SYMPLECTIC CONSTRUCTIONS ON 4-MANIFOLDS

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to my wife

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In this paper we give various conditions under which a rational blow-down can be done symplectically. This result may be used to directly construct symplectic forms on some 4-manifolds and to prove that log transforms of multiplicity 2 and 3 in certain cusp neighborhoods may be done in the symplectic category. We also develop techniques to study tight contact structures on lens spaces. We use these techniques to show that on any lens space there is a class $c \in H^2(L(p,q);\mathbb{Z})$ that is realized as the Euler class of a unique contact structure (if it is realized by one at all).

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CHAPTER 1

Introduction

In recent years there has been much to suggest that symplectic and contact geometry are closely related to low dimensional topology. For example Taubes [**T**] has related diffeomorphism invariants (the Seiberg–Witten invariants) of a closed 4-manifold X to the symplectic geometry of X, if X supports a symplectic structure. Kotschick [**Ko**], using these results of Taubes, has shown that a closed minimal simply connected symplectic 4-manifold is irreducible. Thus symplectic 4-manifolds provide (some) basic building blocks for all 4-manifolds. They do not, however, exhaust the set of all irreducible 4-manifolds, as was recently shown by Fintushel and Stern [**FS3**] and independently by Szabó. From this is should be clear that a fundamental question in 4-dimensional topology (and symplectic geometry) concerns the existence of a symplectic form on a given 4-manifold. A necessary condition for X to admit a symplectic form is that it have an almost-complex structure (i.e. a map $J : TX \longrightarrow TX$ on the tangent bundle of X such that $J^2 = -1$). If X is an open manifold Gromov [**Gr2**] has shown that this is actually sufficient. The picture is not so clear if X is a closed manifold.

There are two main cut-and-paste constructions one can use to make irreducible 4-manifolds. In this paper we study when one of these constructions, called a *rational blowdown*, can be made symplectic (the other construction, normal connected sum, has recently been studied extensively by Gompf [**G2**]). One may briefly describe a rational blowdown as follows: Given a 4-manifold X, one forms the rational blowdown of X, denoted X_p, by removing a neighborhood $C(p)$ of some embedded spheres Σ_i for $i = 0, ..., p-2$ intersecting according to the diagram in Figure 2.11 and replacing it with a rational homology ball $B(p)$ (i.e. a 4-manifold with boundary whose rational homology agrees with that of the 4-ball). There are obstructions to making this operation symplectic; however, in certain circumstances, it is possible to do a rational blowdown symplectically. In Chapter 6 we prove the following result.

Theorem 6.1. *If X is a symplectic 4-manifold, the spheres mentioned above are symplectically embedded (or Lagrangian) and have a neighborhood with an ω-convex boundary, then X_p is also symplectic.*

By ω-convex I mean there is a vector field v defined in a neighborhood of the boundary of $C(p)$ and transversally pointing out of it so that the Lie derivative of ω (the symplectic 2-form on X) in the direction of v is a positive multiple of ω. In most situations it is difficult to verify the ω-convexity condition in this theorem. Thus the theorem is most useful in conjunction with

Theorem 6.2. *A neighborhood of the spheres mentioned above has an ω-convex boundary if the spheres are symplectic and $p \leq 3$ or Σ_0 is symplectic and the other spheres are Lagrangian.*

These two results allow us to show many manifolds have a symplectic structure. In particular, we can see that many of the Gompf–Mrowka manifolds [**GM**] as well as manifolds studied by Fintushel and Stern in [**FS1**] and [**FS2**] have symplectic structures.

In the proof of Theorem 6.1, one encounters *contact structures* on 3-manifolds. A contact structure is a 3-dimensional analog of a symplectic structure. The usual definition of a contact structure is a 2-dimensional distribution ξ in the tangent bundle of the 3-manifold that is completely nonintegrable (one can think of this condition as saying that the planes in ξ cannot be realized, even locally, as the tangent planes to a 2-dimensional foliation). The ω-convexity of $C(p)$ induces a contact structure ξ_0 on the lens spaces $L(p^2, p-1)$, which is the boundary of $C(p)$. Next we note that the rational homology ball $B(p)$ is a Stein manifold, which can be seen by using Gompf's "Stein calculus" [**G3**] for 4-manifolds. Thus $B(p)$ has a symplectic structure and the boundary of $B(p)$ is ω-convex and so the boundary of $B(p)$, which is of course also $L(p^2, p-1)$, has a contact structure induced on it that we call ξ_1. We prove a theorem in Chapter 3 that allows us to glue $B(p)$ to $X \setminus C(p)$ while preserving a symplectic structure if we can verify that ξ_0 and ξ_1 are contactomorphic contact structures on $L(p^2, p-1)$.

Thus to complete the proof of Theorem 6.1 we develop some techniques to classify (tight) contact structures on lens spaces. The main idea is to find a nice CW decomposition of a lens space and then see when a contactomorphism can be build one cell at a time. This results in the the following theorem.

Theorem. *When p is odd there exists an Euler class $e \in H^2(L(p,q);\mathbb{Z})$ that may be represented by at most one tight contact structure.*

There is a similar result with p is even but the statement is more technical and involves the "half-Euler class." Since the Euler classes of ξ_0 and ξ_1 both correspond to the cohomology class mentioned in the theorem they must be contactomorphic, completing the proof of Theorem 6.1.

Understanding contact structures on 3-manifolds is interesting in its own right. Recently there has been a lot of work done on the construction of tight contact structures on 3-manifolds by Eliashberg [**E5**] and Gompf [**G3**] using Stein 4-manifolds and by Eliashberg and Thurston [**ET**] using perturbations of codimension one foliations. However, there is still very little known about the uniqueness of contact structures on 3-manifolds. Eliashberg has shown there is a unique tight contact structure on S^3 [**E4**] and $\mathbb{R}P^3$ (unpublished) and Giroux has classified tight contact structures on T^2-bundles over S^1. The techniques developed in this paper can be used to reprove Eliashberg's result on $\mathbb{R}P^3$ and generalize it to other lens spaces. A particularly interesting application of these techniques is to the nonexistence of tight contact structures. For example the cohomology class $0 \in H^2(L(3,1);\mathbb{Z})$ is not the Euler class of a tight contact structure on $L(3,1)$. These and other results will be explored in a future paper.

In order to make this work a self-contained as possible we have included Chapter 2. In this chapter we review the background necessary for the remainder of the paper. We begin with basic results from symplectic geometry, the most important of which is the Moser–Weinstein theorem on local symplectic geometry. In Section 2 we cover the basic facts from contact geometry. While in Section 3 we discuss the nature of contact geometry in three dimensions. Finally, in the last section of Chapter 2 we recall various methods to construct 4-manifolds. We begin with log transforms and then rational blowdowns and their relationship to log transforms. We then use these constructions to make many irreducible 4-manifolds, some of which will be shown to be symplectic in later chapters. Chapter 3 contains the material we need on convexity. In particular we discuss cut-and-paste operations using ω-convexity. We then show that the manifold $C(p)$ frequently has an ω-convex boundary and construct a Stein structure on $B(p)$. In Chapter 4 we build up the machinery necessary to see that the two contact structures induced on $L(p^2, p-1)$ as the ω-convex boundary of $C(p)$ and $B(p)$ are homotopic as 2-plain fields. We do this by recalling Gompf's definition of a complete set of invariants for 2-plain fields [**G3**] and then computing them for the contact structures of interest to us.

The last two chapters are the core of the paper. In Chapter 5 we develop a general procedure for analyzing tight contact structures on lens spaces, which we apply to prove Theorem 6.1. The techniques developed here have much wider applications, which will be explored in a future paper. In the last chapter we prove our main symplectic result, Theorem 6.1. We then discuss the application of this theorem to the question of the existence of symplectic forms on 4-manifolds.

CHAPTER 2

Background and Preliminary Results

This chapter contains the background needed in the rest of the paper. Section 2.1 contains a review of symplectic geometry with its primary objective a proof of the Moser–Weinstein theorem and some of its corollaries. We also consider the classical obstructions to a 4-manifold admitting a symplectic structure. In Section 2.2 we discuss contact geometry in general and in Section 2.3 we look at contact geometry in 3 dimensions. We will develop a lot of the amazing machinery of Eliashberg on which much of our work rests. Finally, in Section 2.4, we will review a little 4-dimensional topology. In particular, we will consider the two principal cut-and-paste constructions to be examined in subsequent chapters.

1. Symplectic Geometry

The main goal of this section is to prove the Moser–Weinstein theorem. The techniques used to prove this theorem originated in Moser's paper [**Mo**] and were refined by Weinstein in [**W1**] (see also [**GuS**]). It is from this theorem that most standard local uniqueness properties of symplectic structures follow. In subsequent chapters we will use this theorem to derive local properties needed for our constructions. The proof of the Moser–Weinstein theorem and several corollaries appears in in Subsection 1.4. Before this we will review basic symplectic geometry for the convenience of the reader. In Subsection 1.1 we discuss the linear algebra of symplectic forms. Symplectic manifolds and the classical obstructions to the existence of symplectic forms are discussed in Subsection 1.2. Several useful examples of symplectic manifolds are considered in Subsection 1.3.

1.1. Linear Algebra. Before we discuss symplectic manifolds we recall the basics of symplectic linear algebra. A **symplectic vector space** is a pair (V, ω) where V is a finite dimensional real vector space and

$$\omega : V \otimes V \longrightarrow \mathbb{R}$$

is a nondegenerate skew-symmetric bilinear form. Nondegenerate means that for any vector $v \in V$ if $\omega(v, w) = 0$ for all $w \in V$ then $v = 0$. The form ω is called a **symplectic form**. Note that any bilinear form ω induces a map

$$\phi_\omega : V \longrightarrow V^*$$

by $\phi_\omega(v) : V \longrightarrow \mathbb{R}$ with $\phi_\omega(v)(w) = \omega(v, w)$. Saying that ω is non-degenerate is equivalent to saying that ψ_ω is an isomorphism. The standard example of a symplectic vector space is $V = \mathbb{R}^{2n}$ with

$$\omega_0 = \sum_{j=1}^{n} x_j^* \wedge y_j^* \tag{2.1}$$

where $\{x_1, y_1, \ldots, x_n, y_n\}$ is a basis for V and $\{x_1^*, y_1^*, \ldots, y_n^*\}$ is the dual basis for V^*.

Let (V^{2n}, ω) be a symplectic vector space and W a subspace of V. Define the **ω-orthogonal complement** to W to be

$$W^\perp = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

An exercise in linear algebra shows that

$$\dim(V) = \dim(W) + \dim(W^\perp). \tag{2.2}$$

Thus $(W^\perp)^\perp = W$ since clearly $W \subset (W^\perp)^\perp$ and they have the same dimension. The subspace W is said to be **isotropic** if $W \subseteq W^\perp$, **symplectic** if $W \cap W^\perp = \{0\}$ and **Lagrangian** if $W = W^\perp$. From the dimension formula (2.2) above we have $\dim(W) \leq n$ if W is isotropic; moreover, W is Lagrangian if and only if it is a maximal isotropic subspace. W is symplectic if and only if $\omega|_W$ is nondegenerate if and only if $W^\perp$ is symplectic.

We are now ready to show that there is, up to isomorphism, only one symplectic vector space of a given (even) dimension.

