ON CONTACT COSMETIC SURGERY

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ABSTRACT. We demonstrate that the contact cosmetic surgery conjecture holds for all non-trivial Legendrian knots, with the possible exception of Lagrangian slice knots. We also discuss the contact cosmetic surgeries on Legendrian unknots and make the surprising observation that some Legendrian unknots have a contact surgery with no cosmetic pair, while all other contact surgeries are contactomorphic to infinitely many other contact surgeries on the unknot.

1. INTRODUCTION

In this note we establish that a contact analog of the cosmetic surgery conjecture holds for all Legendrian knots except possibly for a small family of Legendrian knots.

We first recall the smooth cosmetic surgery conjecture. Given a knot K in S^3, we say that two surgeries on K, say $S^3_K(r)$ and $S^3_K(r')$, are *cosmetic* if $S^3_K(r)$ and $S^3_K(r')$ are diffeomorphic and *truly cosmetic* if $S^3_K(r)$ and $S^3_K(r')$ are orientation preserving diffeomorphic. There are many examples of cosmetic surgeries on knots, but the only known truly cosmetic surgeries are on the unknot. Thus the cosmetic surgery conjecture postulates that non-trivial knots admit no truly cosmetic surgeries (see [9] Conjecture 6.1). There has been a great deal of work on this conjecture [3, 10, 14, 15, 19, 20, 21], and this paper heavily depends on that work. We recall the specific results we will need in Section 2.

Turning to contact geometry, we recall that given a Legendrian knot L in a contact manifold (M, ξ) contact (r)-surgery on L is the result of removing a standard neighborhood of L from M and replacing it with a solid torus whose meridian has slope $r + \operatorname{tb}(L)$ and extending the contact structure over this torus to be any tight contact structure. See Section 2 for more details, but we note now that for any $r \neq 1/n$ there will be more than one possibility for a contact surgery.

We can now say that two contact surgeries on a Legendrian knot are *cosmetic* if there is a contactomorphism between the resulting manifolds. We note that since a contactomorphism between two contact manifolds is automatically orientation preserving, we do not need to distinguish between "cosmetic" and "truly cosmetic" as in the smooth case. Generalizing the cosmetic surgery conjecture to the contact category, we have the contact cosmetic surgery conjecture.

Conjecture 1.1 (Contact cosmetic surgery conjecture)**.** *Any Legendrian knot in S^3 with its standard tight contact structure that is not smoothly an unknot admits no cosmetic contact surgeries.*

We note that the contact cosmetic surgery conjecture is a natural generalization of the smooth cosmetic surgery conjecture and is particularly interesting in light of trying to understand different contact surgery representations of different contact manifolds. In addition, our main theorem gives a small class of knots on which the conjecture might be false, and thus give prime candidates to consider for the general cosmetic surgery conjecture too.

Our main result is a proof of this conjecture for almost all Legendrian knots.

Theorem 1.2. *The contact cosmetic surgery conjecture holds true for all non-trivial Legendrian knots except possibly for* ± 2 *surgery on a Legendrian knot* L *that is Lagrangian slice and is in a knot type* K *with* $\tau(K) = 0, \overline{\mathrm{tb}}(K) = -1$*, has Seifert genus* 2*, and is quasi-positive.*

This theorem will be proven by constructing specific contact (± 1)-surgery diagrams for specific contact surgeries and then analyzing the d_3-invariant of the corresponding diagrams. See Section 4 for the details and the Appendix for details on the linear algebra necessary for the computations.

A few days before this manuscript was released, the paper [3] appeared on the arxiv. This paper made significant progress on the smooth cosmetic surgery conjecture and simplified some of the arguments in the proof of Theorem 1.2. In particular, our computations involving ±1/n surgeries are no longer necessary. We have left those computations in the paper as they might be of some interest.

Just as in the smooth setting, it is important to exclude Legendrian unknots from this conjecture as they do admit cosmetic surgeries. We can explicitly write out all the contact cosmetic surgeries on Legendrian unknots. To state these results, we first recall the situation for smooth unknots. Given any non-zero rational surgery on an unknot, one may use Rolfsen twists to find a diffeomorphic manifold described by a rational surgery on the unknot with a surgery coefficient less than or equal to -1. So to describe the cosmetic surgeries on the unknot, we start with a rational number $-p/q \leq -1$. Now define

$$CS(-p/q) = \{-p/q' | q' = q + np \text{ or } q' = \overline{q} + np \text{ for some } n \in \mathbb{Z}\}$$

where $\overline{q}$ is the inverse of q mod p if it exists (otherwise, we ignore the second possibility for q'). Since any surgery on the unknot yields a lens space and we know when two lens spaces are diffeomorphic, it is easy to see that given $-q/p \leq -1$ then r surgery on the unknot is orientation preserving diffeomorphic to $-q/p$ surgery on the unknot if and only if $r \in CS(-p/q)$.

Theorem 1.3. *For a Legendrian unknot* L *with* $|\mathrm{rot}(L)| < |\mathrm{tb}(L) + 1|$ *contact* $(r - \mathrm{tb}(L))$*-surgery on* L *is not equivalent to any other contact surgery on* L *if* $r < \mathrm{tb}(L)$*. For any other Legendrian unknot or non-zero contact surgery corresponding to a non-zero smooth surgery, there are infinitely many other contact surgeries on the unknot yielding contactomorphic manifolds.*

Specifically, if L *is the Legendrian unknot with* $\mathrm{rot}(L) = \pm|\mathrm{tb}(L) + 1|$ *and* $-p/q < -1$*, then any contact* $(r - \mathrm{tb}(L))$*-surgery on* L *for* $r \in CS(-p/q)$ *with* $r \notin (\mathrm{tb}(L), 0)$*, is contactomorphic to some contact* $(r' - \mathrm{tb}(L))$*-surgery on* L *for any* $r' \in CS(-p/q)$ *with* $r' \notin (\mathrm{tb}(L), 0)$*. In fact, if* $r' \in (1/(k+1), 1/k)$ *for* k *a non-negative integer, then* $(r - \mathrm{tb}(L))$*-surgery on* L *will be contactomorphic to exactly* $k + 1$ *distinct contact* $(r' - \mathrm{tb}(L))$*-surgeries. Moreover, any contact* $(r - \mathrm{tb}(L))$*-surgery on* L *with* $r \in CS(-p/q)$ *with* $r \in (\mathrm{tb}(L), 0)$*, is contactomorphic to some contact* $(r' - \mathrm{tb}(L))$*-surgery on* L *for any* $r' \in CS(-p/q)$ *with* $r' \in (\mathrm{tb}(L), 0)$*.*

If L is a Legendrian unknot with $|\operatorname{rot}(L)| < |\operatorname{tb}(L)+1|$ and $r \in CS(-p/q)$ is not less than $\operatorname{tb}(L)$, then any contact $(r - \operatorname{tb}(L))$-surgery on L is contactomorphic to a contact $(r' - \operatorname{tb}(L))$-surgery on L for any $r' \in CS(-p/q)$ with r' not less than $\operatorname{tb}(L)$.

Remark 1.4. We note the interesting phenomenon that any smooth surgery on the unknot (other than 0 surgery) is diffeomorphic to infinitely many other surgeries on the unknot, but for Legendrian knots, there are some that have unique contact surgeries, though most have infinitely many cosmetic contact surgeries.

This theorem will be proven using a careful analysis of contact surgery and the classificaiton fo tight contact structures on various simple 3-manifolds. The details can be found in Section 3.

Remark 1.5. Previously, Chatterjee and Kegel [2] had studied surgeries on Legendrian unknots in the tight contact structure on S^3. Specifically, in Section 3 of their paper, they characterized which contact surgeries were tight and which were overtwisted (but did not specifically identify the contact structures obtained), then in Theorem 4.1 they did identify the overtwisted contact structures obtained by some contact surgery corresponding to $-(4m+3)/4$ surgeries on Legendrian unknots. Their proofs involved contact surgery diagrams and computing d_3-invariants of the specific surgeries considered. While it is possible that similar techniques could be used to prove our theorem above, it is not clear how this could be done, given that the methods we have for computing d_3-invariants for arbitrary contact surgeries on the unknot do not have simple closed-form expressions and can get quite complicated for surgery coefficients with long continued fraction expantions. So we will base our proof on the Farey graph and simple classification results on tori and lens spaces.

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2. Background and preliminary result

We assume the reader is familiar with contact geometry and convex surface theory as discussed in [11]. However, we recall some facts about the classification of contact structures in Section 2.1 for the convenience of the reader; more details on this discussion and the relevant notation can be found in [6]. We then recall some facts about contact surgery on Legendrian knots in Section 2.2, while Section 2.3 discusses the d_3-invariant of plane fields. We end this section by recalling several results about the smooth cosmetic surgery conjecture.

2.1. **Contact structures on thickened tori, solid tori, and lens spaces.** A contact structure ξ on $T^2 \times [0,1]$ with convex boundary, where each boundary component has two dividing curves with slope s_i on $T^2 \times \{i\}$ is *minimally twisting* if any convex torus parallel to the boundary has dividing slope clockwise of s_0 and anti-clockwise of s_1 (we denote this by saying its slope is in $[s_0, s_1]$). From [7, 11] we know that any minimally twisting contact structure on $T^2 \times [0,1]$ with boundary conditions as above is determined by a minimal path in the Farey graph from s_0 clockwise to s_1 with signs on each edge in the path.

To discuss contact structures on solid tori we set up some notation. Consider $T^2 \times [0,1]$. If we foliate $T^2 \times \{0\}$ by linear curves of slope r and let S_r be the result of collapsing the leaves of this foliation, then we call S_r a *solid torus with lower meridian* r. Similarly, we can define the *solid torus* S^r *with upper meridian* r by collapsing the same curves on $T^2 \times \{1\}$. We note that the standard way of thinking of the solid torus as $S^1 \times D^2$ is S_∞ in this notation. Moreover, if one performs r Dehn surgery on a knot K in a manifold M then this is equivalent to removing a standard neighborhood of K and replacing it with S_r.

