURE 6. Equating Figures 2 and 4 when $e_0 < -2$.

$D^2 \times S^1 \to \partial X$. That is, it is attached along the neighborhood of two knots in ∂X and the attaching map is determined by framings on the knots. If X' is the result of this attachment, then we note that $\partial X'$ is obtained from ∂X by a normal sum along the attaching knots. We also note that a round 1-handle attachment can be effected by attaching a 1-handle to ∂X so that the attaching sphere is two points, one on each knot, and then attaching a 2-handle along the knot formed by the connected sum of the knots after the 1-handle attachment.

Now consider $X = (L(p,q) \times [0,1]) \cup (S^1 \times D^3)$. We will attach a round 1-handle to this disjoint union along the core of a Heegaard torus in $L(p,q) \times \{1\}$ and a regular fiber in a

Seifert fibered structure on $S^1 \times S^2$. This will give us a cobordism X' from $L(p,q)\#S^1 \times S^2$ to the Seifert fibered space shown in Figure 2. Indeed, after attaching the 1-handle, we see the manifold in Figure 2 without the n-framed 2-handle. The attaching knot of the n-framed 2-handle is exactly the connected sum of a core of a Heegaard torus in $L(p,q)$ and a regular fiber in a Seifert fibration of $S^1 \times S^2$. Thus, attaching the n-framed 2-handle completes the round 1-handle attachment.

Notice that X' is also the result of attaching a 1-handle to $L(p,q) \times [0,1]$ and then a 2-handle that non-trivially runs over that 1-handle. Hence X' is clearly a rational homology cobordism. □

Proof of Proposition 2.3. If the small Seifert fibered space Y with complementary legs is obtained from a lens space $L(p,q)$ bounding a rational homology ball as stated in the proposition, then we can glue the cobordism built in Lemma 2.4 to $L(p,q)$ to obtain a rational homology ball with boundary Y. On the other hand, if Y bounds a rational homology ball, then we may glue the cobordism from Lemma 2.4 (turned upside down) to this rational homology ball to build a rational homology ball with boundary $L(p,q)$. □

2.2. **Contact structures on some small Seifert fibered spaces.** We would now like to see various ways of understanding contact structures on small Seifert fibered spaces with complementary legs so that we can relate notation used in other papers with our perspective here. We start by discussing the standard tight contact structure on $S^1 \times S^2$ and then discuss how to change contact surgery descriptions of contact structures on Seifert fibered spaces. We then discuss tight contact structures on small Seifert fibered spaces with a zero twisting Legendrian fiber.

We begin by considering the standard contact structure on $S^1 \times S^2$. One can think of $S^1 \times S^2$ as $T^2 \times [0,1]$ with curves of slope ∞ on $T^2 \times \{i\}$, for $i = 0, 1$, collapsed to points. The contact structure is then $\ker(\cos(\pi t)\, d\theta + \sin(\pi t)\, d\phi)$. Notice that on $T^2 \times \{i\}$ the characteristic foliation has slope ∞, and thus, we may perform a contact cut, [20], to get a contact structure on $S^1 \times S^2$. This contact structure is the standard tight contact structure ξ_{std}. Notice that $T^2 \times \{1/2\}$ is a Heegaard torus in $S^1 \times S^2$, and it has a linear characteristic foliation of slope 0. We can perturb this torus to be convex with two dividing curves, denote it by T. Then T splits $S^1 \times S^2$ into two solid tori V_1 and V_3, and each V_i is a neighborhood of a Legendrian knot L_i. Moreover, one of the Legendrian dividing curves L_2 is also a Legendrian knot and Legendrian isotopic to both L_1 and L_3. (This is easily seen in the standard model of a Legendrian knot, [11].) It is clear that the contact planes along each L_i twist 0 times with respect to the framing given by the product structure. The L_i are shown in Figure 7.

Let N_i be a standard neighborhood of L_i and assume that all the N_i are disjoint. Notice that $C_2 = (S^1 \times S^2) \setminus (N_1 \cup N_3)$ is a thickened torus, $T^2 \times I$, with an I-invariant contact structure on it. Thus, performing contact surgery on L_1 and L_3 is equivalent to taking the surgery tori and gluing them together along their boundary. Recall that the contact structure on a solid torus with meridional slope m and dividing slope d is determined by a minimal path in the Farey graph from m clockwise to d whose edges (except for the first) are decorated with a sign. As noted above $-r/s$ surgery on L_1 and r/s surgery on L_3 result in $S^1 \times S^2$. (We note that since the contact framing on L_i is 0, the smooth surgery

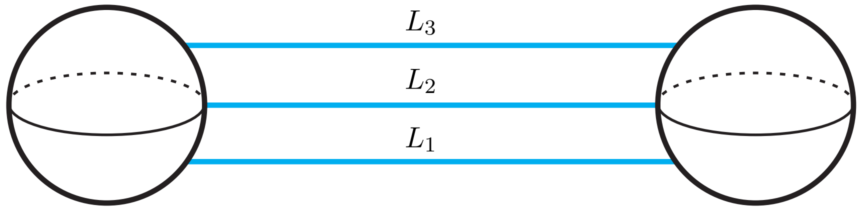

FIGURE 7. The Legendrian knots L_1, L_2, and L_3 in $S^1 \times S^2$.

coefficients and the contact surgery coefficients are the same.) We now recall Lemma 4.2 from [13]. After [13] was released the authors found out that the lemma had previously appeared in Matkovič's work [24].

Lemma 2.5. *Consider contact $(-r/s)$-surgery on L_1 and contact (r/s)-surgery on L_3. If the signs associated to the first surgery are the same and the signs for the second surgery are all opposite those of the first, then the resulting contact structure is ξ_{std} on $S^1 \times S^2$. Otherwise, the contact structure is overtwisted.*

We now move to tight contact structures on small Seifert fibered spaces with zero twisting fibers. It is not hard to show, see [15, 23, 30], that any tight contact structure on a small Seifert fibered space with zero twisting fiber can be obtained by contact surgery on the link in Figure 7. It is known [30, 31] that any small Seifert fibered space with $e_0 \geq 0$ has a zero twisting fiber. Thus, all tight contact structures come from some contact surgery on the link in Figure 7. For $e_0 > 0$ we can arrange that $-1/r_2, -1/r_3 \leq -2$ and $-1/r_1 \in (-2, -1)$. Since these are all negative contact surgeries and the contact twisting along the link components is 0, we know that all the resulting contact structures are tight (and Stein fillable). It was shown in [15, 30] that all these contact structures are distinct. For $e_0 = 0$ we can arrange that all the $-1/r_i \leq -2$. Once again we see that all these contact structures are tight (and Stein fillable), but in [15] we see that they are not all distinct.

Moving to $e_0 = -1$ Seifert fibered spaces, it is not true that all tight contact structures have zero twisting fibers, but the ones that do will again come from a contact surgery on the link in Figure 7. In this case, we can arrange that $-1/r_1, -1/r_3 \leq -2$ and $-1/r_2 > 1$. Notice that in this case, we must do a positive contact surgery on L_2, and hence, we are not guaranteed that all such contact structures are tight. In fact, we will see below that some are not. Note that a small Seifert fibered space with complementary legs and $e_0 = -1$ must have a zero twisting fiber, [31]. Thus, we can use the surgery diagram above to represent all tight contact structures on such a manifold. We note that this surgery diagram seems different from the one used in [23, 24] shown on the right of Figure 8. To see that these diagrams are really the same, we note that if we write $r_i = p_i/q_i$ then $Y(-1; r_1, r_2, r_3)$ will be given in Figure 7 by performing $-1/r_i$ surgery on L_i for $i = 1, 3$ and $q_2/(q_2 - p_2)$ surgery on L_2. It is well known that the surgery diagram in Figure 7 corresponds to the surgery diagram on the left in Figure 8, see for example [9]. In our case, the $s_i = -1/r_i$ for $i = 1, 3$, and $s_2 = q_2/(q_2 - p_2)$. Using the algorithm in [10] to convert a positive contact surgery into contact $(+1)$-surgeries and negative contact surgeries, we see that our diagram is equivalent to the one given on the right of Figure 8.

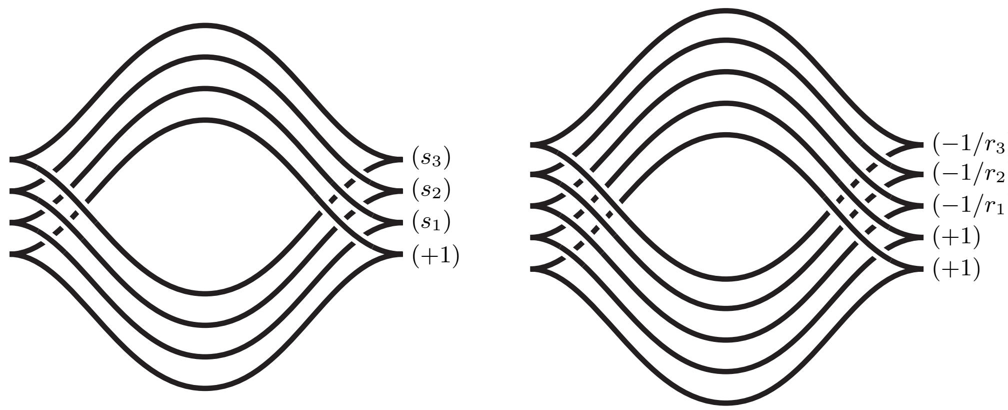

FIGURE 8. The left diagram is equivalent to Figure 7 with contact (s_i)-surgery performed on L_i. On the right is the diagram for $Y(-1; r_1, r_2, r_3)$.

We now focus on $Y = Y(e_0; r_1, r_2, r_3)$ with complementary legs. Specifically, we will assume that $r_1+r_3 = 1$ so that we can write $r_1 = p_1/q_1$ and $r_3 = (q_1-p_1)/q_1$. In the $e_0 = -1$ case, we see a contact surgery diagram for all possible tight contact structures on Y given in the right-hand diagram of Figure 8. Arguing as in the last paragraph, we can combine one of the contact $(+1)$-surgery curves and the curve with contact surgery coefficient $(-1/r_2)$ to obtain the diagram on the left with $s_1 = -q_1/p_1 \leq -2$, $s_2 = -q_2/p_2 \leq -2$, and $s_3 = q_1/p_1$. Thus all contact structures on Y come from surgery on the L_i in Figure 7 with contact surgery coefficients $(-1/r_1), (-1/r_2)$, and $(1/r_1)$.

We will say contact (r)-surgery on L is *consistent* if all of the signs determining the contact structure on the surgery torus have the same sign. We will say a contact surgery presentation for Y in Figure 7 is *balanced* if the contact surgery on L_1 and L_3 are both consistent and their signs are opposite. By Lemma 2.5 above it is clear that if ξ comes from a balanced surgery diagram, then ξ is Stein fillable (since $S^1 \times S^2$ is). We have the following strengthening of this observation.

Theorem 2.6 (Matkovič 2023, [24]). *A contact structure ξ on $Y(-1; r_1, r_2, r_3)$ with complementary legs is symplectically fillable if and only if its contact surgery presentation is balanced.*

Though not necessary for our results we observe the following result.

Lemma 2.7. *Consider $Y = Y(-1; r_1, r_2, r_3)$ with complementary legs. Let ξ be a contact structure on Y with contact surgery diagram given in Figure 7 and the contact surgeries on L_1 and L_3 are consistent and have the same sign. If the signs determining the contact surgery on L_2 in the last continued fraction block have the same sign as L_1 and L_3, then ξ is overtwisted.*

Remark 2.8. We know that some Seifert fibered spaces with $e_0 = -1$ admit tight but not fillable contact structures. We conjecture that all contact surgery presentations that do not meet the criteria in the previous lemma are tight. If the surgery presentation is not balanced this would account for the tight not fillable examples.

Proof. Using the notation described above we take $s_1 = -1/r_1$ to be $-q_1/p_1$ and $s_3 = q_1/p_1$. If we do not perform contact surgery on L_2, we will obtain $S^1 \times S^2$ (since L_1 and L_3 are complementary legs), and doing surgery on L_2 is doing surgery on some torus knot in the $S^1 \times S^2$.