Theorem 2.1. *If (V, ω) is a symplectic vector space, then for some n there is an isomorphism*

$$\psi : V \longrightarrow \mathbb{R}^{2n}$$

such that $\psi^*\omega_0 = \omega$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. In particular, we know that V is even dimensional and has a preferred orientation given by the volume form

$$\omega^n = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}.$$

We can use ψ to pull back the standard basis of $\mathbb{R}^{2n}$ to get a basis of V, this basis is called a **symplectic basis** for V.

PROOF. We will induct on the dimension of V. If $\dim V = 1$ then there are no nondegenerate forms on V. So we start with $\dim V = 2$. In this case take any nonzero vector v. Since ω is nondegenerate there is a vector w' such that $\omega(v, w') \neq 0$. Set $w = \frac{w'}{\omega(v,w')}$ and $W = \operatorname{span}\{v, w\}$. Clearly $V = W$ and the linear map that takes v, w to x_1, y_1 is an isomorphism from $(W, \omega|_W)$ to $(\mathbb{R}^2, \omega_0)$ that preserves the symplectic form. Now if $\dim V > 2$ then construct W as above. We have $V = W \oplus W^\perp$ with both W and $W^\perp$ symplectic and each with smaller dimension than V. Thus,

$$(V, \omega) = (W, \omega|_W) \oplus (W^\perp, \omega|_{W^\perp}) \cong (\mathbb{R}^2, \omega_0) \oplus (\mathbb{R}^{\dim V - 2}, \omega_0) \cong (\mathbb{R}^{2n}, \omega_0).$$

The last equivalence can easily be checked from the definition of ω_0.

The remaining statements in the theorem follow easily by working in $(\mathbb{R}^{2n}, \omega_0)$ and pulling the result back to V by ψ. $\square$

A map between symplectic vector spaces that preserves the symplectic form is called **symplectic**. If the map is also an isomorphism then it is called a **symplectomorphism**.

Let ω_0 be the standard symplectic form on $\mathbb{R}^{2n}$. The group of symplectomorphism of $\mathbb{R}^{2n}$ is denoted by $\mathrm{Sp}(n)$. Before beginning our analysis of $\mathrm{Sp}(n)$ it will be useful to consider $\mathbb{C}^n$ with a Hermitian structure given by

$$h(v, w) = \sum_{j=1}^{n} v_j \overline{w_j}$$

where $v, w \in \mathbb{C}^n$. Now take a complex basis $\{x_1, \ldots, x_n\}$ for $\mathbb{C}^n$ and let $y_j = i x_j$, for $j = 1, \ldots, n$, then $\{x_1, y_1, \ldots, x_n, y_n\}$ is a real basis for $\mathbb{C}^n$ and explicitly identifies $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. It is interesting to note that ω_0 is just the imaginary part of $-h$, as is quite easy to check using the above real basis for $\mathbb{C}^n$. Recall that the unitary group $\mathrm{U}(n)$ is the group of linear transformations of $\mathbb{C}^n$ that preserve the Hermitian form h. Since an element of $\mathrm{U}(n)$ preserves h it must also preserve ω_0 and hence is in $\mathrm{Sp}(n)$. So we have shown

$$\mathrm{U}(n) \subset \mathrm{Sp}(n).$$

We will end this section with the following useful lemma, whose proof may be found in [**McS**].

LEMMA 2.2. $\mathrm{Sp}(n)/\mathrm{U}(n)$ *is a contractable space.*

1.2. Symplectic Manifolds. A **symplectic manifold** is a pair (X, ω) where X is a manifold and ω is a closed nondegenerate 2-form. We say that ω a **symplectic form** on X. By closed we mean $d\omega = 0$, and nondegenerate means that for all $x \in X$, ω_x is a nondegenerate form on the vector space T_xX. Since all symplectic vector spaces are even dimensional and ω induces a symplectic structure on each tangent space to X, a manifold must necessarily be even dimensional to admit a symplectic structure. A submanifold Y of a symplectic manifold (X, ω) is called **symplectic** if $\omega|_Y$ is a symplectic form on Y and called **Lagrangian** if $\omega|_Y = 0$. The standard example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega)$ where

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

and $\{x_1, y_1, \ldots, x_n, y_n\}$ are coordinates on $\mathbb{R}^{2n}$. Indeed, one can easily check that ω is closed. Furthermore, for each point $p \in \mathbb{R}^{2n}$ we have $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ with ω_p exactly ω_0 from Equation (2.1) above. So ω is clearly nondegenerate. We will see many more examples of symplectic manifolds later.

From our discussion of symplectic vector spaces it is clear that if (X^{2n}, ω) is a symplectic manifold then $\Omega = \omega^n$ is a volume form on X. Thus if X is a closed manifold (i.e. compact without boundary) then ω cannot be exact. For if ω were exact then Ω would also be exact, but this is impossible since $H^{2n}(X; \mathbb{R}) \cong \mathbb{R}$ and is generated by the volume form. So ω represents a nonzero class $[\omega]$ in $H^2(X; \mathbb{R})$; moreover,

$$[\omega]^m = \underbrace{[\omega] \cup \ldots \cup [\omega]}_{m} = [\underbrace{\omega \wedge \ldots \wedge \omega}_{m}] = [\omega^m]$$

must also be a nonzero cohomology class for all $m \leq n$ by a similar argument to the one above. So a necessary condition for a closed manifold X to admit a symplectic form is that it have a nonzero cohomology class $\alpha \in H^2(X; \mathbb{R})$ such that α^m is also nonzero for all $m \leq n$. This simple obstruction already tells us that many manifolds cannot admit symplectic structures. For instance the $2n$-sphere for $n \neq 1$ has no symplectic structure since $H^2(S^{2n}; \mathbb{Z}) = 0$.

In order to discuss the other classic obstruction to a manifold admitting a symplectic structure we first must recall a few facts about almost complex manifolds. A manifold X is called **almost complex** if there exists a bundle map

$$J : TX \longrightarrow TX$$

such that $J^2 = -\mathrm{id}_{TX}$. Thus (T_xX, J_x) is a complex vector space for each $x \in X$. Given an almost complex manifold (X, J) we can always turn TX into a Hermitian vector bundle. By this we mean there is a smoothly varying Hermitian inner product on the tangent space. The proof that such a Hermitian structure exists on TX is the standard partition of unity proof also used to show that all manifolds admit a Riemannian metric. A Hermitian structure on TX is just a reduction of the structure group of TX from $\mathrm{SL}(2n, \mathbb{R})$ to $\mathrm{U}(n)$. Thus a manifold X admits an almost complex structure if and only if TX may be reduced to a $\mathrm{U}(n)$-bundle. For the theory of G-bundles the reader is referred to [**Br**].

One may easily check that having a symplectic structure on a manifold X reduces TX to a $\mathrm{Sp}(n)$-bundle. Note that just because TX reduces to a $\mathrm{Sp}(n)$-bundle we cannot conclude that X has a symplectic structure. We do get a nondegenerate form ω on X but there is no guarantee that ω is closed. In Lemma 2.2 above we noticed that $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ and in fact that $\mathrm{Sp}(n)/\mathrm{U}(n)$ is contractable. Thus any symplectic manifold admits an almost complex structure which is unique up to homotopy.

Now if a manifold X^{2n} has an almost complex structure J on its tangent bundle then we may define Chern classes

$$c_j(X) = c_j(TX) \in H^{2j}(X; \mathbb{Z}), \quad \text{for } j = 1, \ldots, n$$

associated to X (and J). We shall not stop to review characteristic classes here, but refer the reader to Milnor and Stasheff's book [**MS**]. For the rest of this section we shall assume that $n = 2$ so that X is a closed four dimensional manifold. Thus we have only two Chern classes: $c_1 = c_1(X)$ and $c_2 = c_2(X)$. It is well known that c_2 is equal to the Euler class $e(X)$ (where TX is thought of as real vector bundle with orientation induced by the almost complex structure J) and hence

$$\chi(X) = \langle e(X), [X] \rangle = \langle c_2, [X] \rangle \tag{2.3}$$

where $[X]$ represents the fundamental homology class of X in $H_4(X; \mathbb{Z})$ and $\langle \cdot, \cdot \rangle$ is the pairing between homology and cohomology. We also have

$$w_2(X) = c_1 \text{ reduced modulo } 2 \tag{2.4}$$

where $w_2(X) \in H^2(X; \mathbb{Z}_2)$ is the second Stiefel-Whitney class of X. The Pontrjagin classes of any complex vector bundle E are determined by its Chern classes. In particular we have

$$p_1(E) = c_1^2(E) - 2c_2(E). \tag{2.5}$$

The final ingredient we need is the Hirzebruch signature theorem. In dimension four it says

$$3\sigma(X) = \langle p_1(TX), [X] \rangle \tag{2.6}$$

where $\sigma(X)$ is the signature of the intersection form Q_X on $H_2(X; \mathbb{Z})$. An elementary proof of this formula my be found in Kirby's book [**K1**].

Now evaluating both sides of Equation (2.5) on the fundamental class of X and using Equation (2.6) and (2.3) to evaluate the result yields

$$\begin{aligned} 3\sigma(X) = \langle p_1(TX), [X]\rangle &= \langle c_1^2(X), [X]\rangle - 2\langle c_2(X), [X]\rangle \\ &= \langle c_1^2(X), [X]\rangle - 2\chi(X) \end{aligned} \tag{2.7}$$

and thus

$$\langle c_1^2(X), [X]\rangle = 3\sigma(X) + 2\chi(X). \tag{2.8}$$

Notice that the right hand side of this equation is completely determined by the homotopy type of X^4 and thus does not depend on the almost complex structure or the symplectic structure. This gives us another, more subtle, obstruction to the existence of a symplectic structure. Specifically, if X^4 admits a symplectic structure then it must also have an almost complex structure and hence a first Chern class c_1. So if X^4 does not have an element $c \in H^2(X; \mathbb{Z})$ that is a candidate for c_1 of an almost complex structure then X does not admit a symplectic structure. For c to be a candidate for c_1 of an almost complex structure it must satisfy Equation (2.4) and (2.8).

From Equation (2.8) we can derive other simple topological obstructions to a four manifold admitting a symplectic form (or even an almost complex structure). We begin by noticing that since $c_1(X)$ reduces modulo 2 to $w_2(X)$ its Poincaré dual is a characteristic element for the intersection form on X. A standard fact form the theory of intersection forms tells us that

$$\langle c_1^2(X), [X]\rangle \equiv \sigma(X) \mod 8.$$

Thus

$$\sigma(X) \equiv 2\chi(X) + 3\sigma(X) \mod 8.$$

A little algebra shows

$$\chi(X) + \sigma(X) \equiv 0 \mod 4. \tag{2.9}$$

Now let $b_i = \dim H^i(X; \mathbb{Z})$ be the i^{th} Betti number of X and let $b_2^\pm$ be the number of ± 1's down the diagonal of a diagonalization (over $\mathbb{R}$) of the intersection form Q_X of X. Notice that $b_2 = b_2^+ + b_2^-$, $\chi(X) = 2 - 2b_1 + b_2$ and $\sigma(X) = b_2^+ - b_2^-$, thus we can write Equation (2.9) as

$$1 - b_1 + b_2^+ \equiv 0 \mod 2. \tag{2.10}$$

So if a simply connected manifold admits a symplectic structure then b_2^+ must be odd. This allows us to easily see that $\#_k\mathbb{C}P^2\#_l\overline{\mathbb{C}P}^2$ cannot have a symplectic structure unless k is odd.