From [7, 11], we know that any tight contact structure on S_r with convex boundary having dividing slope s is determined by a minimal path in the Farey graph from r clockwise to s with a sign on each edge except the first. (We have a similar description for S^r with dividing slope s except the path runs from r anti-clockwise to s.) Moreover, any such path gives a tight contact structure on S_r.

Moreover, as shown in [11], if we glue two contact structures on thickened tori determined by minimal signed paths in the Farey graph, the result will be tight if when shortening the concatenated paths to a shortest path, the signs of any to shortened edges are the same. The same holds when gluing a thickened torus to a solid torus, except that the contact structure remains tight when shortening the path adjacent to the unlabeled edge.

Now one can describe a lens space by taking $T^2 \times [0,1]$ and collapsing a linear foliation of slope s on $T^\times\{0\}$ and of slope r on $T^2 \times \{1\}$, denote this by L^r_s. The standard lens space $L(p,q)$ is $L^0_{-p/q}$ where $-p/q < -1$. Tight contact structures on T^r_s are determined by a minimal path in the Farey graph from s clockwise to r with signs on all the edges except the first and the last, [7, 11].

2.2. **Contact surgery.** Given a Legendrian knot L in a contact manifold (M, ξ) it has a standard neighborhood N_L. The boundary of N_L is convex with dividing curves of slope $\mathrm{tb}(L)$. If r is any rational number not equal to 0 then contact (r)-surgery is defined to be the result of removing N_L from M and replacing it with a tight contact structure on the solid torus $S_{r+\mathrm{tb}(L)}$ with lower meridional $r + \mathrm{tb}(L)$. We know from the previous section that this contact structure is defined by a minimal path in the Farey graph from $r + \mathrm{tb}(L)$ clockwise to $\mathrm{tb}(L)$ with signs on all but the first edge in the path. So "contact surgery" is not uniquely defined unless there is an edge from r to $\mathrm{tb}(L)$, and this will occur when doing contact $(\pm 1/n)$-surgery on L.

There is an algorithm for turning contact (r)-surgery on L into a sequence of contact (±1)-surgeries on some link L' obtained from L. This was first described in [4]. We describe a modified version of this here (see [17] for the connection). When $r < \mathrm{tb}(L)$ then let $r = [c_1, c_2, \ldots, c_n]$ be the continued fraction expansion. Let L' be the link obtained from L by adding a chain of $(n-1)$ unknots linked to L (that is, L_2 is a meridian to L and L_2 is a meridian of L_1 and so on) and then stabilizing the first component $|c_1 + 1|$ times and i^{th} component, for $i > 1$, $|c_i + 2|$ times. Contact (r)-surgery on L is equivalent to contact (-1)-surgery on L'. If $r > \mathrm{tb}(L)$ then let $p/q = r$. There is a smallest positive integer k such that $p/(q-kp)$ is negative. Then let L' be the Legendrian link obtained for L by taking $k+1$ Legendrian push-offs of L and contact (r)-surgery on L is the same as contact $(+1)$-surgery on k components of L' and contact $(p/(q-kp))$-surgery on the $(k+1)^{st}$ component of L'.

2.3. **The d_3-invariant.** In [8], Gompf defined an invariant of homotopy classes of plane fields that obstructed them from being homotopic over the 3-skeleton of a 3-manifold. This invariant goes by many names (differing by multiplicative or additive constants), but we will use the following normalization. Given a plane field ξ on a 3-manifold M one can find a 4-manifold X with an almost complex structure J such that $\partial X = M$ and ξ is the J-tangencies to the boundary (that is $\xi = TM \cap JTM$), see [8]. We define the d_3-invariant of ξ to be

$$d_3(\xi) = \frac{1}{4}(c_1^2(J) - 3\sigma(X) - 2(\chi(X) - 1))^1,$$

where $\sigma(X)$ is the signature of X, $\chi(X)$ is the Euler characteristic of X, and $c_1^2(J)$ is the first Chern class of TM with the almost complex structure J. To make sense of $c_1^2(J)$ we need to assume that the Euler class (or Chern class) of ξ is torsion. In that case, if one lets Q be the intersection matrix of X, then there is a class $c \in H_2(X)$ such that Qc is the Poincaré dual of $c_1(J)$. Then $c_1^2(J)$ is the intersection of c with itself. That is, if one chooses a basis for $H_2(X)$ then Q and c are represented by a matrix and a vector, and $c_1^2(J) = c^T Q c$ where c^T is the transpose of c.

Given a Legendrian link L with components $L_1 \cup \cdots \cup L_k$ and the contact structure ξ is obtained form the standard contact structure on S^3 by contact $(+1)$-surgery on the L_i for $i = 1 \ldots, l$ and contact (-1)-surgery on the L_i for $i = l+1, \ldots, k$, then the d_3 invariant can be computed by

$$d_3(\xi) = \frac{1}{4}(c^2 - 3\sigma(X) - 2(\chi(X) - 1)) + l$$

where X is the 4-manifold obtained by attaching 2-handles to B^4 along the link L with framing $\mathrm{tb}(L_i) + 1$ for $i = 1, \ldots, l$ and framing $\mathrm{tb}(L_i) - 1$ for $i = l+1, \ldots, k$, and c^2 is computed as follows. Let Q be the intersection form of X and let $\mathbf{r}$ be the column vector with i^{th} entry the rotation number of L_i. Now set $\mathbf{r}' = Q^{-1}\mathbf{r}$ and $c^2 = (\mathbf{r}')^T Q \mathbf{r}' = \mathbf{r}'^T \mathbf{r}$.

2.4. **Past results on the smooth cosmetic surgery conjecture.** Recently, Hanselman used Heegaard Floer invariants to prove the cosmetic surgery conjecture for most classes of knots and surgery coefficients.

Theorem 2.1 (Hanselman, 2023 [10])**.** *If K is a non-trivial knot in S^3 and $S^3_K(r) \cong S^3_K(r')$, then we have the following:*

- *The pair of slopes r, r' are either ± 2 or $\pm\frac{1}{n}$ for some positive integer n;*
- *if r, r' are ± 2 then $g(K) = 2$*

The theorem above implies that one only needs to check if 2 and -2 surgery or $1/n$ and $-1/n$ surgery on a knot results in diffeomorphic manifolds.

[1] We note that we use $\chi(X) - 1$ in the formula instead of $\chi(X)$. This is because with this definition, the invariant is additive under connected sum and has nice relations with other invariants.

Plamenevskaya showed that the Ozsváth-Szabó concordance invariant for a knot, τ, gives a bound on classical invariants for Legendrian knots and Ni and Wu gave a restriction on this invariant for cosmetic surgery on the knot. We state these results here.

Theorem 2.2 (Plamenevskaya 2004, [16])**.** *For a Legendrian knot L in (S^3, ξ_{std}),*

$$\mathrm{tb}(L) + |\,\mathrm{rot}(L)| \leq 2\tau(L) - 1.$$

Theorem 2.3 (Ni and Wu, 2013[14])**.** *Suppose $S^3_K(\frac{p}{q}) \cong S^3_K(\frac{p}{q'})$ with $q \neq q'$. Then $\tau(K) = 0$, where τ is the Ozsváth-Szabó concordance invariant.*

3. Cosmetic contact surgeries on unknot

To relate surgeries on the unknot, we recall the Rolfsen twist. Specifically, p/q surgery on an unknot is equivalent to $p/(q + np)$ surgery on the unknot. We will need to know why this is true, so sketch the proof now. If U is a smooth unknot in S^3, we let N_U be the solid torus neighborhood of U and S^3_U its complement in S^3. Now $S^3_U(p/q)$ is the result of removing N_U from S^3 and regluing it to S^3_U by the map

$$f = \begin{bmatrix} q' & q \\ p' & p \end{bmatrix}.$$

Now consider the diffeomorphism $\phi_n : S^3_U \to S^3_U : (\phi, (r, \theta)) \mapsto (\phi, (r, \theta + n\phi))$. Here we are identifying S^3_U with $S^1 \times D^2$ using longitude-meridian coordinates coming from U. On ∂S^3_U this map is given by

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

We can build a diffeomorphism from $S^3_U(p/q) = S^3_U \cup_f N_U$ to $S^3_U \cup_{\phi \circ f} N_U$ by sending S^3_U to S^3_U by ϕ_n and N_U to N_U by the identity map. The latter manifold is $S^3_U(p/(p+nq))$, thus establishing the diffeomorphism claimed by the Rolfsen twist.

Another useful way to see this diffeomorphism is via upper and lower meridians, as discussed in the previous section. That is, we can take $T^2 \times [0, 1]$ and then we obtain $S^3_U(p/q)$ by taking a foliation of $T^1 \times \{1\}$ by linear curves of slope 0 and collapse them to obtain a solid torus with upper meridian 0. Now if we take a foliation of $T^2 \times \{0\}$ by linear curves of slope p/q and collapse them, we get $S^3_U(p/q)$. We see that $T^2 \times \{1/2\}$ splits this manifold into two solid tori, one with upper meridian 0 (that will correspond to S^3_U) and one with lower meridian p/q (that will correspond to the reglued N_U). Now we can see the Rolfsen twist as follows: Apply the map ϕ_n to $T^2 \times [0, 1]$. Notice that this does not affect the 0 sloped curves on $T^2 \times \{1\}$ and send the p/q sloped curves on $T^2 \times \{0\}$ to curves of slope $p/(q + np)$. Thus, after collapsing the leaves of these foliations, we have a diffeomorphism between $S^3_U(p/q)$ and $S^3_U(p/(q + np))$.

One may easily check that given any surgery on U, one may perform Rolfsen twists to get an equivalent surgery with surgery coefficient $-p/q < -1$, and such a surgery coefficient is unique. Now, the first family of equivalent surgeries in the set $CS(-p/q)$ defined in the introduction are all obtained via Rolfsen twists.

To understand the second family of equivalent surgeries in the set $CS(-p/q)$, we note that one can perform (inverse) slam dunk moves to write $-p/q$ surgery on the unknot as surgery on a chain of k unknots with surgery coefficients $-a_1, -a_2, \dots - a_k$ where $[-a_1, \dots, -a_k]$ is the continued fraction of $-p/q < -1$. Now performing slam dunks in the reverse order, one obtains surgery on the unknot with surgery coefficient $[-a_k, \dots, -a_1]$ and one may check by induction on k that this is $-p/\overline{q}$ surgery where $\overline{q}$ is the inverse of q mod p.