We now consider the contact structure on $S^1 \times S^2$ given by contact surgery on L_1 and L_3. It is given by a path in the Farey graph that starts at $-q_1/p_1$ and moves clockwise to $-q_1/p_1$. (Note the flip in the sign from q_1/p_1 to $-q_1/p_1$. To see why this happens, note that when we remove neighborhoods of L_1 and L_3 to perform the surgery, we obtain $T^2 \times [0,1]$. We put coordinates on T^2 coming from the boundary of the neighborhood of L_1. Thus, performing surgery on L_1 is the same as collapsing curves of slope $-q_1/p_1$ on $T^2 \times \{0\}$. When performing q_1/p_1 surgery on L_3 we use coordinates on the boundary of the neighborhood of L_3, but the meridian for this neighborhood has the opposite orientation to the meridian for L_1, thus when interpreting q_1/p_1 surgery on L_3 as collapsing curves on $T^2 \times \{1\}$ we must reverse the sign.) This path will go from $-q_1/p_1$ clockwise to 0 and then from 0 clockwise to $-q_1/p_1$. The first path corresponds to the contact surgery on L_1 and the second path corresponds to the contact surgery on L_3. All the signs on the first path are the same, and all the signs on the second path are the opposite of those on the first path. To see why this claim is true, note that when thinking of these tori as coming from surgery on a Legendrian link in $S^1 \times S^2$, they both have lower meridians, but when thinking about them as describing $S^1 \times S^2$, one has a lower meridian and the other has an upper meridian. When switching between upper and lower meridians, the co-orientation on the boundary of the torus changes. Thus, since both legs were stabilized the same way, which corresponds to the corresponding basic slices being the same when viewed as describing $S^1 \times S^2$ the bypasses have the opposite sign. Thus, we see a mixed torus in the middle of our $S^1 \times S^2$, and the surgery on L_2 is obtained by doing surgery on a torus knot sitting on this torus.

The knot L_2 on which we must perform surgery to get our original Seifert fibered space back is a Legendrian divide on the convex torus with dividing slope 0. We note that since $s_2 = -q_2/p_2 \leq -2$ we must stabilize L_2 at least once to perform the correct surgery. If we consider a ruling curve L_2' on the convex torus of slope -1 in $S^1 \times S^2$ (note from our surgery description of $S^1 \times S^2$ above this exists), then it will be a stabilization of L_2. The sign of the stabilization is determined by the sign on the edge from -1 to 0, and one can check that it will be opposite to that sign. So, this is the stabilization of L_2 that needs to be done to perform the contact surgery to obtain our original contact structure. Thus, in the complement of L_2' we have a basic slice with slopes -1 and 0 and a basic slice with slopes 0 and ∞. These basic slices have opposite signs and do not form a minimal path. Thus, we see that the complement of L_2' is overtwisted and the contact structure obtained by surgery on L_2' is overtwisted. □

Remark 2.9. We can use the ideas in the proof above to see that contact structures on $Y(e_0; r_1, r_2, r_3)$ with $e_0 \geq 0$ are all Stein fillable. We can follow the proof to the point of doing surgery on L_2, but since the surgery coefficient on L_2 must be in $(-1, 0)$ we see that L_2 is not stabilized when performing the desired contact surgery. In particular, the contact surgery on L_2 is equivalent to Legendrian surgery on L_2 followed by Legendrian surgery on possibly stabilized push-offs of L_2. We note that if we just perform Legendrian surgery on L_2 then we see this is the same as cutting $S^1 \times S^2$ along a convex torus with dividing slope 0 (on which L_2 sits as a Legendrian divide) and regluing by a right-handed Dehn twist along L_2. It is easy to check that this results in a minimal path in the Farey graph and thus corresponds to a tight contact structure on a lens space. To complete the contact surgery on L_2 one must perform the Legendrian surgeries on the push-offs and stabilizations of L_2, but since all tight contact structures on lens spaces are Stein fillable, so will the resulting contact structure on Y.

3. Symplectic rational homology ball fillings when $e_0 \geq -1$

Let K be the n-framed unknot in Figure 2. As discussed in the proof of Lemma 2.4, the manifold described by Figure 2 with K removed is $L(p,q)\#(S^1 \times S^2)$ and K is the connected sum of the core of a Heegaard torus in $L(p,q)$ and a regular fiber in a Seifert fibration on $S^1 \times S^2$.

Lemma 3.1. *Let ξ be the connected sum of any tight contact structure on $L(p,q)$ and the tight contact structure on $S^1 \times S^2$. Then there is a Legendrian representative of K with contact twisting given by the blackboard framing in Figure 2.*

Proof. One can find any torus knot in $S^1 \times S^2$ as a leaf in a linear characteristic foliation of a Heegaard torus. See, for example, [13, Lemma 4.2]. So the contact framing of this leaf K agrees with the framing coming from a Heegaard torus. As noted in the proof of Lemma 2.1 this framing agrees with the 0-framing in Figure 3. Now, any tight contact structure on $L(p,q)$ is obtained from some contact $(-p/q+1)$-surgery on the maximal Thurston-Bennequin unknot U. If U' is a Legendrian push-off of U (that is push U slightly along the Reeb vector field), then U' is a core of a Heegaard torus for $L(p,q)$. Clearly, U' has contact twisting (relative to the blackboard framing) -1.

We know that when connecting summing two Legendrian knots, the contact twisting of the connected sum is the sum of the contact twisting of the components plus 1; see [12]. Thus, the contact framing on the connected sum $K\#U'$ has contact framing agreeing with the blackboard framing in Figure 2. $\square$

Proposition 3.2. *There is a rational homology Stein cobordism from any contact structure on the lens space $L(p,q)$ to a contact structure on the small Seifert fibered space whose smooth surgery diagram is depicted in Figure 2 with $n \leq -1$.*

Proof. The rational homology cobordism constructed in Lemma 2.4 can be built with Stein handle attachments given the contact framing on a Legendrian realization of F found in Lemma 3.1. $\square$

Theorem 3.3. *A small Seifert fibered space Y with complementary legs and $e_0 = -1$ bounds a rational homology ball if and only if it is given by the surgery diagram in Figure 2 where $[a_1^2, \dots, a_{n_2}^2] = \frac{m^2}{mh-1}$ for some relatively prime integers $0 < h < m$ or $m = 1$ and $h = 0$, and $n \leq -2$.*

Moreover, given such a Seifert fibered space, it will have exactly $2|n|$ contact structures up to isotopy that have rational homology ball fillings. These contact structures are given by n plane fields with both orientations. For $2|n+1|$ of these contact structures, each will have a unique symplectic rational homology ball filling, and for the other 2 there might be other fillings.

Proof. Lisca [22] has shown that the universally tight contact structure ξ_{can} on $L(m^2, mh-1)$ is the convex boundary of a symplectic rational homology ball and [7, 14] show that these are the only contact structures on any lens spaces admitting symplectic rational homology ball fillings. (The case when $m = 1$ and $h = 0$ corresponds to using the standard tight contact structure on S^3 in the construction.) Thus, using the Stein cobordism from Proposition 3.2 we see that the claimed Y admits rational homology ball fillings.

To see the other implication, we first note that if the contact structure on Y is not balanced, then it is not symplectically fillable by Theorem 2.6. Thus, we can assume the surgery description of Y is balanced. Recall in Section 2.2 (before Theorem 2.6) we saw that all contact structures on Y are obtained from contact surgery on the link in Figure 7. Moreover, for contact structures that might bound symplectic rational homology balls the surgeries on L_1 and L_3 are balanced, and hence we see that all these contact structures come from negative contact surgery on L_2, which after the surgeries on L_1 and L_3 is a torus knot in $S^1 \times S^2$ (that is L_2 is the fiber F in Figure 3). All these surgeries can be effected by attaching a Stein round 1-handle to L_2 and the core of a Heegaard torus in a lens space. Thus, for any contact structure on Y we can build a Stein cobordism (as in the proof of Proposition 3.2) from the corresponding lens space to Y. Since this cobordism is a rational homology cobordism, it is clear that the θ-invariant of the contact structure on Y is the same as the θ-invariant of the contact structure on the lens space. For Y to smoothly bound a rational homology ball the lens space must be of the form $L(p, q)$ where $p/q \in \mathcal{R}$, [21]. In Lemma 9.4 (and Remark 1.8) of [13] we showed that any tight contact structure ξ on $L(p, q)$, where $p/q \in \mathcal{R}$, satisfies $\theta(\xi) > -2$ unless $p/q = m^2/(mh-1)$ and ξ is isotopic to $\pm\xi_{can}$. Thus, if Y has a rational homology ball filling it must be of the form claimed.

From the discussion above, any contact structure on Y that has a symplectic rational homology ball filling is the upper boundary of the symplectic cobordism built from a portion of the symplectization of $(L(m^2, mh-1), \pm\xi_{can})$ union $S^1 \times D^3$ by attaching a round 1-handle to L_2 in $S^1 \times S^2$ and the core of a Heegaard torus in the lens space. After attaching the 1-handle in the round 1-handle we have upper boundary the lens space connect sum $S^1 \times S^2$ and the 2-handle of the round 1-handle is then attached to the connected sum of a fiber in a Seifert fibration on $S^1 \times S^2$ and the core of a Heegaard torus for the lens space. As argued in the proof of Lemma 3.1 the Legendrian has contact twisting 0. Since we are attaching it with framing $n \leq -2$, we see we must stabilize the Legendrian knot $-n-1$ times. Thus, there are $-n$ possible ways to attach this 2-handle. Thus, all the possible contact structures on Y come from the choice of $\pm\xi_{can}$ and the $-n$ choices for the 2-handle. This gives $2|n|$ contact structures on Y that admit symplectic rational homology ball fillings. Finally, using [8, Theorem 1.4] we see that if the 2-handle is attached along a Legendrian knot that has been stabilized both positively and negatively, then any symplectic filling will come from a symplectic filling of the connected sum of the lens space and $S^1 \times S^2$ with this 2-handle attached. Similarly, [8, Theorem 1.4] will show that if the stabilizations of the Legendrian knot are all of one sign and those defining the contact structure on the lens space are of the opposite sign, then the filling will be unique (*cf* [14, Theorem 3.1]). Thus, each of those $2|n+1|$ tight structures will have only one symplectic rational homology ball filling. $\square$

Theorem 3.4. *Let Y be a small Seifert fibered space with complementary legs and $e_0 \geq 0$. A balanced contact structure ξ on Y bounds a symplectic rational homology ball if and only if Y is given by the surgery diagram in Figure 2, where $[a^2_1, \ldots, a^2_{n_2}] = \frac{m^2}{mh-1}$ for some relatively prime integers $0 < h < m$, and $n = -1$.*

If ξ is not a balanced contact structure, then (Y, ξ) might bound a symplectic rational homology ball only if Y is given by the surgery diagram in Figure 2 where $[a^2_1, \ldots, a^2_{n_2}] \in \mathcal{R}$. Given such a Y there is at most one, respectively two, contact structures (up to orientation of the plane field) that can bound a symplectic rational homology ball if $e_0 = 0$, respectively $e_0 > 0$.

Proof. The proof for balanced contact structures is identical to the proof in the $e_0 = -1$ case given in Theorem 3.3.

If ξ is not balanced, then we recall that from the proof of Proposition 5.1 in [13] that if the contact structure ξ on Y admits a rational homology ball filling, it must have consistent legs. Since we are now considering non-balanced legs, there are 4 possible contact structures since the complementary legs all have one sign and the other leg can have any sign. These 4 contact structures consist of two plane fields considered with both orientations. This, coupled with Lecuona's work cited in Theorem 1.1, finishes the case when $e_0 > 0$.

We now consider the case that $e_0 = 0$. In [15] it was shown that for $e_0 = 0$, some contact surgeries realizing $Y(0; r_1, r_2, r_3)$ yield the same contact manifold. Specifically, if $-1/r_i < -2$ for any i and all the legs have the same sign, then one can change a consistent leg into an inconsistent leg (see the proof of Theorem 2.7 in [15]). Thus, these contact structures do not admit a symplectic rational homology ball filling, as noted above. If $-1/r_i = -2$ for all i, then we can rule out the existence of a symplectic rational ball filling multiple different ways. For example, we can use Proposition 2.3 to show that $Y(0; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ does not even bound a smooth rational homology ball. Alternatively, we can show that the relevant contact structures have $\theta = -\frac{4}{3}$. Thus we are left with 2 contact structures verifying the theorem in the case that $e_0 = 0$. $\square$

We finally note that Theorem 1.6 and Theorem 1.7 are now a rephrasing of Theorem 3.3 and Theorem 3.4, respectively. Namely by computing the surgery coefficient associated to $[n, a^2_1, \ldots, a^2_{n_2}]$ where $[a^2_1, \ldots, a^2_{n_2}] = \frac{m^2}{mh-1}$ yield Theorem 1.6 when $-n \geq 2$ and Theorem 1.7 when $-n = 1$.