1.3. Examples of Symplectic Manifolds. We begin with perhaps the most important examples of symplectic manifolds. Given any manifold M let $X = T^*M$ be the cotangent bundle of M and let $\pi : T^*M \longrightarrow M$ be the projection map. In order to construct a symplectic form on X, we recall the definition of the **Liouville 1-form** λ on T^*M: this is a map

$$\lambda : T(T^*M) \longrightarrow \mathbb{R}$$

that is linear on each fiber $T_p(T^*M)$. Given a vector $v \in T(T^*M)$, then $\pi'(v)$ is a linear map $\pi'(v) : T_pM \longrightarrow \mathbb{R}$, where $\pi' : T(T^*M) \longrightarrow T^*M$ is the standard projection and $p = \pi \circ \pi'(v)$. Also note that $d\pi : T(T^*M) \longrightarrow TM$. Thus we may define

$$\lambda(v) = \pi'(v)(d\pi(v)).$$

One may easily check that this is a 1-form.

It is useful to express λ in local coordinates. To this end let $q_1, \ldots, q_n$ be a local coordinate system on $U \subset M$. So at any point $x = (q_1, \ldots, q_n)$ in U an element $z \in T^*_xM$ may be written $z = \sum_{j=1}^n p_j dq_j$. Thus local coordinates on T^*U are $p_1, \ldots, p_n, q_1, \ldots, q_n$ with $\pi(p_1, \ldots, p_n, q_1, \ldots, q_n) = q_1, \ldots, q_n$. By a standard abuse of notation we write $\pi^*(dq_j) \in T^*_z(T^*_xM)$ as dq_j. Now given $v \in T_z(T^*M)$ we have

$$\begin{aligned}
\lambda_z(v) &= \pi'(v)(d\pi(v)) = z(d\pi(v)) \\
&= \sum_{j=1}^n p_j dq_j(d\pi(v)) = \sum_{j=1}^n p_j \pi^* dq_j(v) \\
&= \sum_{j=1}^n p_j dq_j(v).
\end{aligned}$$

Thus we may write

$$\lambda = \sum_{j=1}^n p_j dq_j.$$

Before we go on to the symplectic structure on T^*M let us consider λ further. If $\beta : M \longrightarrow T^*M$ is any 1-form on M then notice $\beta^*\lambda$ will be a 1-form on M as well. In fact,

$$\begin{aligned}
\beta^*\lambda(m) &= \beta^*(\sum_{j=1}^n p_j \pi^* dq_j)(m) = \sum_{j=1}^n p_j(\beta(m))\beta^*\pi^* dq_j \\
&= \sum_{j=1}^n p_j(\beta(m))(\pi \circ \beta)^* dq_j = \sum_{j=1}^n p_j(\beta(m)) dq_j = \beta(m).
\end{aligned}$$

It is not hard to show, though we will not, that this property actually characterizes λ.

Finally, let $\omega = -d\lambda$. We claim that ω is a symplectic structure on X. Indeed since ω is exact it is closed and locally,

$$\omega = -d\lambda = -d\sum_{j=1}^{n} p_j dq_j = -\sum_{j=1}^{n} dp_j \wedge dq_j = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

Thus it is clearly nondegenerate. Note that if $M = \mathbb{R}^n$ then $(X = T^*M, \omega)$ is just the standard example of a symplectic form on $\mathbb{R}^{2n}$ given above. It is interesting to notice that any 1-form β will embed M into X and that

$$\beta^*\omega = \beta^*(-d\lambda) = -d\beta^*\lambda = -d\beta.$$

So $\beta(M)$ is a Legrangian submanifold of X if and only if $d\beta = 0$.

We would now like to construct a symplectic form on $\mathbb{C}P^n$. Recall, that $\mathbb{C}P^n$ is the quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by S^1, where S^1 is thought of as the unit complex numbers and acts on $\mathbb{C}^{n+1}$ by multiplication. Thus we have the S^1-bundle $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$. Take a point z and set $W_z = \{\text{complex span of } z\}^{\perp}$ where we are using the standard Hermitian metric on $\mathbb{C}^{n+1}$ to define orthogonal complements. Identifying $\mathbb{C}^{n+1}$ with $T_z\mathbb{C}^{n+1}$ one may easily check that $d\pi|_{W_z}$ is an isomorphism from W_z to $T_{\pi(z)}\mathbb{C}P^n$ and that this isomorphism is independent of the choice of z in the same orbit. Now W_z is a symplectic subspace of $\mathbb{C}^{n+1}$ with its standard symplectic form ω_0. Thus π gives us a nondegenerate form ω on $T\mathbb{C}P^n$ which is clearly closed since d commutes with π^*. With a little thought one can explicitly write ω down. If v_0, v_1 are two vectors in $T_x\mathbb{C}P^n$ then

$$\omega(v_0, v_1) = \mathrm{Im}\left(\frac{h(w_0, w_1)h(z,z) - h(w_0, z)h(z, w_1)}{h(z,z)}\right)$$

where $z \in S^{2n+1}$ such that $\pi(z) = x$ and $w_0, w_1 \in T_zS^{2n+1}$ with $d\pi(w_j) = v_j$ for $j = 0, 1$.

Next consider a complex manifold X with a Hermitian structure h. If we set $\omega = -\mathrm{Im}(h)$ then ω is clearly a nondegenerate form on X since h is nondegenerate. But ω is not necessarily closed. We call X a **Kähler manifold** if $d\omega = 0$ and then ω is called the **Kähler form** on X. Thus by definition a Kähler manifold is also a symplectic manifold. The following proposition gives us many more examples of symplectic manifolds.

PROPOSITION. *A complex submanifold of a Kähler manifold is also a Kähler manifold.*

For a proof see [**La**]. The analysis of $\mathbb{C}P^n$ above also shows that $\mathbb{C}P^n$ is a Kähler manifold. Thus any complex submanifold of $\mathbb{C}P^n$ is also a symplectic manifold. So for example any nonsingular algebraic variety is a symplectic manifold.

We would like to end this section with a construction (for details see [**McS**]). Let (X^{2n}, ω) be a symplectic manifold. Assume that there is a symplectically embedded $\mathbb{C}P^{n-1}$ in X (where we are giving $\mathbb{C}P^{n-1}$ the symplectic structure constructed above, which we now denote ω_0). We also assume that the normal bundle $\nu(\mathbb{C}P^{n-1})$ in X is isomorphic to the universal line bundle L over $\mathbb{C}P^{n-1}$. It can be shown that there is an ϵ-neighborhood, N_ϵ, of $\mathbb{C}P^{n-1}$ in X such that $N_\epsilon \setminus \mathbb{C}P^{n-1}$ is symplectomorphic to $B^{2n}_{\lambda+\epsilon} \setminus B^{2n}_\epsilon$, where B^{2n}_κ is a ball of radius κ in $\mathbb{C}^n$ endowed with the standard symplectic form on $\mathbb{C}^n$ restricted to B^{2n}_κ. The **blowdown** of X along $\mathbb{C}P^{n-1}$ is obtained by removing N_ϵ form X and gluing in $B^{2n}_{\lambda+\epsilon}$. We now define an inverse construction. We begin with a symplectic embedding e of B^{2n}_λ into X. If we let Z be the zero section of the universal line bundle over $\mathbb{C}P^{n-1}$ then as above we may find some neighborhood N_ϵ of Z in L and a symplectomorphism from a neighborhood of ∂N_ϵ in N_ϵ to a neighborhood of ∂B^{2n}_λ in B^{2n}_λ. Define the **blowup** of X with weight λ, denoted $(\tilde{X}, \tilde{\omega}_e)$, to be the manifold obtained by removing $e(B^{2n}_\lambda)$ from X and gluing in N_ϵ. Topologically, we are gluing X minus a ball to $\overline{\mathbb{C}P}^n$ minus a ball (one may easily check that N_ϵ is diffeomorphic to this). Thus $\tilde{X} \cong X \# \overline{\mathbb{C}P}^n$.

Consider the above constructions on a symplectic 4-manifold (X, ω): to perform a blowdown all we need is a symplectically embedded 2-sphere Σ with self intersection number -1. Such a 2-sphere is called an **exceptional sphere** and X is called **minimal** if it contains no exceptional spheres.

1.4. Local Symplectic Geometry. Most the local geometry of symplectic manifolds will follow from the following theorem.

THEOREM 2.3 (Moser–Weinstein). *Let X^{2n} be a manifold and C a compact submanifold. If ω_0 and ω_1 are two symplectic forms on X that are equal on each T_xX when $x \in C$, then there exists open neighborhoods U_0 and U_1 of C and a diffeomorphism $\phi : U_0 \longrightarrow U_1$ such that $\phi^*\omega_1 = \omega_0$ and ϕ is the identity on C. More generally, ϕ is the identity wherever ω_0 and ω_1 agree.*

PROOF. Let $\omega_t = (1-t)\omega_0 + t\omega_1$ be a path of 2-forms, for $0 \leq t \leq 1$. We will try to find ϕ as the time one map of a flow ϕ_t, generated by a vector field v_t, so that $\phi_t^*\omega_t = \omega_0$. We claim that it will suffice to find a 1-form α such that $\alpha|_{T_CM} = 0$ and $d\alpha = \omega_1 - \omega_0$. Indeed, given such a α we may find a vector field v_t in some neighborhood U of C so that

$$\alpha = -\iota_{v_t}\omega_t,$$

since the ω_t are nondegenerate on some neighborhood of C. Now let ϕ_t be the flow generated by v_t, i.e.

$$\frac{d}{dt}\phi_t = v_t\phi_t.$$

By shrinking U if necessary we may assume that ϕ_t exists for $0 \leq t \leq 1$. Differentiating $\phi_t^*\omega_t$ with respect to t we get

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(\frac{d}{dt}\omega_t + \iota_{v_t}d\omega_t + d\iota_{v_t}\omega_t). \tag{2.11}$$

For a proof of this equation the reader is referred to [**GuS**]. Now $d\omega_t = 0$ and by our choice of α we have

$$\frac{d}{dt}\omega_t = \frac{d}{dt}(\omega_0 + td\alpha) = d\alpha$$

Thus we have shown that $\frac{d}{dt}\phi_t^*\omega_t = 0$, so $\phi_t^*\omega_t = \omega_0$. In particular, $\phi_1^*\omega_1 = \omega_0$. Note that since $\alpha = 0$ on T_CM, $v_t = 0$ on C and hence ϕ_1 is the identity on C (clearly on any subset where $\omega_0 = \omega_1$, ϕ_1 will restrict to the identity).