Proof of Theorem 1.3. Suppose that L is a Legendrian unknot with $|\operatorname{rot}(L)| < |\operatorname{tb}(L) + 1|$. Let N_L be a standard neighborhood of L and S^3_L its complement. So N_L is a solid torus with lower meridian ∞ and convex boundary of slope $\operatorname{tb}(L)$ and S^3_L is a solid torus with upper meridian 0 and convex boundary with dividing slope $\operatorname{tb}(L)$ (we note that all slopes are measured with respect to longitude-meridian coordinates given by L). Now if $r < \operatorname{tb}(L)$, then any contact $(r - \operatorname{tb}(L))$-surgery on L can be achieved by a sequence of Legendrian surgeries and is hence tight. On the other hand, we claim any other contact $(r' - \operatorname{tb}(L))$-surgery for $r' \in CS(-p/q)$ will be overtwisted, thus completing the first claim of the theorem. To see this, we note the contact structure on S^3_L is determined by signs on the path in the Farey graph from $\operatorname{tb}(L)$ clockwise to -1, and because of the condition on $\operatorname{rot}(L)$, the signs cannot all be the same. (To see this, we note that the condition on $\operatorname{rot}(L)$ says that L has been stabilized positively and negatively and thus, positive and negative basic slices are added to its complement [5].) Now if $r' \in CS(-p/q)$ is not less than $\operatorname{tb}(L)$ then contact $(r' - \operatorname{tb}(L))$-surgery on L is the result of removing N_L from S^3 and gluing in a solid torus $S_{r'}$ with a tight contact structure determined by a minimal path in the Farey graph from r' to $\operatorname{tb}(L)$ with signs on all but the first edge. Note that one edge in this path goes from ∞ to $\operatorname{tb}(L)$. Taking the corresponding basic slice in $S_{r'}$ and gluing it to S^3_L we see that there will be a contact structure on $T^2 \times [0,1]$ described by a path in the Farey graph from ∞ to -1, but with edges from ∞ to $\operatorname{tb}(L)$ then $\operatorname{tb}(L) + 1$ and so on until we arrive at -1. Since this path can be shortened to a path with just one edge from ∞ to -1 and the signs on the edges are not all the same, we know that the contact structure is overtwisted.

We now turn to the case of a Legendrian knot L with $\operatorname{rot}(L) = \pm|\operatorname{tb}(L) + 1|$ (this means that L has been only stabilized positively or only stabilized negatively). We will discuss the case when $\operatorname{tb}(L) = -1$ and the other cases follow by a similar argument. Now if N_L is a standard neighborhood of L and S^3_L is its complement, then N_L is a solid torus with lower meridian ∞ and convex boundary with dividing slope -1 and S^3_L will be a solid torus with upper meridian 0 and convex boundary with dividing slope -1. Now contact $(r + 1)$ surgery on L is obtained by removing N_L and gluing in a solid torus S_r with lower meridian r and convex boundary with dividing slope -1. The contact surgery is described by a minimal path in the Farey graph from r clockwise to -1 with decorations on all but the first edge in the path.

We now note the difference between $r \in (-1, 0)$ and $r \notin (-1, 0)$. When $r \in (-1, 0)$ then S_r has a tight contact structure with dividing slope -1 and lower meridian r. So we know we can find a convex torus T in S_r with any dividing slope in $(r, -1]$ (recall this notation means any rational number in the Farey graph that is clockwise of r and anti-clockwise of -1), thus there is a convex torus T with dividend slope 0. Now, a Legendrian divide on T will bound a disk outside the solid torus T bounds in S_r and thus, the contact structure will be overtwisted. However, if $r \notin (-1, 0)$ then some contact $(r + 1)$-surgery on L will be tight (since it will correspond to a minimal decorated path in the Farey graph). Thus, we need to consider the cases separately. We begin with $r \notin (-1,0)$.

Using our description of surgery using upper and lower meridians recalled just before this proof we see that S_r is just $T^2 \times [0,1/2]$ with curves of slope r collapsed on $T^2 \times \{0\}$ and S^3_L is $T^2 \times [1/2,1]$ with curves of slope 0 collapsed on $T^2 \times \{1\}$. See Figure 1. Now if r'

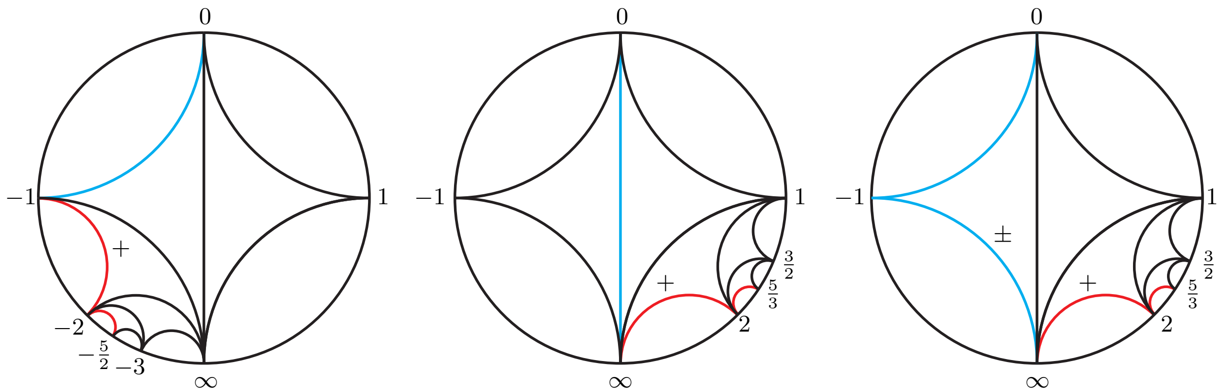

FIGURE 1. On the left we see the result of contact $(-3/2)$-surgery on L. The blue path describes the contact structure on S^3_L and the red path describes the contact structure on S_r. In the middle figure, we see the same manifold after applying the coordinate change ϕ_1. On the right, we see that the image of the S^3_L can be split into a solid torus (with upper meridian 0 and dividing slope -1) and a thickened torus (with dividing slopes -1 and ∞). Attaching the thickened torus to the image of S_r under ϕ_1 shows that this manifold is the result of contact $(8/5)$-surgery on L and different such surgeries are giving this manifold as there are two choices for the sign describing the contact structure on the thickened torus.

is obtained from r by a Rolfsen twist given by ϕ_n and $r' \notin (-1,0)$, then we simply apply ϕ_n to $T^2 \times [0,1]$ and collapse curves of slope 0 on $T^2 \times \{1\}$ and curves of slope r' on $T^2 \times \{0\}$. Now the torus $\partial N_L = T^2 \times \{1/2\}$ will map to a convex torus with dividing slope $1/(n-1)$. So under this diffeomorphism, the torus S_r with dividing slope -1 and meridional slope r will become a solid torus $S_{r'}$ with dividing slope $1/(n-1)$ and meridional slope r'. Its complement, which we call C_L, will be a solid torus with upper meridian 0 and dividing slope $1/(n-1)$. We note that the path in the Farey graph describing the contact structure on S_r becomes, under the diffeomorphism, a path describing a unique contact structure on $S_{r'}$. Moreover, since $1/(n-1)$ and 0 share an edge in the Farey graph, we know there is a unique tight contact structure on C_L and it is a neighborhood of a Legendrian knot L'. Inside of C_L there is a convex torus T with dividing slope -1 (since $-1 \in [1/(n-1), 0)$). The torus T will split C_L into two pieces. One piece is simply S^3_L and the other piece is a thickened torus $T \times I$ with dividing curves of slope -1 and $1/(n-1)$. Notice that there is a path from -1 to $1/(n-1)$ in the Farey graph with n edges and this path forms a continued fraction block. So there are $n+1$ possible contact structures on $T^2 \times I$ but all of them when glued to S^3_L become the same since there is a unique tight contact structure on C_L (note S^3_L

can be thought of as a neighborhood of an n fold stabilization of L', the Legendrian knot determined by C_L). Thus, there are $n+1$ tight contact structures on the complement of S^3_L that will give contact structures on $S^3_L(r')$ that are contactomorphic to the one on $S^3_L(r)$.

Now, if $r' \in CS(-p/q)$ is not obtained from r by a Rolfsen twist, then it is related to r by slam dunks, as discussed just before this proof. (Technically, one might have to perform Rolfsen twists on r to ensure it is less than -1, then perform the slam dunk procedure, and then perform more Rolfsen twists to get r', but since we already understand the Rolfsen twists affect on contact surgery we can ignore this issue.) Since we know all the contact structures on a lens space come from contact surgery on L, we know that under the diffeomorphism from $L(p,q)$ to $L(p,\overline{q})$ any contact structure on $L(p,q)$ comes from some contact $(-p/\overline{q}+1)$ surgery on L. This completes the argument for surgery coefficients $r \notin (-1,0)$.

Now, when $r \in (-1,0)$, we can give the same argument to see there are equivalent contact surgeries for other $r' \in CS(-p/q)$ in $(-1,0)$, just in this case, as noted above, the contact structures will be overtwisted.

We are finally left to consider the case of a Legendrian knot L with $|\operatorname{rot}(L)| < |\operatorname{tb}(L)+1|$ and we are doing contact $(r-\operatorname{tb}(L))$-surgery for r not less than $\operatorname{tb}(L)$. As noted at the start of this proof, all these contact structures are overtwisted and arguments similar to those above will give contactomorphic contact $(r'-\operatorname{tb}(L))$-surgery on L for any $r' \in CS(-p/q)$ ith r' not less than $\operatorname{tb}(L)$. □

4. LIMITING CONTACT COSMETIC SURGERIES

In this section we prove Theorem 1.2. It will be a direct consequence of Propositions 4.1, 4.3, and 4.5, which deal with the case of Legendrian knots with $\operatorname{tb} = -1, -2$, and less than -2, respectively, and the fact that Theorem 2.2 and 2.3 show that the contact cosmetic surgery conjecture holds for Legendrian knots with $\operatorname{tb} \geq 0$.