4. Symplectic rational homology ball fillings when $e_0 \leq -2$

This section is devoted to the proof of Theorem 1.2 which says that a small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$ with complementary legs and $e_0 \leq -2$ does not admit a rational homology ball symplectic filling.

Recall we are assuming that the complementary legs are defined by $r_1 + r_3 = 1$. Set $1/r_k = [a^k_1, \ldots, a^k_{n_k}]$ and $a^2_0 = -e_0$. We notice that the a^3_j terms are determined by the a^1_i terms by the Riemenschneider point rule [27], and thus $Y(e_0; r_1, r_2, r_3)$ is completely determined by the strings $\mathbf{a^1} = (a^1_1, \ldots, a^1_{n_1})$ and $\mathbf{a^2} = (a^2_0, a^2_1, \ldots, a^2_{n_2})$. Hence we denote $Y(e_0; r_1, r_2, r_3)$ by $M_{\mathbf{a^1}, \mathbf{a^2}}$. If we set $p/q = [a^2_0, a^2_1, \ldots, a^2_{n_2}]$ then Lecuona's result discussed in Theorem 1.1 says that $M_{\mathbf{a^1}, \mathbf{a^2}}$ bounds a rational homology ball if and only if $(p-q)/q' \in \mathcal{R}$ where $0 < q' < p - q$ is the reduction of q modulo $p - q$. The equivalence of this criterion with the one given in the caption of Figure 4 can be seen by Lemma 4.1.

Lemma 4.1. *Let $p/q = [a^2_0, a^2_1, \ldots, a^2_{n_2}]$. Suppose that there is some $1 \leq t \leq n_2$ such that $a^2_i = 2$ for $0 \leq i \leq t-1$ and $a^2_t > 2$. If we set $p_t/q_t = [a^2_t, a^2_{t+1}, \ldots, a^2_{n_2}]$, then we have*

$$\frac{p-q}{q'} = \frac{p_t - q_t}{q_t} = [a^2_t - 1, a^2_{t+1}, \ldots, a^2_{n_2}],$$

where $0 < q' < p - q$ is the reduction of q modulo $p - q$.

Proof. Under the given assumptions, we have

$$\frac{p}{q} = [a^2_0, a^2_1, \ldots, a^2_{n_2}] = \frac{(t+1)p_t - tq_t}{tp_t - (t-1)q_t}$$

and hence

$$\frac{p-q}{q} = \frac{p_t - q_t}{tp_t - (t-1)q_t}$$

which implies that

$$\frac{p-q}{q'} = \frac{p_t - q_t}{q_t},$$

since $q' = q_t$. This last term is easily seen to be $[a^2_t - 1, a^2_{t+1}, \ldots, a^2_{n_2}]$. □

We now consider the case when $e_0 \leq -2$. In this case, the plumbing diagram defining $M_{\mathbf{a^1},\mathbf{a^2}}$ is negative definite. Thus, it is the *oriented* link of a normal complex surface singularity, and therefore it has a canonical contact structure ξ_{can} (also known as the Milnor fillable contact structure), which is unique up to contactomorphism [5].

We now recall the Proposition 1.7 from [13].

Proposition 4.2. *If ξ is any tight contact structure on the small Seifert fibered space $M_{\mathbf{a^1},\mathbf{a^2}}$, which is not isotopic to $\pm\xi_{can}$, then we have $\theta(\xi_{can}) < \theta(\xi)$.*

Our main theorem in the $e_0 \leq -2$ case will follow from the following computation.

Proposition 4.3. *The canonical contact structure ξ_{can} on $M_{\mathbf{a^1},\mathbf{a^2}}$, satisfies $-2 < \theta(\xi_{can})$, provided that $(p-q)/q' \in \mathcal{R}$, where $0 < q' < p - q$ is the reduction of q modulo $p - q$.*

Based on Lecuona's characterization, the proof of Theorem 1.2 is obtained by combining Proposition 4.2 and Proposition 4.3 as follows.

Proof of Theorem 1.2. Let $e_0 \leq -2$. Suppose that the small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$ has complementary legs and assume without loss of generality that $r_1 + r_3 = 1$, so that $Y(e_0; r_1, r_2, r_3) = M_{\mathbf{a^1},\mathbf{a^2}}$ with two complementary legs, and $\mathbf{a_1}$ and $\mathbf{a_2}$ as defined above. Lecuona [19] showed that $M_{\mathbf{a^1},\mathbf{a^2}}$ smoothly bounds a rational homology ball if and only if $(p-q)/q' \in \mathcal{R}$. On the other hand, Proposition 4.3 together with Proposition 4.2 imply that $-2 < \theta(\xi)$ for any tight contact structure ξ on $M_{\mathbf{a^1},\mathbf{a^2}}$, provided that $(p-q)/q' \in \mathcal{R}$. Since the θ-invariant of any tight contact structure that admits a rational homology ball symplectic filling is equal to -2, we immediately conclude that no rational homology ball with boundary $M_{\mathbf{a^1},\mathbf{a^2}}$ can symplectically fill any contact structure on $M_{\mathbf{a^1},\mathbf{a^2}}$. □

In order to compute the θ-invariant we recall a useful definition from [21]. Suppose that $r/s = [a_0, a_1, \ldots, a_k]$, where $a_i \geq 2$ for all $0 \leq i \leq k$. We set

$$I(r/s) = \sum_{i=0}^{k}(a_i - 3).$$

The proof of Proposition 4.3 is based on the following result, which is the main technical part of the paper.

Proposition 4.4. *Using the notation above we set $\widetilde{p}/\widetilde{q} = [a_1^1, a_2^1, \ldots, a_{n_1}^1]$. We have the following formula*

$$\theta(\xi_{can}) = 1 - I(p/q) - \frac{1}{[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2 - 1]} + \frac{2(\widetilde{p}-2)}{\widetilde{p}(p-q)} - \frac{(\widetilde{p}-2)^2 q}{(\widetilde{p})^2(p-q)} \tag{1}$$

for the canonical contact structure ξ_{can} on $M_{\mathbf{a^1},\mathbf{a^2}}$.

Remark 4.5. Note that Proposition 4.4 is a generalization of [13, Proposition 9.8], since the spherical 3-manifold $D(p,q)$ is a small Seifert fibered space with two complementary legs with $\mathbf{a_1} = \mathbf{a_3} = (2)$, which implies that $\widetilde{p} = 2$, and therefore the sum of the last two terms in the Formula (1) vanishes for the case of $D(p,q)$.

We will prove this proposition in the next section and now give the proof of Proposition 4.3.

Proof of Proposition 4.3. We consider two cases.

Case (i): Suppose that $p - q > q$, which is equivalent to assuming that $-e_0 = a_0^2 > 2$. It follows that $q' = q$ and the hypothesis $(p-q)/q \in \mathcal{R}$ implies that $I((p-q)/q) \leq 1$ by Lisca's work [21] (see also, [1, Appendix A.2], where the I-values of all rational numbers in $\mathcal{R}$ are listed explicitly with maximum value of 1). One may easily compute that $I((p-q)/q) = I(p/q) - 1$.

If we assume that $I((p-q)/q) \leq 0$ (and thus $I(p/q) \leq 1$), then the inequality $-2 < \theta(\xi_{can})$ follows immediately from Formula (1). This is because the term $\dfrac{2(\widetilde{p}-2)}{\widetilde{p}(p-q)}$ in Formula (1) is always nonnegative, the term $-\dfrac{(\widetilde{p}-2)^2 q}{(\widetilde{p})^2(p-q)}$ is greater than -1 since we assumed that $p - q > q$, and the term $-\dfrac{1}{[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2 - 1]}$ is strictly greater than -1 since $a_0^2 > 2$.

Now suppose that $I((p-q)/q) = 1$. Since we assumed that $(p-q)/q \in \mathcal{R}$, it follows that $(p-q)/q$ belongs to class (V) in [1, Appendix A.2] and the continued fraction expansion of $(p-q)/q$ is obtained from [4] by final-2 expansions. An equivalent characterization of such a rational number $(p-q)/q$ is that $p - q = m^2$ and $q = mh - 1$ for some coprime integers $0 < h < m$ (see, for example, Proposition 4.1 in [29]).

Recall that $p/q = [a_0^2, a_1^2, \ldots, a_{n_2}^2]$ and we assumed that $a_0^2 > 2$. It immediately follows that $[a_0^2 - 1, a_1^2, \ldots, a_{n_2}^2] = (p-q)/q$ and by a well-known fact about continued fraction expansions we have $[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2 - 1] = (p-q)/q^*$, where $0 < q^* < p - q$ is the multiplicative inverse of q mod $(p-q)$, see [25, Lemma A4]. Therefore, we have

$$\frac{1}{[a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_1, a^2_0 - 1]} = \frac{q^*}{p-q}.$$

Since $q^* = m(m-h) - 1$, we have $2 + q + q^* = p - q$. It follows that

$$-\frac{1}{[a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_1, a^2_0 - 1]} - \frac{(\widetilde{p}-2)^2 q}{(\widetilde{p})^2(p-q)} > -\frac{q^*}{p-q} - \frac{q}{p-q} = -1.$$

Since we assumed that $I((p-q)/q) = 1$ (and therefore $I(p/q) = 2$), we have $1 - I(p/q) = -1$. Taking into account the fact that $\dfrac{2(\widetilde{p}-2)}{\widetilde{p}(p-q)}$ is always nonnegative, we again conclude by Formula (1) that $-2 < \theta(\xi_{can})$.

Case (ii): Suppose that $p - q < q$, which is equivalent to assuming that $-e_0 = a^2_0 = 2$. If $a^2_i = 2$, for all $0 \le i \le n_2$, then $p = n_2 + 2$, $q = n_2 + 1$ and $q^* = 1$, which implies that

$$\theta(\xi_{can}) = 1 + (n_2 + 1) - 1 + \frac{2(\widetilde{p}-2)}{\widetilde{p}} - \frac{(\widetilde{p}-2)^2}{(\widetilde{p})^2}(n_2 + 1) > 0.$$

Otherwise, there exists $0 \le t < n_2$ such that $a^2_i = 2$, for all $0 \le i \le t$, but $a^2_{t+1} > 2$. In other words, we have

$$p/q = [\underbrace{2, \ldots 2}_{t+1}, a^2_{t+1}, \ldots, a^2_{n_2}] \text{ and } b_{t+1} > 2.$$

By setting $p_t/q_t = [a^2_{t+1}, \ldots, a^2_{n_2}]$, one may compute that $p_t - q_t = p - q$ and $q_t = q'$. Hence, the condition $(p-q)/q' \in \mathcal{R}$ is equivalent to the condition $(p_s - q_s)/q_s \in \mathcal{R}$, and combined with the observation that $p_t - q_t > q_t$, **Case (i)** of our proof implies that

$$1 - I(p_t/q_t) - \frac{1}{[a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_{t+2}, a^2_{t+1} - 1]} + \frac{2(\widetilde{p}-2)}{\widetilde{p}(p_t - q_t)} - \frac{(\widetilde{p}-2)^2 q_t}{(\widetilde{p})^2(p_t - q_t)} > -2, \tag{2}$$

but note that the $\widetilde{p}$ and $\widetilde{q}$ are different in this equations (as their role below is unimportant, we do not establish new notation for them here).

Next, we observe that

(i) $I(p_t/q_t) = I(p/q) + (t+1)$,
(ii) $[a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_{t+2}, a^2_{t+1} - 1] = [a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_1, a^2_0 - 1]$, and
(iii) $q_t/(p_t - q_t) = q/(p-q) - (t+1)$,

where the last item is proven via induction on t. By substituting these in (2), we deduce

$$\theta(\xi_{can}) = 1 - I(p/q) - \frac{1}{[a^2_{n_2}, a^2_{n_2-1}, \ldots, a^2_1, a^2_0 - 1]} + \frac{2(\widetilde{p}-2)}{\widetilde{p}(p-q)} - \frac{(\widetilde{p}-2)^2 q}{(\widetilde{p})^2(p-q)} > -2$$

using the fact that $0 \le (\widetilde{p}-2)/\widetilde{p} < 1$. □

5. Computing the θ-invariant

The goal of this subsection is to describe a proof of Proposition 4.4, which is the main technical part of the paper. We begin with the following definition.