To finish the proof we must now find α. To this end let $\sigma = \omega_1 - \omega_0$. We may assume that some neighborhood of C is identified with an open normal disk bundle N of C. Let $f_t : N \longrightarrow N$ denote multiplication by t in the fiber of the bundle, for $0 \leq t \leq 1$. Note that f_t is a diffeomorphism for $t > 0$, f_1 is the identity on N and $f_0(N) = C$. Thus $f_0^*\sigma = 0$ and $f_1^*\sigma = \sigma$. Finally let $w_t = \frac{d}{dt}f_t$ and set

$$\alpha = \int_0^1 f_t^*(\iota_{w_t}\sigma)dt.$$

Now we have

$$\begin{aligned} d\alpha = d\int_0^1 f_t^*(\iota_{w_t}\sigma)dt &= \int_0^1 d(f_t^*(\iota_{w_t}\sigma))dt \\ &= \int_0^1 [f_t^*\iota_{w_t}d\sigma + d(f_t^*(\iota_{w_t}\sigma))]dt \\ &= \int_0^1 \frac{d}{dt}(f_t^*\sigma)dt = f_1^*\sigma - f_0^*\sigma \\ &= \sigma - 0 = \omega_1 - \omega_0 \end{aligned}$$

as desired. The second equality follows since $d\sigma = 0$ and the third equality follows from Equation (2.11). Note that σ is zero whenever $\omega_0 = \omega_1$. So σ is zero on T_CM; hence, α is also zero on T_CM. $\square$

This theorem has many significant corollaries.

Corollary 2.4 (Darboux's Theorem). *Any symplectic manifold* (X^{2n}, ω) *is locally symplectomorphic to a neighborhood of the origin in* $\mathbb{R}^{2n}$ *with its standard symplectic form* ω_0.

Proof. Given any point $x \in X$ there is a coordinate chart U and a diffeomorphism $f : \mathbb{R}^{2n} \longrightarrow U$ such that $f(0) = x$. Let $\omega' = f^*\omega|_U$. By Theorem 2.1 there exists a linear map $L : T_0\mathbb{R}^{2n} \longrightarrow T_0\mathbb{R}^{2n}$ such that $L^*(\omega')_0 = (\omega_0)_0$. Now consider the map $g = f \circ L$ and $\omega_1 = g^*\omega$. By construction, ω_1 and ω_0 agree at the origin; hence by Theorem 2.3 there are neighborhoods of the origin U_0 and U_1 and a symplectomorphism $\phi : U_0 \longrightarrow U_1$ such that $\phi^*\omega_1 = \omega_0$. Thus the map $g \circ \phi$ is a symplectomorphism from a neighborhood of the origin in $(\mathbb{R}^{2n}, \omega_0)$ to a neighborhood of x in (X, ω). □

Corollary 2.5 (Symplectic Neighborhood Theorem). *Let* (X_j, ω_j), *for* $j = 0, 1$, *be symplectic manifolds. Assume* Y_j *is a symplectic submanifold of* X_j *and* $\psi : Y_0 \longrightarrow Y_1$ *is a symplectomorphism. If there is a symplectic bundle map* $\Psi : \nu(Y_0) \longrightarrow \nu(Y_1)$ *of the normal bundles that covers* ψ, *then* ψ *extends to a symplectomorphism from a neighborhood of* Y_0 *to a neighborhood of* Y_1.

This corollary is one of the ingredients in Gompf's normal connected sums theorem (see [**G2**]). Given two smooth manifolds X_0 and X_1 with embeddings $j_i : N \longrightarrow X_i$, of a compact oriented manifold N of codimension 2, such that the normal bundles ν_0 and ν_1 are orientation reversing diffeomorphic, one may define the **normal connected sum** of X_0 and X_1 along N as follows:

$$X_0 \#_\phi X_1 = (X_0 \setminus \nu_0) \cup_\phi (X_1 \setminus \nu_1),$$

where $\phi : \partial\nu_0 \longrightarrow \partial\nu_1$ is induced by the aforementioned diffeomorphism from ν_0 to ν_1. In the symplectic case we have the following theorem which we state only in the 4 dimensional case.

Theorem 2.6. *Let* X_i *be a closed symplectic 4-manifold and* Σ_i *a closed connected symplectic surface in* X_i, *for* $i = 0, 1$. *Suppose that* Σ_0 *is diffeomorphic to* Σ_1 *and that they have opposite squares under the intersection pairing in* X_i. *Then* $X_0 \#_\phi X_1$ *admits a symplectic structure for any orientation reversing isomorphism* $\phi : \nu(\Sigma_0) \longrightarrow \nu(\Sigma_1)$. *Moreover, if* Σ_i' *is another closed symplectic surface in* X_i *that intersects* Σ_i *transversely in* l *points all with positive sign, then* $\Sigma_0' \#_{\phi'} \Sigma_1'$ *is a symplectic submanifold of* $X_0 \#_\phi X_1$, *where* ϕ' *is induced from* ϕ.

The proof of this theorem would take us too far afield to be given here. The interested reader is referred to [**G2**].

Proof of Corollary 2.5. Standard differential topology tells us that we may identify $\nu(Y_j)$ with a neighborhood N_j of Y_j in X_j, for $j = 0, 1$. Thus Ψ may be thought of as a map from N_0 to N_1. Now let $\omega_1' = \Psi^*\omega_1$. By hypothesis we know that ω_0 and ω_1' agree on $TX_0|_{Y_0}$. Thus Theorem 2.3 gives us neighborhoods U_0 and U_1 of Y_0 and a symplectomorphism $\phi : (U_0, \omega_0) \longrightarrow (U_1, \omega_1')$. So the desired symplectomorphism is $\Psi \circ \phi|_{U_0 \cap N_0}$. $\square$

Corollary 2.7 (Lagrangian Neighborhood Theorem). *Let (X, ω) be a symplectic manifold with L a Lagrangian submanifold. The exists a neighborhoods U_0 of L in X and U_1 of L in T^*L (where L is the zero section of T^*L) and a symplectomorphism $\phi : U_0 \longrightarrow U_1$ (where T^*L is endowed with its standard symplectic form).*

Proof. We begin by noting that the normal bundle, $\nu(L)$, to L in X is isomorphic to the cotangent bundle T^*L of L. To see this first notice that on the vector space level if $\{x_1, \ldots, x_n\}$ is a basis for a Lagrangian subspace W of V then we may extend this basis by $\{y_1, \ldots, y_n\}$ to get a symplectic basis for V. Since the y_j's span V/W they also span W^* (using the map ϕ_ω from Section 1 above), so $V/W \cong W^*$. Now doing this construction at each point in the tangent bundle $TX|_L$ yields an isomorphism $TX/TL \cong T^*L$. But TX/TL is well known to be isomorphic to the normal bundle to L in X. Thus confirming our observation above.

Now the symplectic form ω on X induces a symplectic structure, denoted ω_0, on T^*L by using the isomorphism between $\nu(L)$ and T^*L. T^*L also has a canonical symplectic form that we will denote ω_1. We would be done if ω_0 and ω_1 agreed on $T(T^*L)|_Z$ where Z is the zero section of T^*L. This will not be true in general; though, we do always have ω_0 and ω_1 agreeing on $TZ \subset T(T^*L)$.

At each point z of Z let M_z be the set of linear maps of $T_z(T^*L)$ that take $(\omega_1)_z$ to $(\omega_0)_z$ and are fixed on $T_zZ \subset T(T^*L)$. One may easily check that the M_z's fit together to from a bundle M over Z. A little linear algebra shows that M_z is isomorphic to the set of $n \times n$ symmetric matrices. Thus there is a section σ of M over Z since M has contractable fibers. Consider the map $f : T^*L \longrightarrow T^*L$ defined by $f(z, v) = (z, \sigma(z)v)$. So f is the identity map on Z and $df = id_{TZ} \oplus \sigma$. Thus $f^*\omega_0$ and ω_1 agree on $T(T^*L)|_Z$ and we may now use Theorem 2.3 to complete the proof. $\square$

Our final result on the "local geometry" of symplectic manifolds involves families of symplectic forms.

THEOREM 2.8 (Moser Stability). *Let X be a closed manifold and ω_t a smooth family of symplectic forms on X, for $0 \leq t \leq 1$. If all the ω_t are cohomologous, then there exists a family of diffeomorphisms ϕ_t such that $\phi_0 = id_X$ and $\phi_t^* \omega_t = \omega_0$.*

PROOF. We reproduce the beautiful proof in [**McS**]. As in the proof of Theorem 2.3 we can construct the desired symplectomorphisms if we can find a family of 1-forms α_t such that $\frac{d}{dt}\omega_t = d\alpha_t$. To this end choose a Riemannian metric on X and let d^* be the formal L^2 adjoint to d. Recall that d is an isomorphism from $d^*(\Omega^2(X))$ to the exact 2-forms. Now $\frac{d}{dt}\omega_t = \frac{d}{dt}(\omega_t - \omega_0)$ is exact (since all the ω_t are cohomologous) so there exists some 1-form α_t such that $d\alpha_t = \frac{d}{dt}\omega_t$ as needed. $\square$

2. Contact Geometry

Contact structures are an odd-dimensional analog of symplectic structures. A **k-dimensional distribution** ξ on an n-manifold M is a subbundle of TM such that $\xi_m \equiv T_m M \cap \xi$ is a k-dimensional subspace of $T_m M$ for every $m \in M$. We say ξ is smooth if we may locally find k linearly independent smooth vector field on M that span ξ. A **contact structure** ξ on a $(2n+1)$-dimensional manifold M is a smooth $2n$-dimensional distribution on M that is "maximally nonintegrable." One may intuitively think of "maximally nonintegrable" as meaning that the planes in ξ cannot be realized, even locally, as the tangent planes to a $2n$-dimensional foliation.

To get a better understanding of "maximally nonintegrable" we first need to understand what it means for a distribution of be integrable. A k-dimensional distribution ξ an n-manifold M is a **integrable** if we may locally find a k-dimensional foliation such that ξ is the tangent bundle to the leaves of the foliation. The following theorem tells us when a distribution is integrable.

FROBENIUS THEOREM. *A k-dimensional distribution ξ on an n-manifold is integrable if and only if ξ is closed under the Lie bracket (i.e. if v and w are vector fields contained in ξ then so is $[v, w]$).*

Note that a codimension one distribution ξ may be defined (at least locally) by a 1-form, say α. By this we mean $\xi = \ker(\alpha)$. Hence a vector field v is a section of ξ if and only if $\alpha(v) = 0$. Thus the Frobenius Theorem may be restated as: ξ is integrable if and only if $\alpha([v, w]) = 0$ whenever $\alpha(v) = 0 = \alpha(w)$. Note that

$$d\alpha(v, w) = v\alpha(w) - w\alpha(v) - \alpha([v, w]) \tag{2.12}$$

implies that ξ is integrable if and only if $d\alpha|_\xi = 0$. Finally, a little thought shows that ξ is integrable if and only if $\alpha \wedge d\alpha = 0$.