Proposition 4.1. *The contact cosmetic surgery conjecture holds for Legendrian knots with Thurston-Bennequin invariant -1 except possibly for ± 2 surgery on a Legendrian knot L in a prime knot type K with $\overline{\operatorname{tb}}(K) = -1$, $\tau(K) = 0$, $g(K) = 2$, and $g_4(K) = 0$. Moreover, L is Lagrangian slice and hence K is quasi-positive.*

Proof. Let L be a Legendrian knot with $\operatorname{tb} = -1$. Because we know that $\tau(L)$ must be zero if L admits a cosmetic surgery by Theorem 2.3 above and we have the Bennequin type bound in Theorem 2.2 above, we see that the rotation number of L must be 0.

We begin by considering $\pm 1/n$ surgery on L. We note that we cannot consider $n = 1$ since -1 surgery on L would correspond to contact (0) surgery and this is not well-defined. So we assume $n \geq 2$. Surgery diagrams for these contact surgeries are shown in Figure 2.

We begin with $n = 2$ and let $X_{\pm 2}$ be the 4-manifolds given by the surgery diagram in the figure for $\pm 1/2$ surgery. The intersection matrix for X_{-2} is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

One may easily compute that the Euler characteristic is $\chi(X_{-2}) = 3$ and the signature is $\sigma(X_{-2}) = 0$. Moreover, since the rotation number of L is zero we see that $c_2^2(X_{-2}) = 0$. To

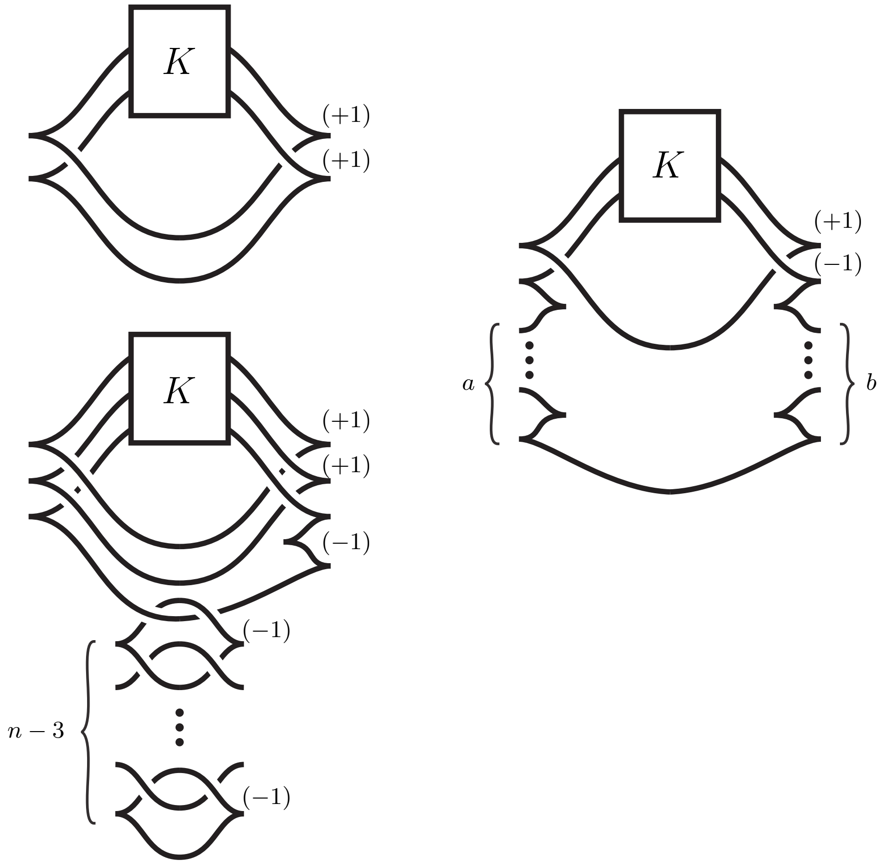

FIGURE 2. For a Legendrian knot K with $\mathrm{tb} = -1$ we see a smooth $-1/2$ surgery (that is contact $(1/2)$ surgery) and $-1/n$ surgery for $n > 2$ (that is contact $((n-1)/n)$ surgery) on the upper and lower left, respectively, and a smooth $1/n$ surgery (that is a contact $((n+1)/n)$ surgery) on the right. On the right a and b are non-negative integers so that $a + b = n$ (that is the second knot is the Legendrian push-off of the first with n stabilizations of one sign or the other).

compute the d_3 invariant, we note that there were two $+1$ contact surgeries in the diagram so

$$\begin{aligned} d_3(\partial X_{-2}) = &\frac{1}{4}\left(c_1^2(X_{-2}) - 3\sigma(X_{-2}) - 2(\chi(X_{-2}) - 1)\right) + 2 \\ &\frac{1}{4}(0 - 0 - 4) + 2 = 1. \end{aligned}$$

We now consider X_2, which has intersection matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & -4 \end{bmatrix}$$

One may easily compute that the Euler characteristic is $\chi(X_{-2}) = 3$ and the signature is $\sigma(X_{-2}) = 0$. The bottom Legendrian knot in the surgery diagram has rotation number either $-2, 0$, or 2. On may check in all cases that $c_1^2(Q) = 0$. Noting that there is a single contact $+1$ surgery in the diagram of X_2 we see that

$$d_3(\partial X_2) = \frac{1}{4}(0 - 0 - 4) + 1 = 0.$$

Thus, $\pm 1/2$ surgery on a Legendrian with $\mathrm{tb} = -1$ can never give a cosmetic surgery.

We now turn to the case of $\pm 1/n$ surgery for $n > 2$. Denote the 4-manifold constructed in the surgery diagram for $\pm 1/n$ surgery by $X_{\pm k}$. The intersection matrix of X_{-n} is given by

$$\begin{bmatrix} 0 & -1 & -1 & & & \\ -1 & 0 & -1 & & & \\ -1 & -1 & -3 & -1 & & \\ & & -1 & -2 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & -2 \end{bmatrix}_{n\times n}$$

One may easily compute that the Euler characteristic is $\chi(X_{-n}) = n + 1$. To compute the signature, we note that the 4-manifold X on the left of Figure 3 has the same signature as X_{-n}. The manifold X' in the middle of the figure is obtained from X by a $+1$ blowup. Thus $\sigma(X') = \sigma(X) + 1$. The manifold X'' on the right of the figure is obtained from X' by two $+1$ blowdowns. Thus $\sigma(X'') = \sigma(X') - 2$. It is clear that $\sigma(X'') = -n + 1$ and thus $\sigma(X_{-n}) = \sigma(X) = -n + 2$. In Appendix A we will prove the following lemma.

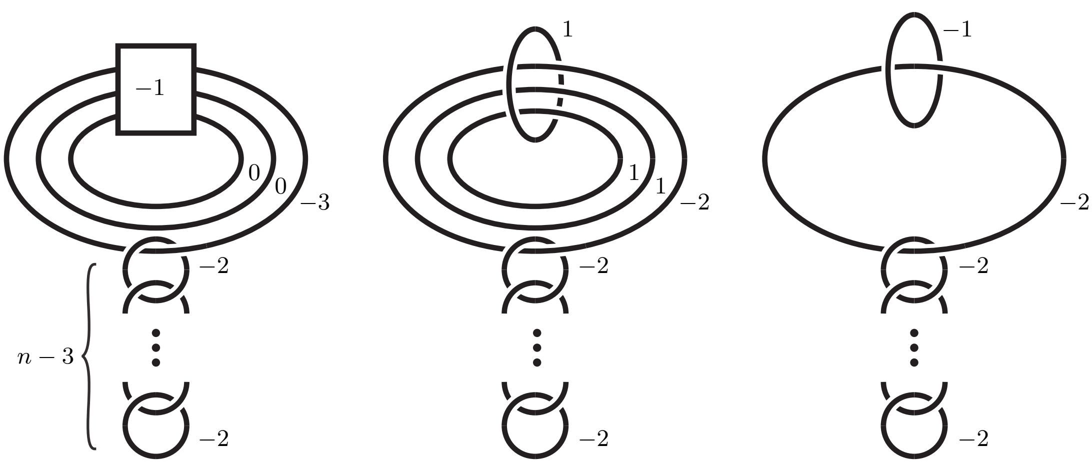

FIGURE 3. Computing the signature for X_n.

Lemma 4.2. *For any choice of stablization in the definition of X_{-n} we have $c_1^2(X_{-n}) = 2 - n$.*

Now noting that there are two contact $(+1)$ surgery in the surgrery diagram for X_{-n} we see

$$d_3(\partial X_{-n}) = \frac{1}{4}((2 - n) - 3(-n + 2) - 2n) + 2 = 1$$

We now consider X_n. To this end, we note the intersection form of X_n is

$$\begin{bmatrix} 0 & -1 \\ -1 & -2-n \end{bmatrix}$$

So we easily see that $\chi(X_n) = 3$, $\sigma(X_n) = 0$, and as above, we see, independent of the rotation number of the bottom Legendrian in the surgery diagram, that $c_1^2(Q) = 0$ and thus

$$d_3(\partial X_n) = \frac{1}{4}(0-0-4)+1 = 0.$$

Since this does not agree with $d_3(\partial X_{-n})$, we see that there are no cosmetic surgeries on L with surgery coefficients $\pm 1/n$.