Definition 5.1. For any integer $n > 0$ and any sequence of integers $m_1, m_2, \ldots, m_n$ with $m_i \geq 2$, we set

$$M = M(m_1, m_2, \ldots, m_n) = \begin{bmatrix} -m_1 & 1 & & \\ 1 & -m_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -m_n \end{bmatrix}.$$

For any $1 \leq i \leq n$, we define u_i^M as the absolute value of the determinant of the top-left $i \times i$ submatrix of M, and we set $u_0^M = 1$. Let $\mathbf{u}^M = [u_0^M \; u_1^M \; \cdots \; u_{n-1}^M]^T \in \mathbb{R}^n$. Similarly, for any $1 \leq j \leq n$, we define v_j^M as the absolute value of the determinant of the bottom-right $j \times j$ submatrix of M, and we set $v_0^M = 1$. Let $\mathbf{v}^M = [v_{n-1}^M \; v_{n-2}^M \; \cdots \; v_0^M]^T \in \mathbb{R}^n$.

Lemma 5.2. *Using the notation as in Definition 5.1, let $s/t = [m_1, m_2, \ldots, m_n]$ for some relatively prime integers $0 < t < s$. Then $\det M = (-1)^n s$, the first column of M^{-1} is given by the vector $-\frac{1}{s}\,\mathbf{v}^M$ and its dot product with the vector $[m_1 - 2 \; m_2 - 2 \; \cdots \; m_n - 2]^T \in \mathbb{R}^n$ is equal to $-1 + \frac{1+t}{s}$. Moreover, the last column of M^{-1} is given by the vector $-\frac{1}{s}\,\mathbf{u}^M$. Furthermore, the first entry of $\mathbf{v}^M$ is t.*

Proof. We first show that $\det M = (-1)^n s$. To prove this claim, we set

$$d_i = \det M(m_1, m_2, \ldots, m_i) \text{ and } \frac{s_i}{t_i} = [m_1, m_2, \ldots, m_i] \text{ for } 1 \leq i \leq n.$$

We also set $d_0 = 1 = s_0$ and $t_0 = 0$. Note that, by definition, $d_1 = -m_1$, and $d_n = \det M$. Computing d_i using the cofactor expansion about the last row shows that the d_i's satisfy the following recursive relation

$$d_i = -m_i d_{i-1} - d_{i-2} \text{ for } 2 \leq i \leq n.$$

On the other hand, it is well-known and easily proven by induction, that

$$s_i = m_i s_{i-1} - s_{i-2} \text{ and } t_i = m_i t_{i-1} - t_{i-2}$$

hold for $2 \leq i \leq n$. Note that, by definition, $s_1 = m_1, t_1 = 1, s_n = s$, $t_n = t$. Comparing the recursive relations for d_i's and s_i's we see that $d_i = (-1)^i s_i$ for $0 \leq i \leq n$. In particular, for $i = n$ we have

$$\det M = d_n = (-1)^n s_n = (-1)^n s,$$

as claimed in the lemma.

Next, we observe that for any $1 \leq j \leq n$, the determinant of the submatrix $M_{1,j}$ of M obtained by deleting its first row and jth column is equal to the determinant of the bottom-right $(n-j) \times (n-j)$ submatrix of M, simply expanding along the first columns as we compute the determinants. Since v_{n-j}^M is defined as the *absolute value* of the determinant of the bottom-right $(n-j) \times (n-j)$ submatrix of M, we see that $\det M_{1,j} = (-1)^{n-j} v_{n-j}^M$. It follows that the $(1, j)$-cofactor of M is

$$(-1)^{1+j} \det M_{1,j} = (-1)^{1+j}(-1)^{n-j} v_{n-j}^M = (-1)^{n+1} v_{n-j}^M.$$

This implies that the $(j,1)$-entry of M^{-1} is equal to

$$\frac{(-1)^{1+j}\det M_{1,j}}{\det M} = \frac{(-1)^{n+1}v^M_{n-j}}{(-1)^n s} = -\frac{v^M_{n-j}}{s}$$

for any $1 \le j \le n$. Therefore, the first column of M^{-1} is given by the vector $-\dfrac{1}{s}\,\mathbf{v}^M$.

Since the first row of M^{-1} is equal to the transpose of its first column, the dot product of $-\dfrac{1}{s}\,\mathbf{v}^M$ with the vector $[m_1-2\ m_2-2\ \cdots\ m_n-2]^T$ is equal to the first component w_1 of the solution $\mathbf{w}$ of the linear system

$$M\mathbf{w} = [m_1-2\ m_2-2\ \cdots\ m_n-2]^T.$$

We set

$$M^1 = M^1(m_1, m_2, \ldots, m_n) = \begin{bmatrix} m_1-2 & 1 & & \\ m_2-2 & -m_2 & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ m_n-2 & & 1 & -m_n \end{bmatrix}$$

which is obtained by replacing the first column of M by $[m_1-2\ m_2-2\ \cdots\ m_n-2]^T$. By Cramer's rule,

$$w_1 = \frac{\det M^1}{\det M}.$$

We claim that $\det M^1 = (-1)^{n+1}(s-t-1)$ so that

$$w_1 = \frac{\det M^1}{\det M} = \frac{(-1)^{n+1}(s-t-1)}{(-1)^n s} = -1 + \frac{1+t}{s},$$

which finishes the proof. To prove this last claim, for $1 \le i \le n$, we set $k_i = s_i - t_i - 1$ and observe that

$$k_i = m_i(k_{i-1}+1) - k_{i-2} - 2$$

for any $2 \le i \le n$, by the above recursive relations for s_i's and t_i's. Now we set

$$d'_i = \det M^1(m_1, m_2, \ldots, m_i),$$

for $1 \le i \le n$ and $d'_0 = 0$. Note that $d'_1 = m_1 - 2$ and $d'_n = \det M^1$, by definition. Expanding along the last columns while computing the determinants, we derive the recursive relation

$$d'_i = -m_i(d'_{i-1} + (-1)^i) - d'_{i-2} + (-1)^i 2$$

for any $2 \le i \le n$. Finally, by comparing the recursive relations for k_i's and d'_i's, we see that $d'_i = (-1)^{i+1}k_i$ and in particular, for $i = n$, we have

$$\det M^1 = d'_n = (-1)^{n+1}k_n = (-1)^{n+1}(s_n - t_n - 1) = (-1)^{n+1}(s-t-1).$$

To prove the next statement in the lemma we set $M' = M(m_n, m_{n-1}, \ldots, m_1)$ and by what we showed above we know that first column of $(M')^{-1}$ is $-\dfrac{1}{s}\,\mathbf{v}^{M'}$. Here we use the fact that $[m_n, m_{n-1}, \ldots, m_1] = s/t^*$, where t^* is the inverse of t mod s, see [25, Lemma A4]. But the last column of M^{-1} is the first column of $(M')^{-1}$ written in reversed order and the result follows since $\mathbf{u}^M$ is equal to $\mathbf{v}^{M'}$ written in reversed order.

Finally, we show that the first entry of $\mathbf{v}^M$ is t, which is the last statement in the lemma. Note that the first entry of $\mathbf{v}^M$ is the absolute value of the determinant of

$$M(m_2, m_3, \ldots, m_n),$$

by definition. Recall that $s/t = [m_1, m_2, \ldots, m_n]$, which implies that

$$\frac{s}{t} = m_1 - \frac{1}{[m_2, m_3, \ldots, m_n]} = m_1 - \frac{\widetilde{t}}{\widetilde{s}} = \frac{m_1\widetilde{s} - \widetilde{t}}{\widetilde{s}}$$

where we set $\widetilde{s}/\widetilde{t} = [m_2, m_3, \ldots, m_n]$. As a consequence, by the first statement of the lemma, we observe that the absolute value of the determinant of $M(m_2, m_3, \ldots, m_n)$ is $\widetilde{s}$, which in turn, is equal to t by the line above. □

Remark 5.3. In Definition 5.1, we assumed that $m_i \geq 2$ for all $1 \leq i \leq n$. Suppose that we relax this condition so that $m_1 = 1$ and $m_i \geq 2$ for all $2 \leq i \leq n$, and let $M = M(1, m_2, \ldots, m_n)$, and the vectors $\mathbf{u}^M$ and $\mathbf{v}^M$ be defined as in Definition 5.1. Then Lemma 5.2 still (partially) holds as follows. We first observe that $[1, m_2, \ldots, m_n] = \dfrac{s}{t}$ for some uniquely determined co-prime integers $0 < s < t$ (as opposed to $0 < t < s$). Then $\det M = (-1)^n s$, the first and last columns of M^{-1} are given by $-(1/s)\,\mathbf{v}^M$ and $-(1/s)\,\mathbf{u}^M$, respectively.

Let $X_{\mathbf{a^1},\mathbf{a^2}}$ denote the 4-manifold obtained by Kirby diagram in Figure 9 so that $\partial X_{\mathbf{a^1},\mathbf{a^2}} =$

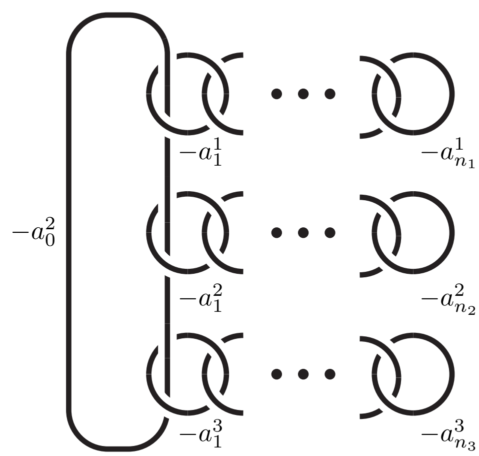

FIGURE 9. Kirby diagram for the manifold $X_{\mathbf{a^1},\mathbf{a^2}}$.

$M_{\mathbf{a^1},\mathbf{a^2}}$. Then the intersection matrix $Q_{\mathbf{a^1},\mathbf{a^2}}$ for $X_{\mathbf{a^1},\mathbf{a^2}}$ can be described as in Figure 10, where

$$A = M(a^1_{n_1}, a^1_{n_1-1}, \ldots, a^1_1), \quad B = M(a^2_0, a^2_1, \ldots, a^2_{n_2}), \text{and} \quad C = M(a^3_1, a^3_2, \ldots, a^3_{n_3}),$$

and in the off-diagonal terms, all entries not specified are 0.

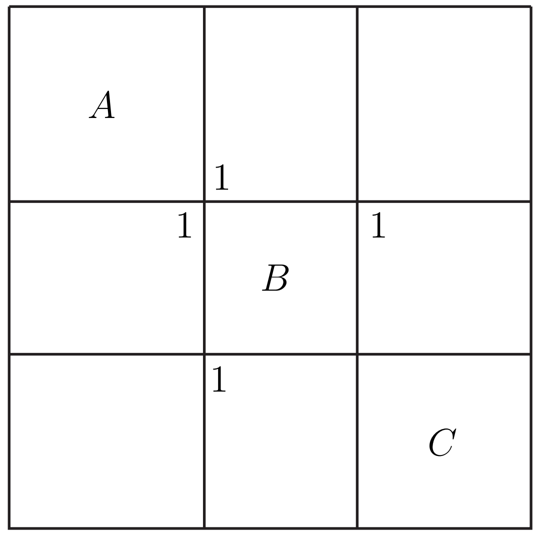

FIGURE 10. Intersection matrix $Q_{\mathbf{a^1},\mathbf{a^2}}$ for $X_{\mathbf{a^1},\mathbf{a^2}}$.

Lemma 5.4. *The inverse $Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}$ can be described as in Figure 11, where the block matrices $\widetilde{B}$, D, E, F, G, H in the figure are defined as follows:*

$$G=-\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{u}^A(\mathbf{u}^A)^T,\quad D=-\frac{1}{\widetilde{p}(p-q)}\mathbf{u}^A(\mathbf{v}^B)^T,\quad E=-\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{u}^A(\mathbf{v}^C)^T$$

$$\widetilde{B}=M(a_0^2-1,a_1^2,\ldots,a_{n_2}^2),\quad F=-\frac{1}{\widetilde{p}(p-q)}\mathbf{v}^B(\mathbf{v}^C)^T,\quad H=-\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{v}^C(\mathbf{v}^C)^T.$$

$$\begin{array}{|c|c|c|}\hline A^{-1}+G & D & E \\ \hline D^T & \widetilde{B}^{-1} & F \\ \hline E^T & F^T & C^{-1}+H \\ \hline\end{array}$$

FIGURE 11. The matrix $Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}$.