We may now rigorously define "maximally nonintegrable". We will say that a $2n$-dimensional distribution ξ on a $(2n+1)$-dimensional manifold M is **maximally nonintegrable** if for any locally defining 1-form α we have $\alpha \wedge d\alpha^n \neq 0$. If ξ can globally be defined by a 1-form α then it is said to be **transversely oriented** (α is usually referred to as a **contact 1-form**). One may equivalently say that ξ is transversely oriented if there is a vector field v on M such that v_m and ξ_m span T_mM for every $m \in M$. Given an contact form α we may choose v such that $\iota_v d\alpha = 0$ and $\alpha(v) = 1$, and then we call v the **Reeb vector field** of α. Notice that the maximal nonintegrability of ξ implies that $d\alpha|_\xi$ is nondegenerate and hence induces a symplectic structure on the vector bundle ξ. Now if α and α' are two different 1-forms defining ξ then there must exist a nonzero function $f : M \longrightarrow \mathbb{R}$ on M such that

$$\alpha' = f\alpha.$$

So $d(f\alpha) = df \wedge \alpha + f d\alpha$ and thus $d(\alpha')|_\xi = f d\alpha|_\xi$. In other words, α and α' define the same symplectic structure on ξ up to a nonzero scaling factor. Moreover, $\alpha' \wedge d\alpha'^n = f^{n+1}\alpha \wedge d\alpha^n \neq 0$, so if n is odd a contact structure ξ gives a natural orientation on M.

The standard example of a contact structure is given by $\mathbb{R}^{2n+1}$ with ξ_0 defined as the kernel of

$$\alpha_0 = dt + \sum_{j=1}^{n} x_i dy_i$$

where $t, x_1, y_1, \ldots, x_n, y_n$ are the standard coordinates on $\mathbb{R}^{2n+1}$. One may easily check that $\alpha_0 \wedge d\alpha_0^n$ gives a volume form on $\mathbb{R}^{2n+1}$ that induces the standard orientation; and thus, $\xi_0 = \ker \alpha_0$ is a transversely oriented contact structure. Another simple example of a contact manifold is given by S^{2n+1}. In order to construct a contact structure on S^{2n+1} we need to think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Now at each point $p \in S^{2n+1}$ let ξ_p be the set of complex tangencies to S^{2n+1}, i.e.

$$\xi_p = T_pS^{2n+1} \cap iT_pS^{2n+1}.$$

To check that ξ is indeed a contact structure on S^{2n+1} we will define a contact form for it. To this end, let $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$ be a set of real coordinates on $\mathbb{C}^{n+1}$ such that $y_j = ix_j$ for $j = 1 \ldots (n+1)$. Set

$$\alpha = \frac{1}{2}\sum_{j=1}^{n+1}(y_j dx_j - x_j dy_j).$$

We leave it as an exercise to the reader to check that ξ is the kernel of $\alpha|_{S^{2n+1}}$ and hence is clearly a nonintegrable $2n$-dimensional distribution.

Consider a contact manifold (M, ξ) with contact form α and a submanifold L such that $T_mL \subset \xi_m$ for all $m \in L$. Given two tangent vector fields v, w to L then $[v, w]$ is also a tangent vector field, so $d\alpha(v, w) = v\alpha(w) - w\alpha(v) - \alpha([v, w]) = 0$. Thus T_mL is an isotropic subspace of the symplectic vector space $(\xi_m, d\alpha_m)$ and by the dimension formula (2.2) we know that $\dim L \leq n$. We call L **Legendrian** if $\dim L = n$.

Two contact structures ξ_0 and ξ_1 on M are **contactomorphic** if there is a diffeomorphism $f : M \longrightarrow M$ such that $df(\xi_0) = \xi_1$. In this case f is called a **contactomorphism**. If α_1 is a contact 1-form for ξ_1 then f is a contactomorphism if and only if $f^*\alpha_1$ is a contact 1-form for ξ_0.

2.1. Local Contact Geometry. As in symplectic geometry there are several powerful local results in contact geometry. We begin with the following theorem.

THEOREM 2.9 (Gray's Theorem). *Let M be a closed manifold. If ξ_t, $0 \leq t \leq 1$, a family of contact structures on M, then we may find a family of diffeomorphisms $\phi_t : M \longrightarrow M$ such that $(\phi_t)_*\xi_0 = \xi_t$. Moreover, we may assume that ϕ_t is the identity map on any subset of M where all the ξ_t's agree.*

PROOF. As in the proof of Theorem 2.3 we would like to find a time dependent vector field v_t whose flow will give us the ϕ_t's. We begin by finding contact forms α_t for our contact structures ξ_t (we may only be able to do this locally but that will not matter in the end). We want to find $\phi_t : M \longrightarrow M$ such that

$$\phi_t^*\alpha_t = f_t\alpha_0 \tag{2.13}$$

for nonvanishings functions $f_t : M \longrightarrow \mathbb{R}$. Since $d\alpha_t|_{\xi_t}$ is nondegenerate on ξ_t we can find a vector field $v_t \in \xi_t \subset TM$ such that

$$\iota_{v_t} d\alpha_t|_{\xi_t} = -\frac{d\alpha_t}{dt}|_{\xi_t}. \tag{2.14}$$

Now let ϕ_t be the flow of v_t. Note that where $\xi_t = \xi_0$ we can assume that the α_t's agree, thus the vector field $v_t = 0$ and hence ϕ_t is the identity. We claim that ϕ_t satisfies Equation (2.13). To see this we compute

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\alpha_t) &= \phi_t^*(\frac{d}{dt}\alpha_t + \iota_{v_t}d\alpha_t + d\iota_{v_t}\alpha_t)\\ &= \phi_t^*(\frac{d}{dt}\alpha_t + \iota_{v_t}d\alpha_t).\end{aligned} \tag{2.15}$$

So by Equation (2.14) we see that the 1-form $\frac{d}{dt}(\phi_t^*\alpha_t) = 0$ on $\phi_t^*\xi_t$. Thus since $\phi_t^*\xi_t = \ker \phi_t^*\alpha_t$ we have $\ker \phi_t^*\alpha_t \subset \ker \frac{d}{dt}(\phi_t^*\alpha_t)$. Which implies

$$\frac{d}{dt}(\phi_t^*\alpha_t) = g_t(\phi_t^*\alpha_t) \tag{2.16}$$

for some smooth function $g_t : M \longrightarrow \mathbb{R}$. This clearly implies that at each point m in M, $\phi_t^*\alpha_t$ lies on a ray in T_m^*M through α_0 and thus the ϕ_t's satisfy Equation (2.13).

In order to finish the proof when the ξ_t's are not transversely oriented, we check that the v_t's above are independent of the α_t's chosen to represent ξ_t (thus we can define the v_t's locally and they will patch together to give a global vector field). So let α_t' be another set of contact forms for ξ_t. Then we know

$$\alpha_t' = g_t\alpha_t$$

for some positive functions g_t. So

$$d\alpha_t' = dg_t \wedge \alpha_t + g_t d\alpha_t$$

and

$$\frac{d\alpha_t'}{dt} = \frac{dg_t}{dt}\alpha_t + g_t\frac{d\alpha_t}{dt}.$$

Now if we contract v_t defined in Equation (2.14) into the first equation and then restrict to ξ_t we get

$$\begin{aligned}\iota_{v_t}d\alpha_t'|_{\xi_t} = g_t\iota_{v_t}d\alpha_t|_{\xi_t} &= -g_t\frac{d\alpha_t}{dt}|_{\xi_t}\\ &= -\frac{d\alpha_t'}{dt}|_{\xi_t}.\end{aligned}$$

Thus we get v_t using the α_t's or the α_t''s. $\square$

We may now easily prove the following corollaries.

Corollary 2.10 (Darboux's Theorem). *Any $(2n+1)$-dimensional contact manifold (M, ξ) is locally contactomorphic to a neighborhood of the origin in $\mathbb{R}^{2n+1}$ with its standard contact structure ξ_0.*

Proof. As in the proof of the symplectic Darboux Theorem, given a point $m \in M$ we can find a coordinate patch $f : \mathbb{R}^{2n+1} \longrightarrow U$, where $m \in U$, such that $f(0) = m$, $df_0(\xi_0) = \xi_m$, f is orientation preserving and df is a symplectomorphism from $(d\alpha_0|_{\xi_0})_0$ to $(d\alpha|_\xi)_m$. Now let $\alpha_1 = f^*\alpha$ and $\alpha_t = (1-t)\alpha_0 + t\alpha_1$. Then $\xi_t = \ker \alpha_t$ agree with ξ_0 at the origin. Moreover, at the origin $d\alpha_t = (1-t)d\alpha_0 + t\, d\alpha_1$ is a symplectic form on $\xi_t = \xi_0$ since α_0 and α_1 agree at the origin. Thus α_t is a contact form near the origin. Gray's Theorem gives us $\phi_t : N \longrightarrow N$, where N is a neighborhood of the origin in $\mathbb{R}^{2n+1}$, such that $\phi_t(0) = 0$ and $(\phi_t)_*\xi_0 = \xi_t$. Finally, $f \circ \phi_t$ is a contactomorphism from a neighborhood of the origin in $\mathbb{R}^{2n+1}$ to a neighborhood of m in M. $\square$

Corollary 2.11. *Let (M, ξ) be a contact manifold and $\psi_t : M \longrightarrow M$ an isotopy of M with each ψ_t a contactomorphism outside a subset $U \subset M$ with compact closure. Then there is an isotopy $\psi'_t : M \longrightarrow M$ through contactomorphisms such that $\psi'_t = \psi_t$ for $t = 0$ and for all m not in U.*

Proof. Let $\xi_t = \psi_t^*\xi$ and apply Gray's theorem to ξ_t to obtain a family of contactomorphisms $\phi_t : M \longrightarrow M$ such that $(\phi_t)_*\xi = \xi_t$. Set $\psi'_t = \psi_t \circ \phi_t$ and notice that

$$(\psi'_t)_*\xi = (\psi_t)_*(\phi_t)_*\xi = (\psi_t)_*\xi_t = \xi.$$

Thus all the ψ'_t's are contactomorphisms. Outside of U we know that $(\xi_t)_m = \xi_m$ so $\psi_t|_{M\setminus U} = \mathrm{id}$ and thus $\psi'_t = \psi_t$ outside U. $\square$

In dimension 3 we get a much stronger result.

Theorem 2.12. *Let M be an oriented 3-manifold and $C \subset M$ a compact subset of M. If ξ_0 and ξ_1 two positive contact structures on M that agree on C, then there exists a neighborhood N of C such that the identity id_N on N is isotopic rel C to a contactomorphism.*

Proof. We will assume that ξ_i is transversely orientable near C (the general case can be handled as in the proof of Theorem 2.9). Let α_i be a contact from for ξ_i and set $\alpha_t = (1-t)\alpha_0 - t\alpha_1$. The plane fields $\xi_t = \ker \alpha_t$ are clearly independent of t on C. Now $d\alpha_0$ and $d\alpha_1$ are both positive volume forms on $\xi_t|_C$ and hence so is $d\alpha_t$ (a convex combination of positive volume forms is a volume form). Thus $d\alpha_t$ is a volume from on ξ_t near C. Therefore, α_t is a contact form near C. We can now use the proof of Gray's Theorem above to find $\phi_t : N \longrightarrow N$, where N is a neighborhood of C, such that $(\phi_t)_*\xi_0 = \xi_t$. All the ϕ_t will be the identity on C since ξ_0 and ξ_1 agree there already. $\square$

2.2. Symplectification. In this subsection we will see that a contact manifold may be realized as a hypersurface in a symplectic manifold in such a way that the contact form and symplectic form are related. Given a transversely oriented contact manifold (M, ξ) chose a contact 1-form α for ξ and consider the submanifold of T^*M

$$X = \{v \in T^*_m M : m \in M, v = t\alpha_m \text{ and } t > 0\}.$$

Clearly, for each $m \in M$, $X \cap T^*_m M$ is the ray in $T^*_m M$ on which α_m lies and so $X = (0, \infty) \times M$. Any other contact from α' for ξ can be thought of as a section of X. Thus the manifold X depends only on (M, ξ) and not on α (α does however provide an embedding of M in X). We now claim that X is a symplectic manifold. To see this let $\omega = \omega_0|_X$ where ω_0 is the canonical symplectic structure on T^*M. Recalling that $\omega_0 = d\lambda$, where λ is the Liouville 1-form on T^*M, we know that

$$\alpha^*\lambda = \alpha.$$

Hence

$$\pi^*\alpha = \lambda|_M,$$

where $\pi : T^*M \longrightarrow M$ is projection and M is thought of as sitting in X by using α as an embedding. Thus

$$t\pi^*\alpha = \lambda|_X$$

and so

$$\omega = d\lambda|_X = d(t\pi^*\alpha) = dt \wedge \pi^*\alpha + t\pi^*(d\alpha).$$

If the dimension of M is $2n-1$ then we may finally compute

$$\omega^n = t^{n-1}[dt \wedge \pi^*(\alpha \wedge (d\alpha)^{n-1})]$$

which is clearly a volume form on X since $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on M and $X = (0, \infty) \times M$. Hence ω is a symplectic form on X. We define the **symplectification** of M, denoted $\mathrm{Symp}(M, \xi)$, to be the manifold X with symplectic form ω.