We now turn to ± 2 surgeries on L. These are given by the surgery diagram in Figure 4. One may easily check that the d_3-invariant for both contact structures in $1/4$, so

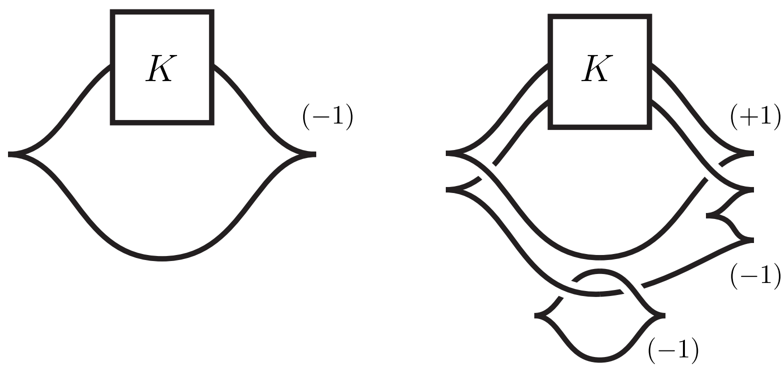

FIGURE 4. For a Legendrian knot K with $\mathrm{tb} = -1$ we see a smooth -2 surgery (that is contact (-1) surgery) on the left and a smooth 2 surgery (that is a contact $(+3)$ surgery) on the right (the stabilization can be either positive or negative.

it is possible from this computation that contact cosmetic surgeries are possible. This is not surprising given that for the $\mathrm{tb} = -1$ unknot, these surgeries both produce the tight contact structure on $L(2,1)$ and hence are contactomorphic (and the d_3 computations for a general Legendrian knot with $\mathrm{tb} = -1$ and rotation number 0 will yield the same result). So the best we can do in this situation is to limit the possible knots that might admit cosmetic surgeries. One may easily check that the d_3-invariant for both contact structures is $1/4$, so it is possible from this computation that contact cosmetic surgeries are the same. This is not surprising given that for the $\mathrm{tb} = -1$ unknot, these surgeries both produce the tight contact structure on $L(2,1)$ and hence are contactomorphic (and the d_3 computations for a general Legendrian knot with $\mathrm{tb} = -1$ and rotation number 0 will yield the same result). So the best we can do in this situation is to limit the possible knots that might admit cosmetic surgeries.

If L is in the knot type K then we know from Theorem 2.3 above that $\tau(K) = 0$ and from that we know from Theorem 2.2 (and our assumption that $\mathrm{tb}(L) = -1$) that $\overline{\mathrm{tb}}(K) = -1$. Moreover, K is prime by [19] and has Seifert genus 2 by Theorem 2.1, and since contact $(+3)$ surgery on L is contactomorphic to contact (-1) surgery on L we know that it must be symplectically fillable and then [13, Corollary 1.3] implies that K is quasi-positive. In addition Proposition 1.7 in [13] says that if a positive contact surgery on L is symplectically fillable, then the minimal possible smooth surgery coefficient that could be symplectically fillable is in $(2g_4(K), 4g_4(K)]$ if $g_4(K) > 0$, where $g_4(K)$ is the minimal genus of a surface in B^4 with boundary K. Since our K has Seifert genus 2 we know $g_4(K)$ is $0, 1$, or 2. Since 2 is not in $(2, 4]$ or $(4, 8]$ we see that we must have $g_4(K) = 0$. Now Corollary 2.10 in [13] implies that L is Lagrangian slice. □

We now turn to Legendrian knots with Thurston-Bennequin invariant -2.

Proposition 4.3. *The contact cosmetic surgery conjecture holds for Legendrian knots with Thurston-Bennequin invariant -2.*

Proof. Let L be a Legendrian knot with Thurston-Bennequin invariant -2. From Theorems 2.3 and 2.2 we see that $\text{tb}(L) = -2$ and that the rotation number of L must be ± 1.

Recall we only need to check that smooth ± 2 and $\pm 1/n$ surgeries do not yield cosmetic surgeries. Note that smooth -2 surgery on L is a contact (0) surgery and so is not well-defined, so we are left to check $\pm 1/n$ surgeries. We see surgery diagrams for these in Figure 5.

We begin by considering the case when $n = 1$. Let $X_{\pm 1}$ denote the manifold obtained from the surgery diagram for ± 1 surgery on L. We see that the intersection matrix of X_{-1} is $\begin{bmatrix} -1 \end{bmatrix}$ and hence $\chi(X_{-1}) = 2$ and $\sigma(X_{-1}) = -1$. Whether or not the rotation number of L is -1 or 1 we see $c_1^2(X_{-1}) = -1$. Given that there is one contact $+1$ surgery in the diagram, we can compute

$$\begin{aligned} d_3(\partial X_{-1}) &= \frac{1}{4}\left(c_1^2(X_{-2}) - 3\sigma(X_{-2}) - 2(\chi(X_{-2}) - 1)\right) + 1 \\ &= \frac{1}{4}(-1 - 3(-1) - 2(1)) + 1 = 1. \end{aligned}$$

Moving to X_1 we note the intersection matrices of all the X_n are similar

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & -4 & -1 \\ 0 & -1 & -n-1 \end{bmatrix}$$

It is clear that $\chi(X_1) = 4$. One may also compute that $\sigma(X_1) = -1$ and the inverse of the matrix is

$$\begin{bmatrix} 4n+3 & -2(n+1) & 2 \\ -2(n+1) & n+1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Using this and the fact that the rotation number of the top Legendrian knot in the diagram is ± 1, of the middle Legendrian knot is ± 2 or 0 (and the sign on the 2 must match that of the 1), and of the bottom Legendrian knot is 0, we can compute that $c_1^2(X_1)$ is 7 or -1 and hence $d_3 = 2$ or 0. In all these cases we see that this does not agree with $d_3(X_{-1})$, and so ± 1 surgery cannot be a contact cosmetic surgery slope.

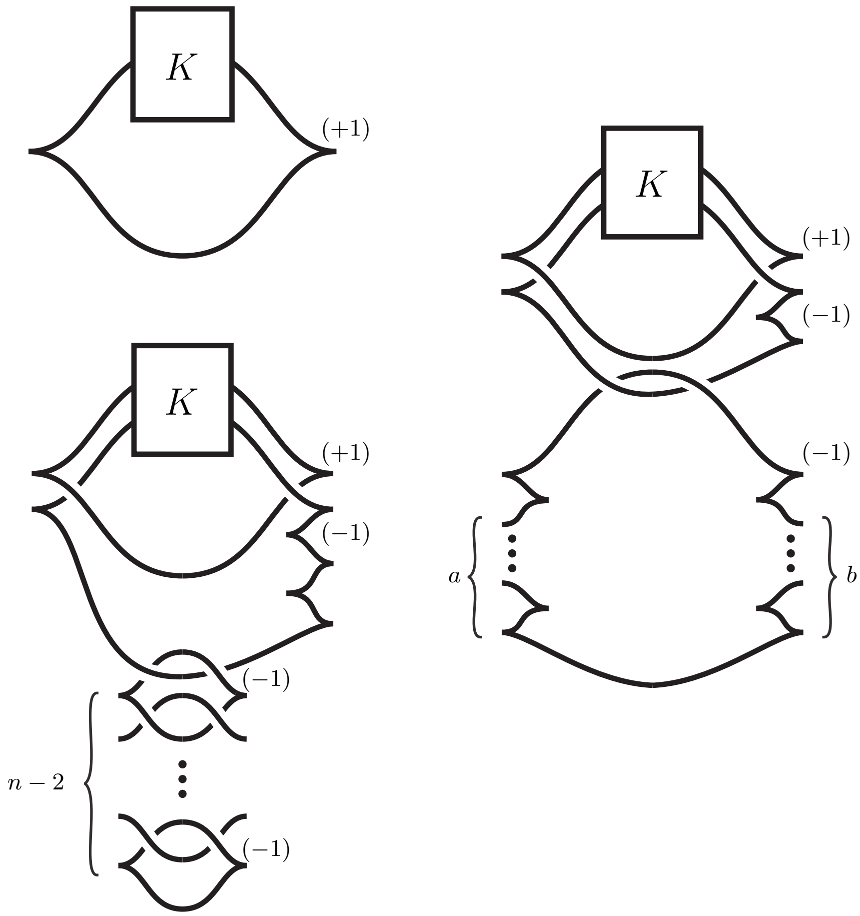

FIGURE 5. For a Legendrian knot K with $\mathrm{tb} = -2$ we see a smooth -1 surgery (that is contact (1) surgery) and $-1/n$ surgery for $n > 1$ (that is contact $((2n-1)/n)$ surgery) on the upper and lower left, respectively (the stabilizations in the lower diagram can be of any sign), and a smooth $1/n$ surgery (that is a contact $((2n+1)/n)$ surgery) on the right. On the right a and b are non-negative integers so that $a+b = n-1$ (that is the second knot is the Legendrian push-off of the first with $n-1$ stabilizations of one sign or the other).

We now turn to the case of $\pm 1/n$ surgery for $n \geq 2$ and let $X_{\pm n}$ be the 4-manifold given in the figure for $\pm 1/n$ surgery on L. The intersection form of X_{-n} is given by

$$\begin{bmatrix} -1 & -2 & & & \\ -2 & -5 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2 \end{bmatrix}_{n\times n}$$

It is clear that $\chi(X_{-n}) = n + 1$, and in Appendix A, we will prove the following lemma.

Lemma 4.4. *The intersection from of X_{-n} is negative definite, so $\sigma(X_{-n}) = -n$ and $c_1^2(X_{-n}) = -9n + 8$ or $-n$.*

Thus, we can compute

$$d_3(X_{-n}) = \frac{1}{4}(-n + 3n - 2n) + 1 = 1$$

or

$$d_3(X_{-n}) = \frac{1}{4}((-9n + 8) + 3n - 2n) + 1 = -2n + 3$$

We now consider X_n. The intersection matrix of X_n is given above, from which we compute that $\chi(X_n) = 4$ and $\sigma(X_n) = -1$. The possible rotation vectors are

$$\mathbf{r} = \begin{bmatrix} \pm 1 \\ \pm 2 \\ i \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \pm 1 \\ 0 \\ i \end{bmatrix}$$

where $i = n - 1, n - 3, \ldots, -n + 1$. We note that in the first vector, the signs must be the same. This leads to $c_1^2(X_n) = -1$ or $3 + 4n \pm 4i$. Thus, we can compute

$$d_3(X_n) = \frac{1}{4}(-1 + 3 - 6) + 1 = 0$$

when $\mathbf{r} = \left[\pm 1, \pm 2, i\right]^T$ and

$$d_3(X_n) = \frac{1}{4}(3 + 4n \pm 4i + 3 - 6)) + 1 = n \pm i + 1$$

when $\mathbf{r} = \left[\pm 1, 0, i\right]^T$.