Remark 5.5. Note that in the definition of the matrix E, for example, $\mathbf{u}^A(\mathbf{v}^C)^T$ is the $n_1 \times n_3$ matrix obtained by multiplying the $n_1 \times 1$ column matrix $\mathbf{u}^A$ with the $1 \times n_3$ row matrix $(\mathbf{v}^C)^T$. Since the first component of $\mathbf{u}^A$ is equal to 1, by definition, this means that the first row of the matrix E is given by $-\dfrac{q}{(\widetilde{p})^2(p-q)}(\mathbf{v}^C)^T$ and the jth row of E is obtained by multiplying its first row by the jth component of $\mathbf{u}^A$. A similar discussion applies to the matrices D, F, G, H and D^T, E^T, F^T as well.

Our proof of Lemma 5.4 is rather long and technical. So, for the reader's convenience we postpone its proof for now and provide it in Appendix A.

By definition (see [17]),

$$\theta(\xi_{can}) = c_1^2(X_{\mathbf{a^1},\mathbf{a^2}}) - 3\sigma(X_{\mathbf{a^1},\mathbf{a^2}}) - 2\chi(X_{\mathbf{a^1},\mathbf{a^2}}),$$

where $\sigma(X_{\mathbf{a^1},\mathbf{a^2}}) = -(n_1+n_2+n_3+1)$ since the Kirby diagram in Figure 9 is negative-definite and $\chi(X_{\mathbf{a^1},\mathbf{a^2}}) = n_1+n_2+n_3+2$ because the 4-manifold $X_{\mathbf{a^1},\mathbf{a^2}}$ consists of a single zero handle and $n_1+n_2+n_3+1$ two handles. Therefore, to finish the proof of Proposition 4.4, we need to calculate the term

$$c_1^2(X_{\mathbf{a^1},\mathbf{a^2}}) = \mathbf{r}_{can}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{r}_{can},$$

where $\mathbf{r}_{can}$ denotes the rotation vector corresponding to the canonical contact structure ξ_{can}. Note that $\mathbf{r}_{can}$ is given by $\mathbf{x}+\mathbf{y}$, where

$$\mathbf{x} = [\underbrace{0 \ \cdots \ 0}_{n_1} \ \ a_0^2-2 \ \ a_1^2-2 \ \ \cdots \ \ a_{n_2}-2 \ \ \underbrace{0 \ \cdots \ 0}_{n_3}]^T$$

$$\mathbf{y} = [a_{n_1}^1-2 \ \ a_{n_1-1}^1-2 \ \ \cdots \ \ a_1^1-2 \ \ \underbrace{0 \ \cdots \ 0}_{n_2+1} \ \ a_1^3-2 \ \ a_2^3-2 \ \ \cdots \ \ a_{n_3}^3-2]^T.$$

Lemma 5.6. *We have*

$$\mathbf{x}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{x} = 2n_2+3-(a_0^2+a_1^2+\cdots+a_{n_2}^2) - \frac{1}{[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2-1]}. \tag{3}$$

Proof. The calculation is the same as for the case of the spherical 3-manifold $D(p,q)$, for which the formula above was derived in [13, Lemma 9.7]. □

Next we set

$$\alpha = -1 + \frac{1+\widetilde{q}}{\widetilde{p}}, \ \text{ and } \ \beta = -1 + \frac{1+\widetilde{p}-\widetilde{q}}{\widetilde{p}} = \frac{1-\widetilde{q}}{\widetilde{p}},$$

and observe that $\alpha+\beta = \dfrac{2-\widetilde{p}}{\widetilde{p}}$.

Lemma 5.7. *We have*

$$\mathbf{x}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{y} = \mathbf{y}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{x} = (\alpha+\beta)\left(1-\frac{1}{p-q}\right) = \frac{2-\widetilde{p}}{\widetilde{p}}\left(1-\frac{1}{p-q}\right). \tag{4}$$

Proof. It is clear by symmetry, that $\mathbf{x}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{y} = \mathbf{y}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1}\mathbf{x}$. Now let

$$\mathbf{y}_{\mathbf{a^1}} = [a_{n_1}^1-2 \ \ a_{n_1-1}^1-2 \ \ \cdots \ \ a_1^1-2 \ \ \underbrace{0 \ \cdots \ 0}_{n_2+1+n_3}]^T$$

and

$$\mathbf{y}_{\mathbf{a}^3} = [\underbrace{0 \ \cdots \ 0}_{n_1+n_2+1} \ \ a^3_1 - 2 \ \ a^3_2 - 2 \ \ \cdots \ \ a^3_{n_3} - 2]^T$$

so that $\mathbf{y} = \mathbf{y}_{\mathbf{a}^1} + \mathbf{y}_{\mathbf{a}^3}$. We claim that

$$\mathbf{y}^T_{\mathbf{a}^1} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{x} = \alpha\left(1 - \frac{1}{p-q}\right) \ \text{ and } \ \mathbf{y}^T_{\mathbf{a}^3} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{x} = \beta\left(1 - \frac{1}{p-q}\right),$$

which finishes the proof of Lemma 5.7. To prove the former of these claims, we observe that only the matrix $D = -\frac{1}{\widetilde{p}(p-q)}\mathbf{u}^A(\mathbf{v}^B)^T$, in Figure 11 is involved in the calculation. We also note that $(p-q)/q = [a^2_0 - 1, a^2_1, \ldots, a^2_{n_2}]$ since $p/q = [a^2_0, a^2_1, \ldots, a^2_{n_2}]$, and $\mathbf{v}^B = \mathbf{v}^{\widetilde{B}}$. Hence it follows from Lemma 5.2 that, the first row of $(\widetilde{B})^{-1}$ is equal to

$$-\frac{1}{p-q}(\mathbf{v}^{\widetilde{B}})^T = -\frac{1}{p-q}(\mathbf{v}^B)^T.$$

Moreover, the dot product of $-\frac{1}{p-q}(\mathbf{v}^{\widetilde{B}})^T$ with $[a^2_0 - 3 \ \ a^2_1 - 2 \ \ \cdots \ \ a^2_{n_2} - 2]^T$ is equal to $-1 + \frac{1+q}{p-q}$ by Lemma 5.2. Hence the dot product of $-\frac{1}{p-q}(\mathbf{v}^B)^T$ with $[a^2_0 - 2 \ \ a^2_1 - 2 \ \ \cdots \ \ a^2_{n_2} - 2]^T$ is equal to $-1 + \frac{1+q}{p-q} - \frac{q}{p-q} = -1 + \frac{1}{p-q}$. This is because the only difference between these dot products is the first entry of $-\frac{1}{p-q}(\mathbf{v}^B)^T$, which is equal to $-\frac{q}{p-q}$ since the first entry of $(\mathbf{v}^B)^T$ is q, which follows by the last statement in Lemma 5.2. As pointed out in Remark 5.5, the rows of D are the multiples of $-\frac{1}{p-q}(\mathbf{v}^B)^T$ by the components of the vector $(-1/\widetilde{p})\mathbf{u}^A$. This implies that

$$\mathbf{y}^T_{\mathbf{a}^1} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{x} = \alpha\left(1 - \frac{1}{p-q}\right),$$

by the following argument. Recall that $A = M(a^1_{n_1}, a^1_{n_1-1}, \ldots, a^1_1)$, by definition and let $A' = M(a^1_1, a^1_2, \ldots, a^1_{n_1})$. Then the vector $\mathbf{v}^{A'}$ is obtained from $\mathbf{u}^A$ by reversing the order of its components. Thus, the dot product of $[a^1_{n_1} - 2 \ \ a^1_{n_1-1} - 2 \ \ \cdots \ \ a^1_1 - 2]^T$ (which is obtained by truncating $\mathbf{y}_{\mathbf{a}^1}$ by removing the zeros at the end) with the vector $(1/\widetilde{p})\ \mathbf{u}^A$ is equal to the dot product of $[a^1_1 - 2 \ \ a^1_2 - 2 \ \ \cdots \ \ a^1_{n_1} - 2]^T$ with the vector $(1/\widetilde{p})\ \mathbf{v}^{A'}$, which in turn, is equal to $-\left(-1 + \frac{1+\widetilde{q}}{\widetilde{p}}\right) = -\alpha$ by Lemma 5.2, since $\widetilde{p}/\widetilde{q} = [a^1_1, a^1_2, \ldots, a^1_{n_1}]$.

A similar argument, involving the matrix F^T in Figure 11, proves that

$$\mathbf{y}^T_{\mathbf{a}^3} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{x} = \beta\left(1 - \frac{1}{p-q}\right),$$

again by Lemma 5.2. □

Lemma 5.8. *We have*

(i) $\mathbf{y}^T_{\mathbf{a}^1} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{y}_{\mathbf{a}^3} = \mathbf{y}^T_{\mathbf{a}^3} Q^{-1}_{\mathbf{a}^1,\mathbf{a}^2}\mathbf{y}_{\mathbf{a}^1} = -\alpha\beta\frac{q}{p-q}$,

(ii) $\mathbf{y}_{\mathbf{a^1}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^1}} = -2\alpha - \alpha^2 \dfrac{q}{p-q} - (n_3 - 1) + \dfrac{\widetilde{q} - (\widetilde{q})^*}{\widetilde{p}}$, *and*

(iii) $\mathbf{y}_{\mathbf{a^3}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^3}} = -2\beta - \beta^2 \dfrac{q}{p-q} - (n_1 - 1) + \dfrac{(\widetilde{q})^* - \widetilde{q}}{\widetilde{p}}$,

where $(\widetilde{q})^*$ *is the multiplicative inverse of* $\widetilde{q}$ *mod* $\widetilde{p}$.

Proof. To prove the equalities stated in Item (i), we first observe by symmetry that

$$\mathbf{y}_{\mathbf{a^1}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^3}} = \mathbf{y}_{\mathbf{a^3}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^1}},$$

and hence it suffices to prove that $\mathbf{y}_{\mathbf{a^1}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^3}} = -\alpha\beta \dfrac{q}{p-q}$. It is clear that only the matrix E in Figure 11 is involved in the calculation. Note that the first row of E is given by $-\dfrac{q}{(\widetilde{p})^2(p-q)}(\mathbf{v}^C)^T$ and moreover, by Lemma 5.2, the dot product of $(\mathbf{v}^C)^T$ with $[a_1^3 - 2 \;\; a_2^3 - 2 \;\; \cdots \;\; a_{n_3}^3 - 2]^T$ (which is obtained by truncating $\mathbf{y}_{\mathbf{a^3}}$ by removing the zeros at the beginning) gives $-1 + \widetilde{q}$. Therefore, the product of the first row of E with $[a_1^3 - 2 \;\; a_2^3 - 2 \;\; \cdots \;\; a_{n_3}^3 - 2]^T$ is equal to $\dfrac{1-\widetilde{q}}{(\widetilde{p})^2} \dfrac{q}{p-q}$. Since all the rows of E are given by multiples of the first row by the components of the vector $\mathbf{u}^A$, and the dot product of $[a_{n_1}^1 - 2 \;\; a_{n_1-1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]^T$ (which is obtained by truncating $\mathbf{y}_{\mathbf{a^1}}$ by removing the zeros at the end) with the vector $\mathbf{u}^A$ is equal to $-(1 + \widetilde{q} - \widetilde{p})$ as observed in the proof of Lemma 5.7, it follows that

$$\mathbf{y}_{\mathbf{a^1}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^3}} = -(1 + \widetilde{q} - \widetilde{p}) \frac{1-\widetilde{q}}{(\widetilde{p})^2} \frac{q}{p-q} = -\alpha\beta \frac{q}{p-q},$$

which finishes the proof of Item (i).