Notice that the vector field $t\frac{\partial}{\partial t}$ is transverse to $\alpha(M) \subset X$ and

$$\begin{aligned} L_{t\frac{\partial}{\partial t}}\omega &= L_{t\frac{\partial}{\partial t}}(dt \wedge \pi^*\alpha + t\pi^*(d\alpha)) \\ &= d\iota_{t\frac{\partial}{\partial t}}(dt \wedge \pi^*\alpha + t\pi^*(d\alpha)) = d(t\pi^*\alpha) \\ &= \omega, \end{aligned}$$

where $L_{t\frac{\partial}{\partial t}}$ is the Lie derivative. Thus we say M is a hypersurface of contact type. More generally, a hypersurface H in a symplectic manifold (X, ω) is said to be of **contact type** if there is a vector field v defined in a neighborhood of H that is everywhere transverse to H and a **symplectic dilation**, this means that $L_v\omega = \omega$. Note

$\alpha = \iota_v \omega|_H$ is a contact form on H. Indeed, $d\alpha = \omega|_H$ so we can find a vector $w \in TH$ such that $\omega(v, w) \neq 0$. Thus $\ker \alpha$ is the symplectic complement of $\{v, w\}$ in $TX|_H$ and so $d\alpha$ is nondegenerate on $\ker \alpha$.

PROPOSITION 2.13. *Given a compact hypersurface M in a symplectic manifold (X, ω) of contact type with symplectic dilation v there is a neighborhood of M in X symplectomorphic to a neighborhood of $\alpha(M)$ in* $\mathrm{Symp}(M, \xi)$ *where $\alpha = \iota_v \omega|_M$ and $\xi = \ker \alpha$.*

PROOF. Let $\omega' = d(t\alpha)$ be the symplectic form on $\mathrm{Symp}(M, \xi)$. By the tubular neighborhood theorem we can find a neighborhood of M in X that is diffeomorphic to a neighborhood of $\alpha(M)$ in $\mathrm{Symp}(M, \xi)$ and sends the flow lines of v to the flow lines of $\frac{\partial}{\partial t}$. Now ω' on $\alpha(M)$ is just $d\alpha$ and ω on $M \subset X$ is also $d\alpha$. Finally, choosing the above diffeomorphism between tubular neighborhood correctly, we can arrange that ω' on $T(\mathrm{Symp}(M, \xi))|_M$ agrees with ω on $TX|_M$. Hence by Theorem 2.3 our diffeomorphism may be isotoped into a symplectomorphism. $\square$

3. Contact Geometry on 3-Manifolds

Let M be a 3-manifold. A transversely oriented contact structure ξ on M gives a preferred orientation on M by $\alpha \wedge d\alpha$ where α is a contact 1-form for ξ. We will always assume that M is given this preferred orientation.

If Σ is a surface embedded in M then we may assume (after possibly a small isotopy) that there are a finite number of points $\{p_1, \ldots, p_n\}$ where Σ is tangent to ξ, i.e.

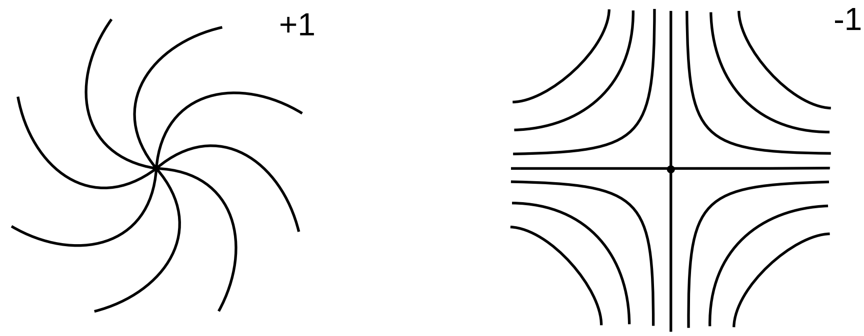
$$T_{p_j}\Sigma = \xi_{p_j} \quad \text{for } j = 1, \ldots, n.$$

On the rest of Σ we have $T\Sigma$ transverse to ξ so if $m \in \Sigma \setminus \{p_1, \ldots, p_n\}$ then

$$T_m\Sigma \cap \xi_m = l_m \text{ a line in } T_m\Sigma.$$

This line field on $\Sigma \setminus \{p_1, \ldots, p_n\}$ integrates to give a foliation on Σ with singularities at the p_j's called the **characteristic foliation** of Σ, denoted by Σ_ξ. We may always locally orient the foliation Σ_ξ thus the index of a the line field l at a singularity is always defined. Generically, the index will be ± 1. A singular point p_j with index $+1$ will be called an **elliptic** point (see Figure 2.1 (a)) and one with index -1 will be called a **hyperbolic** point (see Figure 2.1 (b)).

Since M is oriented and ξ is transversely oriented there is an orientation of each of the ξ_m's. So if Σ is also oriented then we may ask if the orientations of $T_{p_j}\Sigma$ and ξ_{p_j} agree or not. If they agree then we call the point p_j **positive** and if they disagree it is called **negative**. The orientation on Σ also allows us to orient the line field l on



(a) elliptic singularity (b) hyperbolic singularity

FIGURE 2.1

$\Sigma \setminus \{p_1, \ldots, p_n\}$. Thus we get more than a foliation, we actually get a flow. The positive elliptic points of Σ_ξ are sources of the flow and the negative elliptic points are sinks for the flow. The sign of a hyperbolic singularity does not have such an easy interpretation.

We now consider two fundamental examples of contact structures on $\mathbb{R}^3$. Let ρ, ϕ, z be cylindrical coordinates on $\mathbb{R}^3$ and set

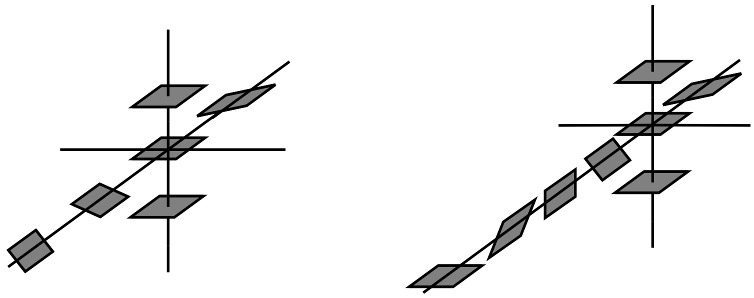
$$\alpha_0 = dz + \rho^2 d\phi.$$

The first contact structure we wish to consider is $\xi_0 = \ker \alpha_0$. In Figure 2.2 (a) we have drawn a few representative hyperplanes in ξ_0. One should notice that the hyperplanes in ξ_0 are horizontal along the z-axis and along any ray perpendicular to the z-axis the hyperplanes twist 90 degrees to the left (from horizontal on the z-axis to vertical "at infinity") as they move out to infinity along the ray. Now let

$$\alpha_1 = \cos \rho \, dz + \rho \sin \rho \, d\phi$$

and consider the contact structure $\xi_1 = \ker \alpha_1$. A few of the hyperplanes in ξ_1 may be seen in Figure 2.2 (b). Once again we see that along the z-axis the hyperplanes are horizontal but as we travel along a ray perpendicular to the z-axis the hyperplanes twist to the left infinitely often.

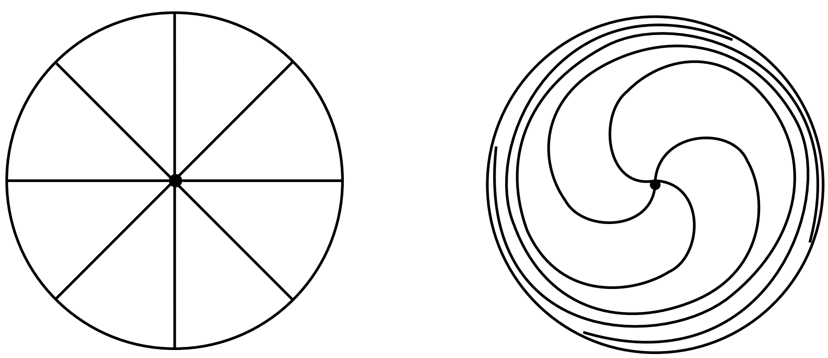
We would now like to consider the characteristic foliation induced on the disk $D = \{(\rho, \phi, z) : z = 0 \text{ and } \rho \leq \pi\}$ for each of the above contact structures. The characteristic foliation D_{ξ_0} is shown in Figure 2.3 (a). Notice that D is not in general position with respect to ξ_1 so we will bump the boundary of D up (here D is actually $\{(\rho, \phi, z) : z = \epsilon\rho^2 \text{ and } \rho \leq \pi\}$). The characteristic foliation D_{ξ_1} is shown in Figure 2.3 (b). Notice that D_{ξ_0} just looks like the standard



(a) the contact structure ξ_0 (b) the contact structure ξ_1

FIGURE 2.2

picture of an elliptic singularity but D_{ξ_1} has an elliptic singularity and its boundary is a limit cycle. We call D_{ξ_1} the **standard overtwisted disk**.



(a) Standard Disk (b) Overtwisted Disk

FIGURE 2.3

A contact 3-manifold (M, ξ) is called **overtwisted** if there exists a disk D embedded in M with D_ξ diffeomorphic to D_{ξ_1} (i.e. D_ξ contains a limit cycle and exactly one elliptic point). We call (M, ξ) **tight** if for any embedded disk D, D_ξ contains no limit cycles. Clearly, an overtwisted contact structure is not tight. In [**E4**] Eliashberg showed that a contact structure that is not overtwisted is tight. Thus contact structures fall into two disjoint classes: overtwisted and tight.