We note that these invariants are always even (recall the restriction on i), thus they do not agree with the d_3-invariants for X_{-n} and there are not $\pm 1/n$ cosmetic surgeries on a Legendrian knot with $\mathrm{tb} = -2$. □

We finally consider Legendrian knots with $\mathrm{tb} < -2$.

Proposition 4.5. *The contact cosmetic surgery conjecture holds for Legendrian knots with Thurston-Bennequin invariant less than -2.*

Proof. Recall we only need to consider the smooth surgeries ± 2 and $\pm 1/n$ by Theorem 2.1. We begin with the ± 2 case. The contact surgery diagrams for these surgeries are given in Figure 6. We denote the 4-manifold in the diagram by $X_{\pm 2}$, where the Thurston-Bennequin invariant of the Legendrian knot will be clear from context.

It will be convenient to consider the $\mathrm{tb} = -3$ case first. Recall from the bound in Theorem 2.2 we know that the rotation number of the Legendrian knot L must be ± 2 or 0. In this case the intersection matrix of X_{-2} is

$$\left[-2\right].$$

From this we see that $\chi(X_{-2}) = 2$ and $\sigma(X_{-2}) = -1$. One may also compute $c_1^2(X_{-2}) = \frac{-r^2}{2}$ and hence $c_1^2(X_{-2}) = -2$ or 0. This gives

$$d_3(\partial X_{-2}) = 3/4 \text{ or } 5/4.$$

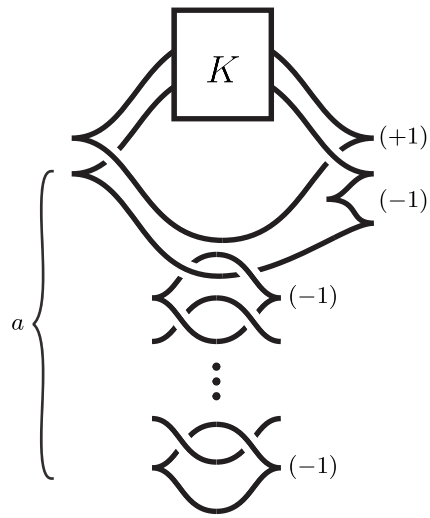

FIGURE 6. For a Legendrian knot K with $\mathrm{tb} = -k < -2$ we see a smooth -2 surgery (that is contact $(k-2)$ surgery) when $a = k-3$ and smoth 2 surgery (that is contact $(k+2)$ surgery) when $a = k+1$.

Now the intersection from of X_2 is

$$\begin{bmatrix} -2 & -3 & & & \\ -3 & -5 & -1 & & \\ & -1 & -2 & -1 & \\ & & -1 & -2 & -1 \\ & & & -1 & -2 \end{bmatrix}.$$

One may easily compute that $\sigma(X_2) = -3$ and $\chi(X_2) = 6$. One may also compute that $c_1^2(X_2) = -2, 4$, or 14 and hence

$$d_3(\partial X_2) = 1/4, 7/4, \text{ or } 17/4.$$

Thus, there is no contact cosmetic surgery in this case.

We now consider ± 2 surgery on a Legenrian knot L with $\mathrm{tb}(L) = -k < -3$. We first note that Theorems 2.2 and 2.3 imply that the rotation number of L must be in the range $k-1, k-3, \ldots, -k+1$. Let $X_{\pm 2}$ be the 4-manifold in the surgery diagram for ± 2 surgery on L given in Figure 6. The intersection matrix for X_{-2} is

$$\begin{bmatrix} -k+1 & -k & & & \\ -k & -k-2 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2 \end{bmatrix}_{k-2\times k-2}$$

So we see that $\chi(X_{-2}) = k-1$. Denote the rotation number of the link L by i, for $i = k-1, k-3\ldots, -k+1$ and the rotation number of the stabilized knot in Figure 6 is $i \pm 1$

and the other knots in the surgery diagram have rotation number 0. In Appendix A, we will prove the following lemma.

Lemma 4.6. *The signature of X_{-2} is $\sigma(X_{-2}) = -k+2$ and*

$$c_1^2(X_{-2}) = -\frac{1}{2}i^2 \pm (k-3)i + \frac{1}{2}(-k^2+4k-3),$$

where $i = k-1, k-3, \dots, -k+1$.

From this we can compute

$$d_3(\partial X_{-2}) = \frac{1}{4}\left(-\frac{1}{2}i^2 \pm (k-3)i + \frac{1}{2}(-k^2+6k-7)\right) + 1$$

We now turn to 2 surgery on L. The intersection matrix for X_2 is

$$\begin{bmatrix} -k+1 & -k & & & \\ -k & -k-2 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2 \end{bmatrix}_{k+2 \times k+2}$$

So we see that $\chi(X_2) = k+3$. Denote the rotation number of the link L by i, for $i = -k+1, \dots, k-1$ and the rotation number of the stabilized knot in Figure 6 is $i \pm 1$ and the other knots in the surgery diagram have rotation number 0. In Appendix A we will prove the following lemma.

Lemma 4.7. *The signature of X_2 is $\sigma(X_2) = -k$ and*

$$c_1^2(X_2) = \frac{1}{2}i^2 \pm (k+1)i + \frac{1}{2}(k^2-1),$$

where $i = k-1, k-3, \dots, -k+1$.

From this, we can compute

$$d_3(\partial X_2) = \frac{1}{4}\left(\frac{1}{2}i^2 \pm (k+1)i + \frac{1}{2}(k^2+2k-9)\right) + 1$$

Using the quadratic equation (or Mathematica) to slove $d_3(\partial X_{-2}) = d_3(\partial X_2)$ for i yields on integer solutons. So there are no contact cosmetic surgeries with smooth surgery coefficients ± 2 when $\mathrm{tb}(L) < -3$.

We now turn to $\pm 1/n$ surgeries on L. They are indicated in Figure 7. It will be convenient to consider ± 1 surgeries first and denote the 4-manifolds given in the figure by $X_{\pm 1}$.

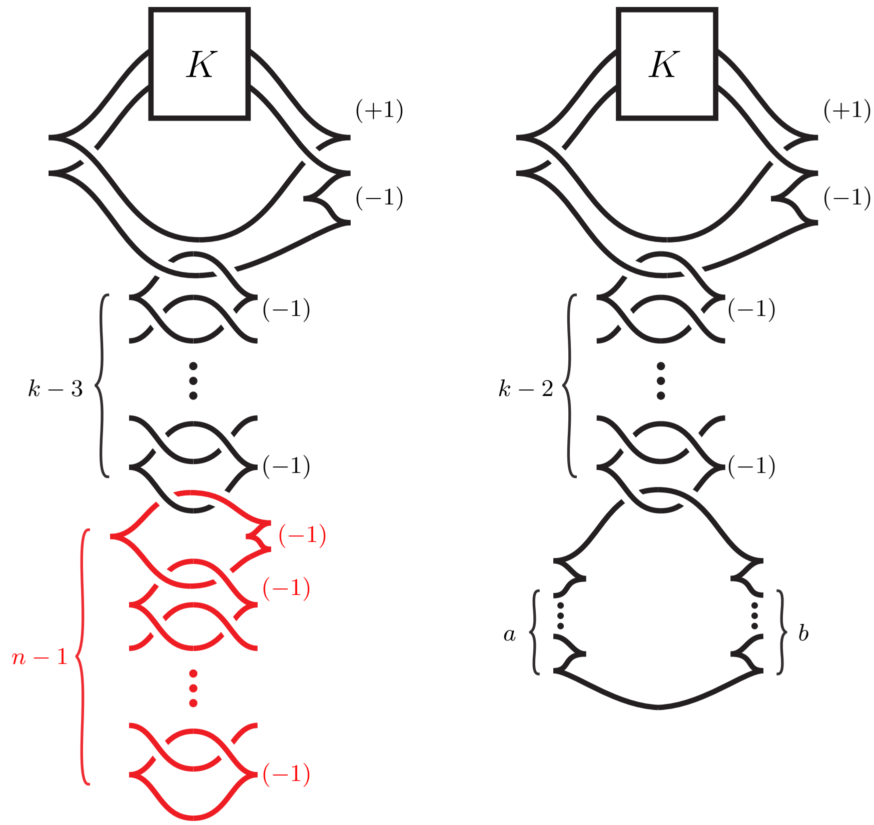

FIGURE 7. For a Legendrian knot K with $\mathrm{tb} = -k < -2$ we see a smooth $-1/n$ surgery (that is contact $((kn-1)/n)$ surgery) on the left, (the stabilizations in the lower diagram can be of any sign). The red portion of the diagram should be ignored for $n = 1$. On the right, we see smooth $1/n$ surgery (that is a contact $((kn+1)/n)$ surgery) on the right. On the right, a and b are non-negative integers so that $a + b = n - 1$ (that is the second knot is the Legendrian push-off of the first with $n - 1$ stabilizations of one sign or the other).

We begin with -1 surgeries. From the surgery diagram, we see that the intersection matrix is

$$\begin{bmatrix} -k+1 & -k & & & \\ -k & -k-2 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2 \end{bmatrix}_{l\times l}$$

where $l = k - 1$. From this we can easily determine that $\chi(X_{-1}) = k$ and in Appendix A we will prove the following lemma.

Lemma 4.8. *The signature of X_{-1} is $\sigma(X_{-1}) = -k+1$ and*

$$c_1^2(X_{-1}) = -i^2 \pm 2(k-2)i - k^2 + 3k - 2,$$

where $i = k-1, k-3, \ldots, -k+1$.

Thus we can compute

$$d_3(\partial X_{-1}) = \frac{1}{4}\left(-i^2 \pm 2(k-2)i - k^2 + 4k - 3\right) + 1.$$

Turning to $+1$ surgery we see the intersection matrix of X_1 is given by the above matrix with $l = k+1$. From this we see that $\chi(X_1) = k+2$ and in Appendix A we will prove the following lemma.

Lemma 4.9. *The signature of X_1 is $\sigma(X_1) = -k+1$ and*

$$c_1^2(X_1) = i^2 \mp 2ki + k^2 - k,$$

where $i = k-1, k-3, \ldots, -k+1$.