To prove Item (ii), it is clear that only the top-left block $A^{-1} + G$ in Figure 11 is involved in the calculation. We have

$$\begin{aligned} \mathbf{y}_{\mathbf{a^1}}^T Q_{\mathbf{a^1},\mathbf{a^2}}^{-1} \mathbf{y}_{\mathbf{a^1}} &= [a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2](A^{-1} + G)[a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]^T \\ &= [a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]A^{-1}[a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]^T \\ &\quad + [a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]G[a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]^T. \end{aligned} \tag{5}$$

We claim that the first term on the right in Equation (5) is equal to

$$-2\alpha - (n_3 - 1) + \frac{\widetilde{q} - (\widetilde{q})^*}{\widetilde{p}},$$

and the second term is equal to

$$-\alpha^2 \frac{q}{p-q},$$

which together proves the formula in Item (ii). To prove the first claim, we use the well-known fact that $\widetilde{p}/\widetilde{q} = [a_1^1, a_2^1, \ldots, a_{n_1}^1]$ if and only if $[a_{n_1}^1, a_{n_1-1}^1, \ldots, a_1^1] = \widetilde{p}/(\widetilde{q})^*$, where $(\widetilde{q})^*$ is the multiplicative inverse of $\widetilde{q}$ mod $\widetilde{p}$, see [25, Lemma A4]. Next we observe that the first term

$$[a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]A^{-1}[a_{n_1}^1 - 2 \;\; \cdots \;\; a_1^1 - 2]^T$$

on the right of Equation (5) can be calculated as follows. By Proposition 9.3 in our earlier work [13], we know that

$$\theta(\xi_{can}) = -\frac{2+\widetilde{q}+(\widetilde{q})^*}{\widetilde{p}} - I(\widetilde{p}/\widetilde{q})$$

for the canonical contact structure ξ_{can} on the lens space $L(\widetilde{p},(\widetilde{q})^*)$. By definition of $\theta(\xi_{can})$, we have

$$c_1^2 = \theta(\xi_{can}) + 3\sigma + 2\chi = \theta(\xi_{can}) + 3(-n_1) + 2(n_1+1) = \theta(\xi_{can}) - n_1 + 2.$$

We observe that $c_1^2 = [a^1_{n_1}-2 \ \ \cdots \ \ a^1_1-2]A^{-1}[a^1_{n_1}-2 \ \ \cdots \ \ a^1_1-2]^T$ and combining the last two equations above, and using the definition of $I(\widetilde{p}/\widetilde{q})$, we see that the first term on the right of Equation (5) is equal to

$$-\frac{2+\widetilde{q}+(\widetilde{q})^*}{\widetilde{p}} - \sum_{i=1}^{n_1}(a^1_i-3) - n_1 + 2. \tag{6}$$

Next, we observe the general fact that

$$I(\widetilde{p}/\widetilde{q}) = \sum_{i=1}^{n_1}(a^1_i-3) = n_3 - n_1 - 1,$$

where n_3 is the length of the continued fraction $\widetilde{p}/(\widetilde{p}-\widetilde{q}) = [a^3_1, a^3_2, \ldots, a^3_{n_3}]$ dual to $\widetilde{p}/\widetilde{q} = [a^1_1, a^1_2, \ldots, a^1_{n_1}]$, which can be proved by induction on n_1 using the Riemenschneider point rule, [27]. Plugging this back in the Equation (6), we see that the first term in Equation (5) is equal to

$$-\frac{2+\widetilde{q}+(\widetilde{q})^*}{\widetilde{p}} + 3 - n_3. \tag{7}$$

On the other hand,

$$\begin{aligned} -2\alpha - (n_3-1) + \frac{\widetilde{q}-(\widetilde{q})^*}{\widetilde{p}} &= -2\left(-1+\frac{1+\widetilde{q}}{\widetilde{p}}\right) - (n_3-1) + \frac{\widetilde{q}-(\widetilde{q})^*}{\widetilde{p}} \\ &= -\frac{2+q+(\widetilde{q})^*}{\widetilde{p}} + 3 - n_3. \end{aligned} \tag{8}$$

By comparing Equation (7) and Equation (8), we conclude that the first term in the Equation (5) is as claimed.

To prove the second claim, we observe that the first row of G is $-\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{u}^A$ and the dot product of $\mathbf{u}^A$ with $[a^1_{n_1}-2 \ \ a^1_{n_1-1}-2 \ \ \cdots \ \ a^1_1-2]^T$ is equal to $1+\widetilde{q}-\widetilde{p}$, as observed in the proof of Lemma 5.7. It follows that the second term

$$[a^1_{n_1}-2 \ \ \cdots \ \ a^1_1-2]G[a^1_{n_1}-2 \ \ \cdots \ \ a^1_1-2]^T$$

in the Equation (5) is equal to

$$-(1+\widetilde{q}-\widetilde{p})^2\frac{q}{(\widetilde{p})^2(p-q)} = -\alpha^2\frac{q}{p-q}.$$

The proof of Item (iii) is very similar to Item (ii), where only the bottom-right block $C^{-1}+H$ in Figure 11 is involved in the calculation. $\square$

Lemma 5.9. *We have*

$$\mathbf{y}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} = 2\left(\frac{\widetilde{p}-2}{\widetilde{p}}\right) - (n_1+n_3-2) - \left(\frac{\widetilde{p}-2}{\widetilde{p}}\right)^2 \frac{q}{p-q}$$

Proof. It follows immediately from Lemma 5.8 that

$$\begin{aligned}
\mathbf{y}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} &= (\mathbf{y_{a^1}}+\mathbf{y_{a^3}})^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}(\mathbf{y_{a^1}}+\mathbf{y_{a^3}}) \\
&= \mathbf{y}^T_{\mathbf{a^1}} Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y_{a^1}} + \mathbf{y}^T_{\mathbf{a^1}} Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y_{a^3}} + \mathbf{y}^T_{\mathbf{a^3}} Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y_{a^1}} + \mathbf{y}^T_{\mathbf{a^3}} Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y_{a^3}}. \\
&= -2(\alpha+\beta) - (n_1+n_3-2) - (\alpha+\beta)^2\frac{q}{p-q} \\
&= 2\left(\frac{\widetilde{p}-2}{\widetilde{p}}\right) - (n_1+n_3-2) - \left(\frac{\widetilde{p}-2}{\widetilde{p}}\right)^2 \frac{q}{p-q}.
\end{aligned} \tag{9}$$

□

Proof of Proposition 4.4. By combining Lemma 5.6, Lemma 5.7, and Lemma 5.9, we obtain

$$\begin{aligned}
c_1^2(X_{\mathbf{a^1},\mathbf{a^2}}) &= \mathbf{r}^T_{can} Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{r}_{can} \\
&= (\mathbf{x}+\mathbf{y})^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}(\mathbf{x}+\mathbf{y}) \\
&= \mathbf{x}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{x} + \mathbf{x}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} + \mathbf{y}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{x} + \mathbf{y}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} \\
&= \mathbf{x}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{x} + 2\mathbf{x}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} + \mathbf{y}^T Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}\mathbf{y} \\
&= 2n_2 + 3 - (a_0^2 + a_1^2 + \cdots + a_{n_2}^2) - \frac{1}{[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2 - 1]} \\
&\quad + \frac{2(2-\widetilde{p})}{\widetilde{p}}\left(1 - \frac{1}{p-q}\right) - \left(\frac{\widetilde{p}-2}{\widetilde{p}}\right)^2\frac{q}{p-q} - (n_1+n_3-2) + \frac{2(\widetilde{p}-2)}{\widetilde{p}}.
\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}
\theta(\xi_{can}) &= c_1^2(X_{\mathbf{a^1},\mathbf{a^2}}) - 3\sigma(X_{\mathbf{a^1},\mathbf{a^2}}) - 2\chi(X_{\mathbf{a^1},\mathbf{a^2}}) \\
&= c_1^2(X_{\mathbf{a^1},\mathbf{a^2}}) + n_2 + n_1 + n_3 - 1 \\
&= 1 - I(p/q) - \frac{1}{[a_{n_2}^2, a_{n_2-1}^2, \ldots, a_1^2, a_0^2 - 1]} + \frac{2(\widetilde{p}-2)}{\widetilde{p}(p-q)} - \frac{(\widetilde{p}-2)^2 q}{(\widetilde{p})^2(p-q)},
\end{aligned} \tag{11}$$

as claimed. □

6. Non-balanced contact structures and spherical 3-manifolds

Recall that Proposition 1.9, in light of Theorem 1.7, says that if ξ is not a balanced contact structure on the small Seifert fibered space $Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$ with $e_0 \geq 0$, then it is not filled by a symplectic rational homology ball. We depicted the plumbing diagram of $Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$ in Figure 12, for the convenience of the reader, where $p/q = [a_1, a_2, \ldots, a_k]$.

We observe that there are exactly two tight contact structures, which we denote by $\xi^{\pm}$, on $Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$ that are not balanced. These are the contact structures where in their contact surgery presentation, the complementary legs are stabilized once consistently (say both

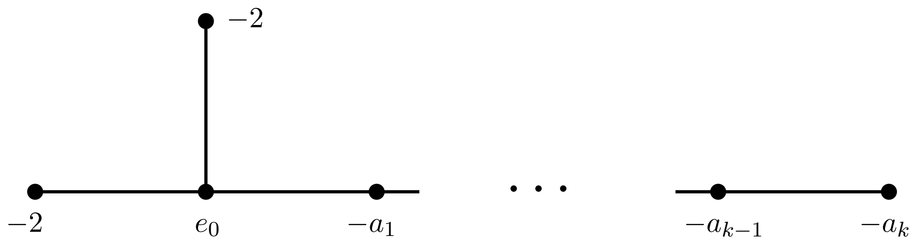

FIGURE 12. The plumbing diagram of the small Seifert fibered space $Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$, where $p/q = [a_1, a_2, \ldots, a_k]$.

positive) and the non-complementary leg is also stabilized consistently, either all positive or all negative. The contact structure ξ^-(resp ξ^+) is when the non-complementary leg has opposite (resp. the same) signs as the complementary legs. Our goal is then to prove that $(Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2}), \xi^{\pm})$ does not admit a rational homology ball symplectic filling. (If one makes the opposite choice for the sign of the complementary legs, then we would get $-\xi^{\pm}$, but these have the same θ-invariant so we will not consider them.) This will be a corollary of the following formulas for $\theta(\xi^{\pm})$.

Lemma 6.1. *Let $\frac{p'}{q'} = [a_1, \ldots, a_{k-1}]$, and assume that $p' = 1, q' = 0$ when $k = 1$. Then*

(1) $\theta(\xi^-) = -(a_1 + \cdots + a_k - (3k + e_0 - 2)) - \frac{(e_0+1)p'+q'}{(e_0+1)p+q}$, and
(2) $\theta(\xi^+) = -(a_1 + \cdots + a_k - (3k + e_0 - 1)) - \frac{(e_0+1)p'+q'}{(e_0+1)p+q} - \frac{(e_0-3)p+q+4}{(e_0+1)p+q}$.

Proof. The proof of Lemma 6.1 is an easier version of the calculations that were done in Section 5, and therefore, we just provide the setup and leave some straightforward details to the interested reader.

We start by describing $\xi^{\pm}$ by appropriate contact surgery diagrams. Note that the plumbing diagram of $Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$ depicted in Figure 12 is not immediately suitable for describing contact surgery diagrams of $\xi^{\pm}$. To remedy this we apply a sequence of blow-ups between the central vertex and the one standing next to it on its right-hand side, until the central curve has framing zero. This adds an additional e_0 many vertices (2 handles) to the leg corresponding to the singular fiber. After this, we can realize the manifold as a Stein handlebody diagram by converting the 0-framed curve to a Stein 1-handle and attaching to it $k + e_0 + 2$ Stein 2-handles with framings

$$-2, -2, \underbrace{-1, -2, \ldots, -2}_{e_0}, -a_1 - 1, -a_2, \ldots, -a_k,$$

respectively. To obtain contact surgery diagrams describing $\xi^{\pm}$, we convert the Stein 1-handle into a contact $(+1)$-surgery (this still describes Stein fillable contact structures). Let $(X, J^{\pm})$ denote the Stein manifold where the complementary legs are stabilized the same way (say positive) and the non-complementary leg includes stabilizations that are either all positive or all negative. In this description, the Stein structures $J^{\pm}$ induce $\xi^{\pm}$ on the boundary.