In [**L**] Lutz showed that given any 2-dimensional distribution on a 3-manifold there is an overtwisted contact structure homotopic to it. Later, in [**E1**], Eliashberg showed that homotopy classes of 2-plane fields and overtwisted contact structures (up to homotopy through contact structures) are in one to one correspondence. Thus questions about the existence and uniqueness of overtwisted contact structures reduces to questions about 2-plane fields, which are well understood in terms of algebraic topology (see Chapter 4 for more details on this). Note, in particular, that the above results imply that any 3-manifold admits a contact structure.

Tight contact structures are much more subtle. Until recently there was very little known about the existence or uniqueness of tight contact structures. The first existence result was due to Bennequin. In his 1983 paper [**Be**] he showed that the standard contact structure (described above) on S^3 is tight. A few years later, in [**Gr1**], Gromov showed that a symplectically fillable contact structure is tight. A contact 3-manifold (M, ξ) is called **symplectically fillable** if there is a symplectic 4-manifold (X, ω) that M bounds so that $\omega|_\xi$ does not vanish and the orientation M inherits as the boundary of X agrees with the one induced by ξ. Using a result of Eliashberg's (see [**E2**] and Chapter 3 below), Gompf [**G3**] and Eliashberg [**E5**] have managed to construct many examples of tight contact structures.

Even less is known about the uniqueness of tight contact structures than about the existence. In [**E4**] Eliashberg proved the following theorem:

THEOREM 2.14 (Eliashberg, 1992). *Two tight contact structures on the ball B^3 which induce the same characteristic foliations on ∂B^3 are isotopic relative to ∂B^3.*

From this and Darboux's theorem one may easily show that S^3 has a unique tight contact structure. Eliashberg also managed to show that $\mathbb{R}^3$ has only one tight structure. In Chapter 5 we shall prove of the following unpublished result of Eliashberg's:

COROLLARY 2.15. *There is only one tight contact structure on $\mathbb{R}P^3$.*

3.1. Legendrian and Transversal Curves. There are two particularly interesting types of curves in a contact 3-manifold (M, ξ). A closed curve $\gamma : S^1 \longrightarrow M$ is called **Legendrian** if the vectors tangent to γ, $\gamma'(t)$, are in $\xi_{\gamma(t)}$. We call γ **transversal** if $\gamma'(t)$ is always transverse to $\xi_{\gamma(t)}$. A transverse curve is positive or negative according to whether or not the vectors $\{\gamma'(t), \text{oriented basis for } \xi_{\gamma(t)}\}$ form a positively oriented basis for $T_{\gamma(t)}M$. The nonintegrability of ξ allows us to show the following lemma:

LEMMA 2.16. *Any curve in a contact 3-manifold may be made Legendrian by a C^0-small isotopy.*

To get a better idea of what Legendrian curves look like and to get an idea of how to prove this lemma, consider Legendrian knots in the standard contact structure ξ_0 on $\mathbb{R}^3$. This structure is given by the from $\alpha = dz + xdy$ (the reader should check that this form induces the same contact structure as α_0 above). So if γ is a Legendrian curve then $\alpha(\gamma') = 0$ or $x = -\frac{dz}{dy}$. Thus the x coordinate of $\gamma(t)$ is determined by the slope of $\gamma(t)$ projected into the yz-plane. In other words given a curve drawn in the yz-plane with no vertical tangencies, we may construct a Legendrian curve in $\mathbb{R}^3$ that projects to it. Figure 2.4 is a typical example of a yz-projection of a Legendrian knot (the cusps denote places where the curve is parallel to the x-axis). Note that we

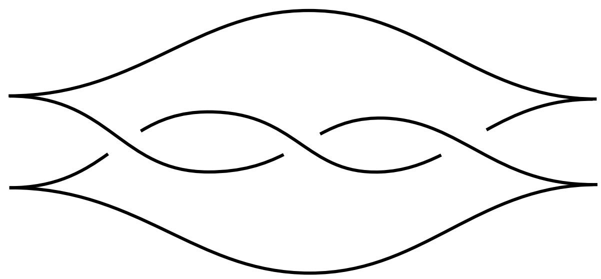


Figure 2.4. Legendrian Knot

do not have to draw the crossings in the projection a Legendrian knot since they are determined by the slope of the two curves involved; we shall, however, continue to draw the crossings to avoid confusion. The idea behind the lemma is now quite simple. We begin by isotoping the curve (in $\mathbb{R}^3$) a little to make its projection onto the yz-plane nice (only isolated double points). We now leave it to the reader to see that by adding (C^0-small) zigzags to the projection the resulting Legendrian curve will be C^0-close to the original curve. In a general manifold we perform the above procedure in Darboux charts covering the curve γ.

A framing on a knot γ is given by a vector field along γ that is transverse to it. Now given a Legendrian curve γ in M the contact structure gives γ a canonical framing by taking the vector field in $\xi|_\gamma$ transverse to γ. If γ is null-homologous in M then it bounds a surface Σ in M and we define the **Thurston-Bennequin invariant** of γ to be

$$\mathrm{tb}(\gamma) = I(\gamma', \Sigma)$$

where γ' is obtained by pushing γ off itself using the canonical framing and $I(\gamma', \Sigma)$ is the oriented intersection number of γ' and Σ. There is another invariant of Legendrian knots called the **rotation number** of γ, denoted $r(\gamma, \Sigma)$. To define r pick a trivialization of $\xi|_\Sigma$ and let T be a vector field tangent to γ. We define $r(\gamma, \Sigma)$ to be the degree of T

with respect to this trivialization. To calculate tb and r we used a surface Σ. In general one must keep track of Σ but when the choice of Σ is irrelevant (e.g. for knots in S^3) or obvious, we will usually drop it from our notation.

If γ is a Legendrian knot in $(\mathbb{R}^3, dz + x dy)$ then it is quite easy to compute these two invariants for γ from the yz-projection. To compute the Thurston-Bennequin invariant note that $\frac{\partial}{\partial z}$ gives the canonical framing on γ and that this framing differs from the blackboard framing only at the cusps of the yz-projection. The linking number of the blackboard framing on γ is given by the writhe $w(\gamma)$. If $\lambda(\gamma)$ is the number of left cusps of the yz-projection of γ then it is easy to convince oneself that

$$\text{tb}(\gamma) = w(\gamma) - \lambda(\gamma). \tag{2.17}$$

To compute the rotation number note that $\frac{\partial}{\partial x}$ trivializes ξ on all of $\mathbb{R}^3$ hence on any surface Σ bounding γ. Thus the degree of T with respect to this trivialization is just the number of times T rotates with respect to $\frac{\partial}{\partial x}$. To calculate this let t_+ (t_-) be the number of upward (downward) cusps of the yz-projection of γ. Then we have

$$r(\gamma) = \frac{1}{2}(t_+ - t_-). \tag{2.18}$$

For more details on these formulas we refer the reader to [**G3**].

If γ is a Legendrian knot in (M, ξ) then we may associate two transversal knots to it. To this end, notice that we may embed $S = [-\epsilon, \epsilon] \times S^1$ into M so that $\{0\} \times S^1$ is mapped to γ and S_ξ is shown in Figure 2.5 (we may do this by generically embedding S so that it twists according to the canonical framing of γ). It is now clear that

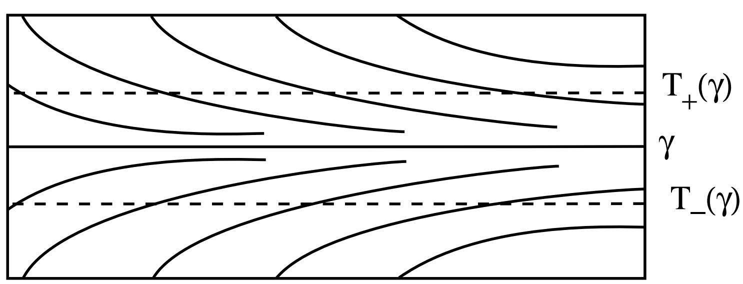


FIGURE 2.5. S_ξ

$\{\epsilon\} \times S^1$ is a positive transverse knot, denoted $T_+(\gamma)$, and $\{-\epsilon\} \times S^1$ is a negative transverse knot, denoted $T_-(\gamma)$.

Now given a transversal knot γ in (M, ξ) that bounds a surface Σ we may define the **self linking number**, $l(\gamma)$, of γ as follows: take a nonvanishing vector field v in ξ over Σ and let γ' be γ slightly pushed along v. Define

$$l(\gamma, \Sigma) = I(\gamma', \Sigma),$$

where again $I(\cdot, \cdot)$ is the oriented intersection number.

LEMMA 2.17. *If γ is a null homologous Legendrian curve on a contact 3-manifold, then*

$$l(T_{\pm}(\gamma), \Sigma') = \mathrm{tb}(\gamma, \Sigma) \mp r(\gamma, \Sigma),$$

where Σ is any surface bounding γ and Σ' is the obvious surface associated to Σ bounding $T_{\pm}(\gamma)$.

A beautiful result of Eliashberg's [**E5**] tells us a great deal about transversal unknots.

PROPOSITION 2.18. *Let γ and γ' be two transversal unknots in a tight contact 3-manifold. If $l(\gamma) = l(\gamma')$ then there is an ambient isotopy of the 3-manifold through contactomorphisms taking γ to γ'.*

A similar result is also true for Legendrian unknots. We end this section by noting that the contact structure near a transversal knot is unique.

PROPOSITION 2.19. *Any two transverse knots have contactomorphic neighborhoods.*

The proof of this proposition follows immediately from Theorem 2.12.

3.2. Surfaces in Contact 3-Manifolds. To understand 3-dimensional manifolds it is quite useful to cut them up along 2-dimensional submanifolds and study the pieces. Thanks to many powerful tools recently developed (mainly by Eliashberg and Giroux) we can also study contact structures in this way. In this section we develop these tools. We begin with the following key lemma which implies that knowing the characteristic foliation on a surface is the same as knowing the contact structure in a neighborhood of the surface.

LEMMA 2.20. *Assume ξ_0 and ξ_1 are two contact structures that induce the same (generic) characteristic foliation on a surface Σ in M (if Σ has nonempty boundary then assume ξ_0 and ξ_1 agree in some neighborhood of $\partial\Sigma$). Then there exists some neighborhood U of Σ and an isotopy $\phi_t : M \longrightarrow M$ fixed outside U (and also where ξ and ξ' already agree) such that ϕ_0 is the identity on M, $\phi_t(\Sigma) = \Sigma$, ϕ_t preserves the characteristic foliation Σ_{ξ_0} and $(\phi_1)_*(\xi_0) = \xi_1$ on some neighborhood $N \subset U$ of Σ.*

REMARK: Makar-Liminov [**ML**] has shown that one can require that ϕ_t is fixed on Σ.