Thus we can compute

$$d_3(\partial X_1) = \frac{1}{4}\left(i^2 \mp 2ki + k^2 - 5\right) + 1.$$

Using the quadratic equation (or Mathematica) to slove $d_3(\partial X_{-1}) = d_3(\partial X_1)$ for i yields on integer solutons. So there are no contact cosmetic surgeries with smooth surgery coefficients ± 1 when $\mathrm{tb}(L) < -2$.

We finally turn to $\pm 1/n$ surgery when $n > 1$. Let L be a Legendrian knot with $\mathrm{tb}(L) = -k < -2$. We let $X_{\pm n}$ denote the 4-manifold described in Figure 7 with boundary $\pm 1/n$ surgery on L.

The intersection matrix of X_{-n} is given by

$$\begin{bmatrix} -k+1 & -k & & & & & & & & \\ -k & -k-2 & -1 & & & & & & & \\ & -1 & -2 & -1 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & -1 & -2 & -1 & & & & \\ & & & & -1 & -3 & -1 & & & \\ & & & & & -1 & -2 & -1 & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & -1 & -2 & -1 \\ & & & & & & & & -1 & -2 \end{bmatrix}_{k+n-2 \times k+n-2}$$

where the -3 occurse in the (k,k) entry. So it is clear that $\chi(X_{-n}) = k+n-1$. From Theorem 2.3 and 2.2 we know that the rotation number of a Legendrian with $tb = -k < -2$ is $i = k-1, k-3, \ldots, -k+1$. So the rotation vector for the surgery diagram is $\mathbf{r} = \left[i, i \pm 1, 0, \cdots 0, j, 0 \cdots, 0\right]^T$ where the second j is in the k^{th} entery and takes teh values ± 1. In Appendix A we will prove the following lemma.

Lemma 4.10. *The signature of X_{-n} is $\sigma(X_{-n}) = -k - n + 2$ and*

$$c_1^2(X_{-n}) = -n+1+(1-k)(-1+(k-1)n)-j2(-1)^k(n-1)(-i\pm k\mp 1)\pm 2i(-1+n(k-1))-ni^2.$$

Thus we can compute, for $j = 1$

$$\begin{aligned} d_3(\partial X_n) =& \frac{1}{4}\Big(-n+1+(1-k)(-1+(k-1)n) - j2(-1)^k(n-1)(-i\pm k\mp 1)+ \\ & \pm 2i((k-1)n-1) - ni^2 + k + n - 2\Big) + 1 \end{aligned}$$

The intersection matrix of X_n is given by

$$\begin{bmatrix} -k+1 & -k & & & & \\ -k & -k-2 & -1 & & & \\ & -1 & -2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & -2 & -1 \\ & & & & -1 & -n-1 \end{bmatrix}_{k+1\times k+1}$$

So it is clear that $\chi(X_n) = k+2$. Notice that the rotation vector for X_n is

$$\mathbf{r} = \big[i, i\pm 1, 0, \cdots, 0, s\big]^T.$$

In Appendix A we will prove the following lemma.

Lemma 4.11. *The signature of X_n is $\sigma(X_n) = -k + 1$ and*

$$c_1^2(X_n) = -1 + k - 2kn\left(2i^2 \pm 3i + 1\right) + nk^2(1\pm 2i)^2 + 2s(-1)^k(i \mp k \pm 1) + n(i\pm 1)^2 \mp 2i.$$

Thus, we can compute d_3

$$\begin{aligned} d_3(\partial X_n) =& \frac{1}{4}\Big(-1 + k - 2kn\left(2i^2 \pm 3i + 1\right) + nk^2(1\pm 2i)^2 + 2s(-1)^k(i\mp k\pm 1)+ \\ & n(i\pm 1)^2 \mp 2i + k + 5\Big) + 1 \end{aligned}$$

We use Mathematica to slove $d_3(\partial X_{-n}) = d_3(\partial X_n)$. For ease, we split the computation into cases based on the parity of k, the sign of j, and the sign of the stabilizations of the push-off of the Legendrian knot (these are the $\pm$ in the formulas). Consider the case when k is even, $j = 1$, and the stabilizations are positive. Solving for n we get

$$n = \frac{ks + k - is - i - s + 1}{2k^2i^2 + 2k^2i + k^2 - 2ki^2 - 4ki - k + i^2 + i}.$$

We then impose the condition that $-n < s < n$ (recall that s is the rotation number of a Legendrian unknot with $\text{tb} = -n$ in the surgery diagram and for Legendrian unknots $\text{tb} < \text{rot} < -\text{tb}$). There are no integral solutions in our range. For the other cases, a similar analysis also shows there are no solutions.

So there are no contact cosmetic surgeries with smooth surgery coefficients $\pm\frac{1}{n}$ when $\text{tb}(L) < -2$. $\square$

Appendix A. Linear algebra computations

We recall a few facts about matrices. Given an invertible $n \times n$ matrix A with entries $a_{i,j}$, denote its inverse by B with entries $b_{i,j}$. We can compute the entry $b_{i,j}$ as follows

$$b_{i,j} = (-1)^{i+j} \frac{\det A_{j,i}}{\det A}$$

where $A_{i,j}$ is the (i, j) minor of A, that is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column, see [12, Section 0.8.2].

The following matrices will frequently appear in our discussion:

$$I_n = \begin{bmatrix} -2 & -1 & & \\ -1 & -2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -2 \end{bmatrix}_{n\times n} \quad \text{and} \quad I'_n = \begin{bmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & -2 & -1 & & \\ 0 & -1 & -2 & \ddots & \\ \vdots & & \ddots & \ddots & -1 \\ 0 & & & -1 & -2 \end{bmatrix}_{n\times n}$$

One may easily compute that $\det I'_n = -\det I_{n-1}$.

Lemma A.1. *The determinant of I_n is*

$$\det I_n = (-1)^n (n+1).$$

Proof. Clearly, $\det I_1 = -2$ and $\det I_2 = 3$. Using the formula

$$\det I_n = -2 \det I_{n-1} - (-1) \det I'_{n-1},$$

One may easily establish the formula via induction. □

We will need to use the following lemma.

Lemma A.2. *Given a matrix of the form*

$$M = \begin{bmatrix} A & B \\ B^T & I_n \end{bmatrix}$$

where I_n is the matrix above

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 \end{bmatrix}$$

is a $2 \times n$ matrix. Then

$$\det M = (-1)^n ((a(c+1) - b^2)n + (ac - b^2)).$$

Thus, if $(a(c+1) - b^2)n + ac - b^2$ is positive, then M is negative definite.

Proof. If D is an invertible matrix, then the determinant of a matrix of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is $\det(D)\det(A - BD^{-1}C)$, where A and D are square matrices of size i and $n-i$ and B is $(i \times n-i)$ matrix and C is $(n-i \times i)$ matrix, see [1, Page 114]. From this, we see that

$$\det M = \det I_n \det(A - BI_n^{-1}B^T).$$

One may easily see that $B^T I_n B$ is the matrix $\begin{bmatrix} 0 & 0 \\ 0 & a_{1,1} \end{bmatrix}$ where $a_{1,1}$ is the upper-left entry in I_n^{-1}. From the formula recalled at the beginning of this appendix, we see that $a_{1,1} = \det I_{n-1}/\det I_n = -n/(n+1)$. Thus

$$\det M = (-1)^n(n+1)[(a(c+n/(n+1)) - b^2] = (-1)^n((a(c+1)-b^2)n + (ac-b)n)$$

as claimed.

For the second claim, we recall Sylvester's criterion [12, Theorem 7.2.5] says that a matrix M is negative definite if and only if $(-1)^k \det M_k > 0$ where M_k is the k^{th} principal minor, that is the $k \times k$ submatrix in the upper left corner of A. □

We now move to the proof of Lemma 4.2. Recall this lemma claims that the intersection matrix of X_{-n} shown on Page 11 has signature $n-2$ and $c_1^2(X_{-n}) = 2-n$.

Proof of Lemma 4.2. We begin by showing that $\det Q = (-1)^{n-1}$. We use the formula for the determinate of a block matrix as in Lemma A.2, where

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & -3 \end{bmatrix}$$

$D = I_{n-3}$ and B is as in the lemma.

We note the rotation vector for X_{-n} is $\mathbf{r} = \begin{bmatrix} 0, 0, \pm 1, 0, \cdots, 0 \end{bmatrix}^T$. For any $n \times n$ matrix A it is clear that $\mathbf{r}^T A \mathbf{r}$ is simply $(\det A)^{-1} a_{3,3}$ where $a_{i,j}$ is the (i,j) entry of A. If we denote the intersection matrix of X_{-n} by Q then $c_1^2(X_{-n})$ is the $(3,3)$ entry of Q^{-1} and from the formula above this is simply $(-1)^{n-1}\det Q_{3,3}$. Since $Q_{3,3}$ is a block diagonal matrix we easily see that $\det Q_{3,3} = (-1)\det I_{n-3} = (-1)^n(n-2)$ and thus $c_1^2(X_{-n}) = 2-n$. □

We now turn to the proof of Lemma 4.4 which says the intersection matrix of X_{-n} (for $\mathrm{tb} = -2$ Legendrian knots) shown on Page 15 is negative definite and $c_1^2(X_{-n}) = -9n+8$ or $-n$.

Proof of Lemma 4.4. The fact that the intersection matrix fo X_{-n} is negative definite follows directly from Lemma A.2.