We observe that $\chi(X) = k + e_0 + 4$, since the handle-decomposition of X consists of $k + e_0 + 3$ two-handles and a zero-handle. As for the signature we first note that

$Y = Y(e_0; \frac{1}{2}, \frac{q}{p}, \frac{1}{2})$ is an L-space, and the plumbing graph corresponding to $-Y$ is negative definite, since it arises as the *oriented* link of quotient surface singularity. (Indeed, $-Y$ will be the dihedral-type spherical 3-manifold $D(\widetilde{p}, \widetilde{q})$, canonically oriented as the link of a complex surface singularity, where $\widetilde{p} = (e_0+2)p + q$ and $\widetilde{q} = p$.) So, as argued in [29, Section 8.1], it must be that $b^+(X) = 1$, and hence $\sigma(X) = -(k + e_0 + 1)$. The main ingredient of the proof of Lemma 6.1 is the calculation of the square of the first Chern class

$$c_1^2(X, J^{\pm}) = \mathbf{r}_{\pm}^T I_X^{-1} \mathbf{r}_{\pm}, \tag{12}$$

where $\mathbf{r}_{\pm} = \begin{bmatrix} 0 & 1 & 1 & 0\cdots 0 & \pm(a_1-1) & \pm(a_2-2)\cdots & \pm(a_k-2) \end{bmatrix}^T$ is the rotation vector, and I_X is the intersection matrix of X described above. Instead of computing I_X^{-1} similar to our calculations in Section 5, it is much easier here to solve the linear system $I_X\mathbf{x} = \mathrm{rot}_{\pm}$ for $\mathbf{x}$. This will result in the computations

$$c_1^2(X, J^-) = -(a_1 + \cdots + a_k - (2k-1)) - \frac{1}{[a_k, \ldots, a_2, a_1+1, \underbrace{2, \ldots, 2}_{e_0}]}$$

where we note that

$$\frac{(e_0+1)p' + q'}{(e_0+1)p + q} = \frac{1}{[a_k, \ldots, a_2, a_1+1, 2, \ldots, 2]},$$

and

$$c_1^2(X, J^+) = -(a_1 + \cdots + a_k - 2k) - \frac{(e_0+1)p' + q'}{(e_0+1)p + q} - \frac{(e_0-3)p + q + 4}{(e_0+1)p + q},$$

which in turn, yields the formulas in the lemma. The details of this calculation are left to the reader. □

Proof of Proposition 1.9. As we mentioned above, our proof will rely on Lemma 6.1. First, for any $e_0 \geq 0$ and $\frac{p}{q} > 1$, one can easily see that, $c_1^2(X, J^-) \notin \mathbb{Z}$ (and hence $\theta(\xi^-) \notin \mathbb{Z}$), since

$$\frac{(e_0+1)p' + q'}{(e_0+1)p + q} = \frac{1}{[a_k, \ldots, a_2, a_1+1, 2, \ldots, 2]} < 1.$$

We note a similar equality was established in the proof of Proposition 4.3. In particular, ξ^- cannot be symplectically filled by a rational homology ball.

Although $\theta(\xi^+)$ may take integer values, with the help of Theorem 1.1, we will show that $\theta(\xi^+) \neq -2$. Recall that Theorem 1.1 characterizes exactly which small Seifert fibered spaces with complementary legs admit smooth rational homology ball fillings. To apply the theorem we first need to perform a sequence of $(-e_0 - 1)$ Rolfsen twists along the singular fiber with framing $-\frac{p}{q}$. The new framing for the singular fiber will be

$$-\frac{p}{(e_0+1)p + q} = -1 + \frac{1}{[\underbrace{2, \ldots, 2}_{e_0}, a_1+1, a_2, \ldots, a_k]}.$$

Now Theorem 1.1 says this Seifert fibered space bounds a smooth rational homology ball exactly when $r/s = [\underbrace{2, \ldots, 2}_{e_0}, a_1+1, a_2, \ldots, a_k] \in \mathcal{R}$. According to Lisca [21] the integer

$I(r/s)$ satisfies

$$I(r/s) = \sum_{i=1}^{e_0}(2-3) + (a_1+1-3) + \sum_{j=2}^{k}(a_j-3) \leq 1.$$

Indeed Lisca also proves that $I(r/s)=1$ exactly when $\frac{r}{s}=\frac{m^2}{mh-1}$ for some relatively prime integers $0<h<m$.

We can expand and calculate this integer as $I(r/s) = -3k - e_0 + 1 + (a_1+\cdots+a_k)$, so that Item (2) of Lemma 6.1 reads as

$$\theta(\xi^+) = -I(r/s) - \frac{(e_0+1)p'+q'}{(e_0+1)p+q} - \frac{(e_0-3)p+q+4}{(e_0+1)p+q}. \tag{13}$$

Note that the fraction terms in the formula for $\theta(\xi^+)$ are both strictly less than one. In particular, if $I(r/s)<1$, then $\theta(\xi^+)>-2$. If $I(r/s)=1$, then we know that $\frac{r}{s}=\frac{m^2}{mh-1}$ for some relatively prime integers $0<h<m$. Since the continued fraction for r/s must start with a 2, we see that that $\frac{m}{2}<h$. We now compute p/q explicitly by using the equality $[2,\ldots,2,a_1+1,a_2,\ldots,a_k]=\frac{m^2}{mh-1}$. We can rewrite this as

$$[\underbrace{2,\ldots,2}_{e_0-1},a_1+1,a_2,\ldots,a_k] = (2-\frac{m^2}{mh-1})^{-1} = \frac{mh-1}{2(mh-1)-m^2}$$

and repeating this e_0-1 more times gives that

$$[a_1+1,a_2,\ldots,a_k] = \frac{e_0(mh-1)-(e_0-1)m^2}{(e_0+1)(mh-1)-e_0m^2}.$$

We point out that if $e_0=0$ (that is when there are no 2's at the beginning) the proof still works. Finally, we "move" the $+1$ in the first term in the continued fraction to the right-hand side to obtain

$$[a_1,a_2,\ldots,a_k] = \frac{m^2-(mh-1)}{(e_0+1)(mh-1)-e_0m^2}$$

which implies that $p=m^2-(mh-1)$ and $q=(e_0+1)(mh-1)-e_0m^2 = mh-1-pe_0$.

We next explicitly calculate p'/q'. Since $[a_1,a_2,\ldots,a_k]=\frac{m^2-(mh-1)}{(e_0+1)(mh-1)-e_0m^2}$, we have $[a_k,a_{k-1},\ldots,a_1]=\frac{m^2-(mh-1)}{t^*}$ where $0<t^*<p$ is the inverse of $(e_0+1)(mh-1)-e_0m^2=(mh-1-pe_0) \bmod p = m^2-mh+1$, and by [25, Lemma A4] we know that $t^*=p'$. Indeed, one can easily check that $p'=(m-h)^2$. To see this, we first observe that $p'=(m-h)^2=p^2+h^2-mh-1$, and calculate

$$\begin{aligned} p'((mh-1-pe_0)) &= (p^2+h^2-mh-1)(mh-1-pe_0) \\ &\equiv (h^2-mh-1)(mh-1) \mod p \\ &= -h^2(m^2-mh+1)+1 \\ &\equiv 1 \mod p. \end{aligned}$$

since $p=m^2-mh+1$.

Similarly one can calculate that $q' = (2e_0+1)mh - (e_0+1)h^2 - e_0m^2 - 1$. We emphasize a surprising observation

$$p - 2 = (e_0+1)p' + q', \tag{14}$$

that will be crucial below.

Recall that we want to prove that $\theta(\xi^+) \neq -2$. Since we are assuming $I(r/s) = 1$, Equation (13) reduces to

$$\theta(\xi^+) = -1 - \frac{(e_0+1)p' + q'}{(e_0+1)p+q} - \frac{(e_0-3)p+q+4}{(e_0+1)p+q}.$$

So, assuming $\theta(\xi^+) = -2$, we obtain the equation

$$(e_0+1)p' + q' + 4 - 4p = 0,$$

which is a impossible, since by using Equation 14, we can explicitly calculate that

$$(e_0+1)p' + q' + 4 - 4p = -3p + 2 < 0.$$

Thus, $\theta(\xi^+) \neq -2$, and we conclude that ξ^+ cannot be symplectically filled by a rational homology ball, which finishes our proof. □

We now show the classification of spherical 3-manifolds (with either orientation) with symplectic rational homology ball fillings, given in Theorem 1.11 follows from Proposition 1.9.

Proof of Theorem 1.11. As noted in the introduction, we only need to determine which of the "oppositely oriented" dihedral-type spherical 3-manifolds admit contact structures having symplectic rational homology ball fillings. We recall that any dihedral-type spherical 3-manifold with its canonical orientation is of the form $Y(e_0; 1/2, s, 1/2)$ with $e_0 \leq -2$. So any "oppositely oriented" dihedral-type spherical 3-manifold has the same form but with $e_0 \geq -1$. The classification of fillings when e_0 is -1 was given in Theorem 1.6. Converting the notation in that theorem to describe a dihedral-type spherical 3-manifold yields Item (2) in the theorem when $n > 1$. Similarly, Theorem 1.7 and Proposition 1.9 classify which contact structures on $Y(e_0; 1/2, s, 1/2)$ with $e_0 \geq 0$ have symplectic rational homology ball fillings. This gives Item (2) in the theorem with $n = 1$. □

APPENDIX A. PROOF OF LEMMA 5.4.

In this appendix we give a proof of Lemma 5.4 which recall calculates the inverse matrix $Q^{-1}_{\mathbf{a^1}, \mathbf{a^2}}$.

Proof of Lemma 5.4. We have $\widetilde{p}/\widetilde{q} = [a^1_1, a^1_2, \ldots, a^1_{n_1}]$, $p/q = [a^2_0, a^2_1, \ldots, a^2_{n_2}]$, and $\widetilde{p}/(\widetilde{p} - \widetilde{q}) = [a^3_1, a^3_2, \ldots, a^3_{n_3}]$, with all continued fraction coefficients a^j_i being greater than or equal to 2, by definition. Note that we also have, $\widetilde{p}/(\widetilde{q})^* = [a^1_{n_1}, a^1_{n_2}, \ldots, a^1_1]$, where $(\widetilde{q})^*$ is the inverse of $\widetilde{q}$ mod $\widetilde{p}$, and $(p-q)/q = [a^2_0 - 1, a^2_1, \ldots, a^2_{n_2}]$ (see Remark 5.3, when $a^2_0 = 2$).

We observe the fact that $\mathbf{v}^B = \mathbf{v}^{\widetilde{B}}$, which immediately follows from the definitions of the matrices B and $\widetilde{B}$. By Lemma 5.2, we know that the last column of A^{-1} is $-(1/\widetilde{p})\ \mathbf{u}^A$, the first column of $(\widetilde{B})^{-1}$ is $-(1/(p-q))\ \mathbf{v}^B$ (and hence the first row of $(\widetilde{B})^{-1}$ is given by $-(1/(p-q))\ (\mathbf{v}^B)^T$) and the first column of C^{-1} is $-(1/\widetilde{p})\ \mathbf{v}^C$.

The proof of the lemma will be achieved by showing that the product of the matrix $Q_{\mathbf{a^1},\mathbf{a^2}}$, depicted in Figure 10, and the matrix depicted in Figure 11 is equal to the identity matrix. Towards that goal, we first define an auxiliary matrix $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ by deleting A^{-1} and C^{-1} from the matrix depicted in Figure 11, and calculate the product of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$.

Let $\widetilde{Q}_A$ be the submatrix of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ obtained by juxtaposition of the blocks G, D, and E, let $\widetilde{Q}_{\widetilde{B}}$ be the submatrix of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ obtained by juxtaposition of the blocks D^T, $(\widetilde{B})^{-1}$ and F and let $\widetilde{Q}_C$ be the submatrix of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ obtained by juxtaposition of the blocks E^T, F^T, and H.

By Remark 5.5, we know that each column of $\widetilde{Q}_A$ is a multiple of $\mathbf{u}^A$ (because the definition of G, D, E involves $\mathbf{u}^A$), that each column of $\widetilde{Q}_C$ is a multiple of $\mathbf{v}^C$ (because the definition of E^T, F^T, H involves $\mathbf{v}^C$) and that the columns of $\widetilde{Q}_{\widetilde{B}}$ belonging to the blocks D^T and F is a multiple of $\mathbf{v}^B$ (because the definition of D^T and F involves $\mathbf{v}^B$). Note that the first column of the block $(\widetilde{B})^{-1}$ is also a multiple of $\mathbf{v}^B$.

Now we calculate the dot product of each row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ with each column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ case by case below. Note that for each dot product there are at most four terms to calculate. So, the calculations are not very difficult but tedious.