The rotation vector for X_{-n} is $\mathbf{r} = \begin{bmatrix} i, j, 0, \ldots, , 0 \end{bmatrix}^T$ where $i = \pm 1$ and if $i = 1$ then j could be $3, 1$, or -1 while if $i = -1$ then j could be $-3, -1$, or 1. For any symmmetric matrix A the quantity $\mathbf{r}^T A \mathbf{r} = i^2 a_{1,1} + 2ij a_{1,2} + j^2 a_{2,2}$. Thus if Q is the intersection matrix for X_{-n} and Q' is its inverse then $c_1^2(X_{-n}) = i^2 q'_{1,1} + 2ij q'_{1,2} + j^2 q'_{2,2}$. Now we compute

$$\begin{aligned} q'_{1,1} &= (-1)^2 (detQ)^{-1} \begin{vmatrix} -5 & -1 & & \\ -1 & -2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -2 \end{vmatrix}_{n-1 \times n-1} \\ &= (-1)^n \left(-5 \det I_{n-2} + \det I'_{n-2}\right) \\ &= (-1)^n((-1)^{n-1}5(n-1) - (-1)^{n-1}(n-2)) \\ &= -(5n - 5 - n + 2) = 3 - 4n \end{aligned}$$

and

$$q'_{2,1} = (-1)^3(detQ)^{-1}\begin{vmatrix}-2 & -1 & & \\ & -2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -2\end{vmatrix}$$
$$= (-1)^{n+1}(-2 \cdot \det I_{n-2}) = (-1)^{n+1}(-1)^{n-2}(2-2n)$$
$$= 2n - 2$$

and

$$q'_{2,2} = (-1)^4(detQ)^{-1}\begin{vmatrix}-1 & & & & \\ & -2 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2\end{vmatrix}$$
$$= (-1)^n(-1 \cdot \det I_{n-2}) = 1 - n.$$

From this we compute that $c_1^2(X_{-n}) = -9n + 8$ or $-n$. □

We next prove Lemma 4.6 that says the signature of X_{-2} is $\sigma(X_{-2}) = -k + 2$ and $c_1^2(X_{-2}) = -\frac{1}{2}i^2 \pm (k-3)i + \frac{1}{2}(-k^2 + 4k - 3)$, where $i = k-1, k-3, \ldots, -k+1$.

Proof of Lemma 4.6. The intersection matrix of X_{-2} is negative definite by Lemma A.2.

The rotation vector for X_{-2} is $\mathbf{r} = \left[i, i \pm 1, 0, \ldots, , 0\right]^T$ where $i = \pm -k+1, -k+3, \ldots, k-1$. As in the proof of Lemma 4.4 above we see that if the intersection matrix is Q and its inverse is Q' then $c_1^2(X_{-n}) = i^2 q'_{1,1} + 2i(i \pm 1)q'_{1,2} + (i \pm 1)^2 q'_{2,2}$. Computing the $q'_{i,j}$ as in the proof of Lemma 4.4 we see that

$$q'_{1,1} = \frac{1}{2}(-k^2 + 2k + 2), \quad q'_{1,2} = \frac{1}{2}(k^2 - 3k), \text{and} \quad q'_{2,2} = \frac{1}{2}(-k^2 + 4k - 3).$$

Form this one can easily compute the claimed value for $c_1^2(X_{-2})$. □

The proof of Lemma 4.7 will require a new technique to compute the signature. Before starting the proof we recall the lemma says that the signature of X_2 is $\sigma(X_2) = -k$ and $c_1^2(X_2) = \frac{1}{2}i^2 \pm (k+1)i + \frac{1}{2}(k^2 - 1)$, where $i = k-1, k-3, \ldots, -k+1$.

Proof of Lemma 4.7. To compute the signature, we show that the intersection matrix Q of X_2 has only one positive eigenvalue. To achieve this, we recall that the eigenvalues are the zeros of the characteristic polynomial of Q. So, we will prove that the characteristic polynomial has only one positive solution. We use Descartes' rule of sign, which says that if a single-variable polynomial with real coefficients has j sign change when ordered by descending variable exponent, then there are $j - 2l$ positive roots where l is a non-negative number [18, pp. 89-93]. Thus, if there is only one sign change, then there is exactly one positive root.

We compute the characteristic polynomial using the principle minors. Recall that a principal minor of a $n \times n$ matrix M of size i is the determinant of the $i \times i$ matrix obtained by deleting some number of rows of Q and the corresponding columns. Let $E_i(M)$ denote the sum of principal minors of size i. Then, the characteristic polynomial of matrix M is

$$P_M(\lambda) = \lambda^n - E_1(M)\lambda^{n-1} + \cdots + (-1)^{n-1}E_{n-1}(M)\lambda + (-1)^n E_n(M),$$

see [12, Section 1.2].

Thus, to show Q has only one positive eigenvalue, we must show that the signs of the $E_i(Q)$ alternate as i increases except at one place. In fact, we will show that the $E_i(Q)$ alternate, except $E_{k+1}(Q)$ and $E_{k+2}(Q)$ will have the same sign. We prove this inductively. Clearly $E_1(Q)$ is just the trace of Q and so has sign $(-1)^1$. For $E_2(Q)$, we note that the 2×2 principal submatrices are either block matrices (with 1×1 blocks) or 2×2 matrices with all non-zero entries. In the former case, the determinant is the product of two negative numbers and, hence, is positive. In the latter case, the possible matrices are

$$\begin{bmatrix} -k+1 & -k \\ -k & -k-2 \end{bmatrix}, \begin{bmatrix} -k-2 & -1 \\ -1 & -2 \end{bmatrix}, \text{or } \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

and each of these matrices has a determinant of sign $(-1)^2$.

We now inductively assume that we have shown that all the principle $l \times l$ minors have sign $(-1)^l$ for $l < k$. We now consider the $(l+1)\times(l+1)$ principal minors. Each principal $(l+1)\times(l+1)$ submatrix is either a block matrix with blocks of size $l_1, \ldots, l_m$ such that $l_1 + \cdots + l_m = l+1$, or is not. In the former case, the sign of the minor will be the product of the signs of the blocks, that is $(-1)^{l_1}\cdots(-1)^{l_m} = (-1)^{l+1}$ as claimed. In the latter case, we must consider the matrices

$$\begin{bmatrix} -k+1 & -k & & & \\ -k & -k-2 & -1 & & \\ & -1 & -2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -2 \end{bmatrix}_{l+1\times l+1} \quad or \quad \begin{bmatrix} -k-2 & -1 & & \\ -1 & -2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -2 \end{bmatrix}_{l+1\times l+1}$$

We may use Lemma A.2 to show that the first matrix has determinant $(-1)^{l-1}(k^2 + (2k - l) - 2)$ which clearly has sign $(-1)^{l+1}$ (recall that l is less than k). The second matrix is easily seen to be negative definite, so it must have determinant $(-1)^{l+1}$ too by Sylvester's criterion mentioned above. This completes the induction for l up to k. When l is k, then the inductive step is the same except for the two non-block matrices of size $(k+1)\times(k+1)$. These matrices are as in the previous equations except $l+1 = k+1$ in this case. One may easily compute that the determinant of the first matrix is $(-1)^k$, which has the wrong sign, but the determinant of the second matrix is $(-1)^{k+1}(k^2+2k+2)$. So their sum has sign $(-1)^{k+1}$ as desired. Finally, one may compute that $E_{k+2}(Q) = \det Q = (-1)^{k+1}$.

The computation of $c_1^2(X_2)$ is identical to the proof of Lemma 4.6 except for the $q'_{i,j}$ we have

$$q'_{1,1} = \frac{1}{2}(k^2+2k+2), \quad q'_{1,2} = -\frac{1}{2}(k^2+k), \text{and} \quad q'_{2,2} = \frac{1}{2}(k^2-1).$$

□

Recall Lemma 4.10 says that the signature of X_{-n} is $\sigma(X_{-n}) = -k-n+2$ and

$$c_1^2(X_{-n}) = -n+1+(1-k)(-1+(k-1)n)-j2(-1)^k(n-1)(-i\pm k\mp 1)\pm 2i(-1+n(k-1))-ni^2$$

Proofs of Lemmas 4.10. The intersection matrix of X_{-n} is negative definite by Lemma A.2. The computation of c_1^2 is as follows:

$$i^2q'_{1,1} + 2i(i\pm 1)q'_{1,2} \pm 2iq'_{1,k} + (i\pm 1)^2q'_{2,2} \pm 2(i\pm 1)q'_{2,k} + q'_{k,k}.$$

The required $q'_{i,j}$ are:

$$q'_{1,1} = -k^2n+k+1, \quad q'_{1,2} = (k)(kn-1-n), \quad q'_{2,2} = (-k+1)(kn-1-n),$$

$$q'_{1,k} = (-1)^k(k)(-1+n), \quad q'_{2,k} = (-1)^k(-k+1)(-1+n) \quad \text{and} \quad q'_{k,k} = -n+1.$$

$\mathbf{r} = \left[i, i\pm 1, 0, \cdots 0, j, 0\cdots, 0\right]^T$. This gives

$$c_1^2(X_{-n}) = -n+1+(1-k)(-1+(k-1)n)-j2(-1)^k(n-1)(-i\pm k\mp 1)\pm 2i(-1+n(k-1))-ni^2$$

as claimed.

□

We finally turn to the proof of Lemma 4.11 which claims that the signature of X_n is $\sigma(X_n) = -k+1$ and

$$c_1^2(X_n) = -1+k-2kn\left(2i^2\pm 3i+1\right)+nk^2(1\pm 2i)^2+2s(-1)^k(i\mp k\pm 1)+n(i\pm 1)^2\mp 2i.$$

Proof of Lemmas 4.11. The intersection matrix of X_n has only one positive eigenvalue. The proof for this is very similar to the one given in the proof of Lemma 4.7 and is left to the reader.

The computation of c_1^2 is as follows:

$$i^2q'_{1,1} + 2i(i\pm 1)q'_{1,2} + 2isq'_{1,k+1} + (i\pm 1)^2q'_{2,2} + 2(i\pm 1)sq'_{2,k+1}.$$

The required $q'_{i,j}$ are:

$$q'_{1,1} = k^2n+k+1, \quad q'_{1,2} = -k^2n+kn-k, \quad q'_{2,2} = (k-1)^2n+k-1,$$

$$q'_{1,k+1} = (-1)^k(k), \quad q'_{2,k+1} = (-1)^k(-k+1) \quad \text{and} \quad q'_{k+1,k+1} = 0.$$

We note there is one difference in the computation. The rotation vector for X_n is $\mathbf{r} = \left[i, i\pm 1, 0, \cdots, 0, s\right]^T$. But since we can compute that $q'_{k+1,k+1} = 0$ the last term will not contribute. This gives

$$c_1^2(X_n) = -1+k-2kn\left(2i^2\pm 3i+1\right)+nk^2(1\pm 2i)^2+2s(-1)^k(i\mp k\pm 1)+n(i\pm 1)^2\mp 2i$$

□

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