For $1 \leq i \leq n_1 - 1$, and $1 \leq j \leq n_1 + n_2 + n_3 + 1$, the dot product of the ith row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is the same as the dot product of the ith row of A and the jth column of $\widetilde{Q}_A$. Hence this is the dot product of the ith row of A and a multiple of the last column of A^{-1} since the last column of A^{-1} is a multiple of $\mathbf{u}^A$ and each column of $\widetilde{Q}_A$ is a multiple of $\mathbf{u}^A$. Therefore this dot product is zero, because we are indeed taking the dot product of any of the first $n_1 - 1$ rows of A with a multiple of the last column of A^{-1}.

We want to show that for $1 \leq j \leq n_1 + n_2 + n_3 + 1$, the dot product of the n_1th row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ (the last row of $\widetilde{Q}_A$) and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is zero. To see this, we first claim that the (n_1+1)st row of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ (the first row of $\widetilde{Q}_{\widetilde{B}}$) is $(\det A = \widetilde{p})$ multiple of the first row of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$. To prove the claim we observe that the first row of $(\widetilde{B})^{-1}$ is $\frac{-1}{p-q}(\mathbf{v}^B)^T = \frac{-1}{p-q}(\mathbf{v}^{\widetilde{B}})^T$ and the first entry of $\mathbf{v}^B$ is q. Then the claim follows by simply comparing the first rows of $D^T, (\widetilde{B})^{-1}, F$ with the first rows of G, D, E, respectively as follows. The first row of

$$D^T = -\frac{1}{\widetilde{p}(p-q)}\mathbf{v}^B(\mathbf{u}^A)^T \text{ is given by } -\frac{q}{\widetilde{p}(p-q)}(\mathbf{u}^A)^T$$

and the first row of

$$G = -\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{u}^A(\mathbf{u}^A)^T \text{ is given by } -\frac{q}{(\widetilde{p})^2(p-q)}(\mathbf{u}^A)^T.$$

As pointed out above, the first row of

$$(\widetilde{B})^{-1} \text{ is given by } \frac{-1}{p-q}(\mathbf{v}^B)^T$$

and the first row of

$$D = -\frac{1}{\widetilde{p}(p-q)}\mathbf{u}^A(\mathbf{v}^B)^T \text{ is given by } -\frac{1}{\widetilde{p}(p-q)}(\mathbf{v}^B)^T.$$

Similarly, the first row of

$$E = -\frac{q}{(\widetilde{p})^2(p-q)}\mathbf{u}^A(\mathbf{v}^C)^T \text{ is given by } -\frac{q}{(\widetilde{p})^2(p-q)}(\mathbf{v}^C)^T$$

and the first row of

$$F = -\frac{1}{\widetilde{p}(p-q)}\mathbf{v}^B(\mathbf{v}^C)^T \text{ is given by } -\frac{q}{\widetilde{p}(p-q)}(\mathbf{v}^C)^T,$$

which finishes the proof of our claim. Note that the n_1th row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ can be seen as the juxtaposition of the last row of A and the row vector $[1, 0, \dots, 0] \in \mathbb{R}^{n_2+n_3+1}$. So, in the dot product at hand we only have to consider the first (n_1+1)st entries in the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$. Our analysis above therefore implies that for $1 \leq j \leq n_1+n_2+n_3+1$, the dot product of the n_1th row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is equal to a multiple of

$$(\text{last row of } A) \cdot \mathbf{u}^A + \det A$$

but $\mathbf{u}^A$ is $-\det A$ times the last column of A^{-1}, and therefore the dot product is a multiple of

$$-\det A(\text{last row of } A) \cdot (\text{last column of } A^{-1}) + \det A = 0$$

in this case as well.

Now we turn our attention to the last $n_3 - 1$ rows of $Q_{\mathbf{a^1},\mathbf{a^2}}$. For

$$n_1 + n_2 + 3 \leq i \leq n_1 + n_2 + n_3 + 1 \text{ and } 1 \leq j \leq n_1 + n_2 + n_3 + 1,$$

the dot product of the ith row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is the same as the dot product of the $(i - n_1 - n_2 - 1)$st row of C and the jth column of $\widetilde{Q}_C$. Hence this is the dot product of the $(i - n_1 - n_2 - 1)$st row of C and a multiple of the first column of C^{-1} since the first column of C^{-1} is a multiple of $\mathbf{v}^C$ and each column of $\widetilde{Q}_C$ is a multiple of $\mathbf{v}^C$. Therefore this dot product is zero, because we are indeed taking the dot product of any of the last $n_3 - 1$ rows of C with a multiple of the first column of C^{-1}.

For $1 \leq j \leq n_1 + n_2 + n_3 + 1$, the dot product of the $(n_1 + n_2 + 2)$nd row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is zero because of the fact that the (n_1+1)st row of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ (the first row of $\widetilde{Q}_{\widetilde{B}}$) is $(\det A = \widetilde{p} = \det C)$ multiple of the first row of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$, as we showed above. Hence, the dot product at hand is equal to a multiple of

$$(\text{first row of } C) \cdot \mathbf{v}^C + \det C$$

but $\mathbf{v}^C$ is $-\det C$ times the first column of C^{-1}, and therefore the dot product is zero in this case as well.

Now we turn our attention to the middle rows of $Q_{\mathbf{a^1},\mathbf{a^2}}$. For $n_1 + 2 \leq i \leq n_1 + n_2 + 1$ and ($1 \leq j \leq n_1 + 1$ and $n_1 + n_2 + 2 \leq j \leq n_1 + n_2 + n_3 + 1$) the dot product of the ith row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is the same as the dot product of the $(i - n_1)$th row of B and the jth column of $\widetilde{Q}_{\widetilde{B}}$. Hence this is the dot product of the $(i - n_1)$th row of B and a multiple of the first column of $(\widetilde{B})^{-1}$ since the first column of $(\widetilde{B})^{-1}$ is a multiple of $\mathbf{v}^{\widetilde{B}} = \mathbf{v}^B$ and every column except the ones enumerated from $n_1 + 2$ to $n_1 + n_2 + 1$ of $\widetilde{Q}_{\widetilde{B}}$ is a multiple of $\mathbf{v}^B$. Therefore this dot product is zero, because we are taking the dot product of any of the last n_2 rows of B (same as the last n_2 rows of $\widetilde{B}$) with a multiple of the first column of $(\widetilde{B})^{-1}$.

For $n_1+2 \leq i \leq n_1+n_2+1$ and $n_1+2 \leq j \leq n_1+n_2+1$, the dot product of the ith row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ gives a $n_2 \times n_2$ identity matrix because these dot products are the same as the dot products of last n_2 rows of $\widetilde{B}$ and the last n_2 columns of $(\widetilde{B})^{-1}$.

In order to move forward with our analysis, we prove a general fact that will be used below. Since each column of $\widetilde{Q}_C$ is a multiple of $\mathbf{v}^C$, and the first entry of $\mathbf{v}^C$ is $(\widetilde{p}-\widetilde{q})$, while its last entry is 1, we see that the first row of $\widetilde{Q}_C$ is $(\widetilde{p}-\widetilde{q})$ multiple of its last row. Recall that we showed above that the first row of $\widetilde{Q}_{\widetilde{B}}$ is $\widetilde{p}$ multiple of the first row of $\widetilde{Q}_A$. Since each column of $\widetilde{Q}_A$ is a multiple of $\mathbf{u}^A$, and the first entry of $\mathbf{u}^A$ is 1, while its last entry $\widetilde{q}$, we see that the last row of $\widetilde{Q}_A$ is $\widetilde{q}$ multiple of its first row. By simply comparing the definitions of the block matrices involved, we also observe that the first row of $\widetilde{Q}_A$ is the same as the last row $\widetilde{Q}_C$. Putting all these observations together, we conclude that

$$\text{last row of } \widetilde{Q}_A + \text{first row of } \widetilde{Q}_C = \text{first row of } \widetilde{Q}_{\widetilde{B}} \tag{15}$$

which is a key result for the rest of our proof.

Next we want to see that the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the (n_1+1)st column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is 1. The equality (15) can be rephrased as

$$(n_1)\text{th row} + (n_1+n_2+2)\text{nd row} = (n_1+1)\text{th row} \tag{16}$$

for the matrix $Q_{\mathbf{a^1},\mathbf{a^2}}$. Note that in the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$, the nonzero terms appear on the n_1th, (n_1+1)st , (n_1+2)nd and (n_1+n_2+2)nd entries as $1, -a_0^2, 1, 1$. Because of the equality (16), the dot product at hand is exactly the dot product of the first row of $\widetilde{B}$ (which is the vector $[-(a_0^2-1)\ 1\ 0\cdots 0] \in \mathbb{R}^{n_2+1}$ and the first column of $(\widetilde{B})^{-1}$, which is equal to 1.

What is left to consider is the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ for $j \neq n_1+1$. If we take $1 \leq j \leq n_1$, the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is not zero. The computation here is very similar to the case $j = n_1+1$ we discussed in the paragraph above. In fact the dot product here is equal to some multiples (for $1 \leq j \leq n_1$) of 1 (that we computed in the previous paragraph), and as a matter of fact these dot products are exactly given by the row vector $(1/\widetilde{p})\ (\mathbf{u}^A)^T$. Therefore, when we compute the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of claimed $Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}$ (where we have to take into account A^{-1} at the top left block) we just have to add $-(1/\widetilde{p})\ (\mathbf{u}^A)^T$ (the last row of A^{-1}) and hence the dot product will be zero.

Similarly, if we take $n_1+n_2+2 \leq j \leq n_1+n_2+n_3+1$, the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is not zero and in fact equal to some multiples of 1 we computed above. These multiples are exactly given by the row vector $(1/\widetilde{p})\ (\mathbf{v}^C)^T$. Hence , for $n_1+n_2+2 \leq j \leq n_1+n_2+n_3+1$, when we compute the dot product of the (n_1+1)st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of claimed $Q^{-1}_{\mathbf{a^1},\mathbf{a^2}}$ (where we have to take into account C^{-1} at the bottom right block) we just have to add $-(1/\widetilde{p})\ (\mathbf{v}^C)^T$ (the first row of C^{-1}) and hence the dot product will be zero.

Finally, we can see that, for $n_1 + 2 \leq j \leq n_1 + n_2 + 1$, the dot product of the $(n_1 + 1)$st row of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and the jth column of $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is zero. The key point is that these are some multiples of the dot products of the first row of $\widetilde{B}$ with the last n_2 columns of $(\widetilde{B})^{-1}$. Here we again use the equality (16).

As a consequence of our case by case analysis above, we can explicitly describe the product $Q_{\mathbf{a^1},\mathbf{a^2}}$ and $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ as follows. It is a square matrix of size $n_1+n_2+n_3+1$ such that the first n_1 and last n_3 rows are zero. The middle rows can be described as the juxtaposition of three block matrices of sizes $(n_2+1)\times n_1$, $(n_2+1)\times(n_2+1)$, $(n_2+1)\times n_3$, respectively. The first row of the first block is $(1/\widetilde{p})\ (\mathbf{u}^A)^T$ and all other rows are zero. The second block is the $(n_2+1)\times(n_2+1)$ identity matrix and the first row of the third block is $(1/\widetilde{p})\ (\mathbf{v}^C)^T$ and all other rows are zero.

Recall that $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$ is obtained by deleting A^{-1} and C^{-1} from the matrix depicted in Figure 11. Therefore, based on the previous paragraph and the fact that

$$AA^{-1} = I \text{ and } CC^{-1} = I,$$

we see that the product of the matrix $Q_{\mathbf{a^1},\mathbf{a^2}}$ depicted in Figure 10 and the matrix depicted in Figure 11 is equal to the identity matrix. This is because the nonzero row $(1/\widetilde{p})\ (\mathbf{u}^A)^T$ in the previous paragraph will be cancelled out by the last row of A^{-1}, which is equal to $-(1/\widetilde{p})\ (\mathbf{u}^A)^T$ and similarly, the nonzero row $(1/\widetilde{p})\ (\mathbf{v}^C)^T$ in the previous paragraph will be cancelled out by the first row of C^{-1}, which is equal to $-(1/\widetilde{p})\ (\mathbf{v}^C)^T$. We would like to emphasize that we have also used the fact that $\widetilde{B}(\widetilde{B})^{-1} = I$, while calculating the product of $Q_{\mathbf{a^1},\mathbf{a^2}}$ and $\widetilde{Q}_{\mathbf{a^1},\mathbf{a^2}}$. □

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