Proof. We will assume that ξ_0 and ξ_1 are transversely oriented (the general case follows as in the proof of Theorem 2.9). Let α_i be a contact form for ξ_i. For the moment let us assume that $\alpha_1|_\Sigma = f\alpha_0|_\Sigma$ for some positive function f defined on Σ. We can extend f over all of M, then by rescaling α_1 we may assume that $\alpha_0|_\Sigma = \alpha_1|_\Sigma$. Now take $\alpha_t = (1-t)\alpha_0 + t\alpha_t$ and note that $\alpha_t|_\Sigma = \alpha_0|_\Sigma$. We now claim that the α_t's are all contact forms on a neighborhood of Σ. To see this, identify a neighborhood of Σ with $\Sigma \times (-\epsilon, \epsilon)$ and let z denote the coordinate in $(-\epsilon, \epsilon)$. Then

$$\alpha_t = \beta_{t,z} + u_{t,z}\, dz,$$

where $\beta_{t,z}$ is a 1-form on Σ and $u_{t,z}$ is a function on Σ for each t and z. Now $\beta_{t,0} = \beta_{0,0}$ and hence $d\beta_{t,0} = d\beta_{0,0}$ since $\alpha_t|_\Sigma = \alpha_0|_\Sigma$. One may readily compute

$$\alpha_t \wedge d\alpha_t = \beta_{t,z} \wedge d\beta_{t,z} + [\beta_{t,z} \wedge (du_{t,z} - \frac{\partial \beta_{t,z}}{\partial t}) + u_{t,z}\, d\beta_{t,z}] \wedge dz.$$

But since $\beta_{t,z}$ is a 1-from on a surface $\beta_{t,z} \wedge d\beta_{t,z} = 0$ and we get

$$\begin{aligned}
(\alpha_t \wedge d\alpha_t)|_\Sigma &= [\beta_{t,0} \wedge (du_{t,0} - \frac{\partial \beta_{t,0}}{\partial t}) + u_{t,0} d\beta_{t,0}] \wedge dz \\
&= [\beta_{0,0} \wedge ((1-t)du_{0,0} - t du_{1,0}) - \frac{\partial \beta_{0,0}}{\partial t}) \\
&\quad + ((1-t)u_{0,0} + t u_{1,0}) d\beta_{0,0}] \wedge dz \\
&= ((1-t)\alpha_0 \wedge d\alpha_0 + t\alpha_1 \wedge d\alpha_1)|_\Sigma.
\end{aligned}$$

So $(\alpha_t \wedge d\alpha_t)|_\Sigma$ is a positive volume from on $TM|_\Sigma$ and thus on $TM|_U$, where U is a neighborhood of Σ. Hence α_t is a contact form on U for all t. Now Gray's Theorem gives us a vector field $v_t \in \xi_t = \ker \alpha_t$ that satisfies

$$\iota_{v_t}\, d\alpha_t|_{\xi_t} = -\frac{d\alpha_t}{dt}|_{\xi_t}.$$

Let $N \subset U$ be a neighborhood of Σ and taper v_t to 0 outside of U. Let ϕ_t be the flow generated by v_t. Then inside N we have $(\phi_1)_*\xi_0 = \xi_1$. Now on Σ we have

$$\iota_{v_t}\, d\alpha_t|_{\xi_t} = -\frac{d\alpha_t}{dt}|_{\xi_t} = -\frac{\partial u_{t,z}}{\partial t}\, dz,$$

which vanishes on Σ (since $\Sigma = \{z = 0\}$). This and the fact that ξ_t is two dimensional imply that $v_t \in \xi_t \cap T\Sigma$ and hence ϕ_t preserve Σ and Σ_{ξ_0}.

To complete the proof we are left to see that $\alpha_1|_\Sigma = f\alpha_0|_\Sigma$ for some positive function f defined on Σ. Since α_0 and α_1 have the same kernel it is easy to see that we can define a unique such f on $\Sigma \setminus \{\text{singular points}\}$. Let $U = \mathbb{R}^2$ be a coordinate patch (in Σ) about a singular point. So $\alpha_i|_U : U \longrightarrow \mathbb{R}^2$ with an isolated zero at the origin. We can choose our local coordinates so that $\alpha_0|_U : U \longrightarrow \mathbb{R}^2$ is just the identity map $\mathrm{id} : U \longrightarrow \mathbb{R}^2$ (since Σ_{ξ_0} is generic). Now $\alpha_1 = f\alpha_0 = f(\mathrm{id})$. So f extends smoothly across the origin and must be nonzero since α_1 is a local diffeomorphism (again by the fact that Σ_{ξ_1} is generic). Thus we may extend f to a positive function on all of Σ. $\square$

In [**Gi**], Giroux explores the notion of convex hypersurface in a contact manifold. In our examination of tight contact structures on lens spaces in Chapter 5 convex hypersurfaces will play a crucial role. In a contact manifold (M^3, ξ) a vector field is called a **contact vector field** if its flow preserves ξ. A surface Σ in M is called **convex** if there is a contact vector field transverse to it. We refer the reader to [**Gi**] to see that a contact vector field that is defined on part of a manifold may be extended over the entire manifold. Thus in our definition of convex surface we might as well assume that the transverse contact vector field is globally defined. We have the following characterization of convex surfaces:

LEMMA 2.21. *Let Σ be a compact surface in the contact manifold (M^3, ξ). Then Σ is a convex surface if and only if there is a tubular neighborhood N of Σ in M that is contactomorphic to $(\Sigma \times (-\epsilon, \epsilon), \beta + u\,dt)$ taking Σ to $\Sigma \times \{0\}$, where β is a 1-form on Σ and u is a function on Σ and $\epsilon > 0$.*

PROOF. First, suppose there is a tubular neighborhood N of Σ in M and a contactomorphism $\psi : (N, \xi|_N) \longrightarrow (\Sigma \times (-\epsilon, \epsilon), \beta + u\,dt)$. Clearly $\frac{\partial}{\partial t}$ is a contact vector field on $\Sigma \times (-\epsilon, \epsilon)$ that is transverse to $\Sigma \times \{0\}$. Thus $(\psi^{-1})_* \frac{\partial}{\partial t}$ is a contact vector field defined on N and transverse to Σ. Hence Σ is a convex surface.

Conversely, assume that Σ is convex and let v be the globally defined transverse vector field on M. Let $\psi_t : M \longrightarrow M$ be the flow generated by v for $t \in (-\epsilon, \epsilon)$ (note: we may take ϵ small enough that $\psi_t(\Sigma) \cap \Sigma = \emptyset$ for all $t \in (-\epsilon, \epsilon)$). Now let α be a contact 1-form on M generating ξ. Set $\beta = \alpha|_\Sigma$ and $u = (\iota_v \alpha)|_\Sigma$. The map $\psi : \Sigma \times (-\epsilon, \epsilon) \longrightarrow M$ is a diffeomorphism onto its range, N, which is a tubular neighborhood of Σ. Moreover, we claim that it is actually a contactomorphism from $(\Sigma \times (-\epsilon, \epsilon), \alpha')$ to (M, α), where $\alpha' = \beta + u\,dt$. Indeed, clearly

$$\psi^* \alpha = \alpha'$$

along $\Sigma \times \{0\}$. Now since v is a contact vector field it is easy to check that $\psi^* \alpha = g\,\alpha'$ on all of $\Sigma \times (-\epsilon, \epsilon)$, for some function $g : (-\epsilon, \epsilon) \longrightarrow (0, \infty)$. $\square$

In Chapter 5 we shall need the following lemma to identify convex surfaces.

LEMMA 2.22. *A surface Σ in a contact manifold (M^3, ξ) is convex if the flow of the characteristic foliation Σ_ξ satisfies:*

i) *All the singularities and closed orbits are hyperbolic (in the dynamical systems sense),*
ii) *No trajectory of Σ_ξ travels from a negative singularity to a positive singularity, and*
iii) *The orbit of any point under the flow limits to either a singular point or a closed orbit.*

A characteristic foliation that satisfies i)–iii) is sometimes referred to as an **almost Morse-Smale** foliation. Below we will give an English translation of Giroux's original proof of this lemma that appeared in [**Gi**]

PROOF. The last two lemmas tell us that it suffices to construct a vertically invariant structure on $\Sigma \times \mathbb{R}$ such that the characteristic foliation on $\Sigma \times \{0\}$ is diffeomorphic to Σ_ξ. Denote the foliation Σ_ξ by F. The strategy of the proof will be to find an area form ω on Σ, a function $u : \Sigma \longrightarrow \mathbb{R}$ and a vector field v that points along F such that

$$u \operatorname{div}_\omega v - v \cdot u > 0. \tag{2.19}$$

Given these let $\beta = \iota_v \omega$ and define the 1-form $\alpha' = \beta + u\, dt$ on $\Sigma \times \mathbb{R}$. Note

$$\begin{aligned} \alpha' \wedge d\alpha' &= (\beta \wedge du + u\, d\beta) \wedge dt \\ &= (\iota_v \omega \wedge du + u\, d(\iota_v \omega)) \wedge dt \\ &= (-v \cdot u + u \operatorname{div}_\omega v)\, \omega \wedge dt \neq 0. \end{aligned}$$

Thus α' is a contact 1-form on $\Sigma \times \mathbb{R}$ which is clearly vertically invariant. Moreover, the characteristic foliation on $\Sigma \times \{0\}$ is given by the kernel of β which is spanned by v when $v \neq 0$ and is everything when $v = 0$. Hence, F is the characteristic foliation on $\Sigma \times \{0\}$.

Now for the construction of v. Choose small disjoint disks around the elliptic points in F and small disjoint annuli around each periodic orbit (also disjoint from the disks). Next, choose small bands about the stable separatrices of the positive hyperbolic points and about the unstable separatrices of the negative hyperbolic points (these will in general intersect the disks and annuli). Let Σ_0 be the union of these disks, bands and annuli. See Figure 2.6. We may now find a vector field v in a neighborhood of each singularity that has positive divergence at the singularity (recall that the divergence of a vector field at

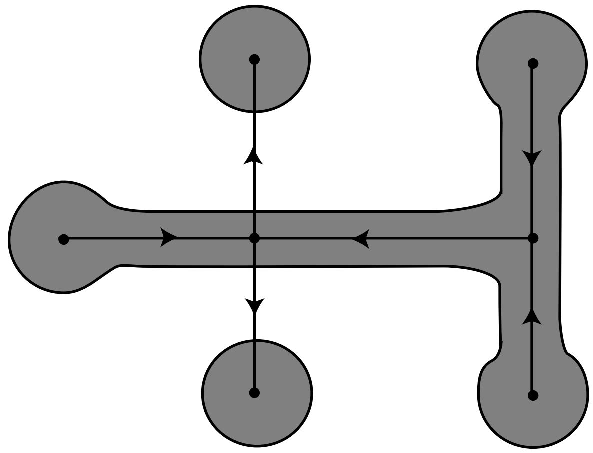


FIGURE 2.6. Gray part is Σ_0

a singular point does not depend on the volume form) and is tangent to F. Property ii) insures that we can extend v to a vector field on a neighborhood U of Σ_0 that is tangent to F and points transversely out of Σ_0. One may easily construct a volume from ω on Σ such that $\operatorname{div}_\omega v > 0$. Now if we take (for the moment) $u = 1$ on U then on U we have v, ω and u satisfying Equation (2.19). We would now like to extend v and u over the rest of Σ. To this end, set $\Sigma_1 = \overline{(\Sigma \setminus \Sigma_0)}$. Notice that Σ_1 is a surface with boundary that supports a nonsingular foliation. In addition, property iii) says that we can take the foliation on Σ_1 to be transverse to the boundary. Thus the components of Σ_1 must be annuli foliated by arcs intersecting the boundary transversely. Let v' be a nonzero vector field on Σ_1 tangent to F and agreeing with $\pm v$ on $U' = U \cap \Sigma_1$. Finally set v'' equal to v' on Σ_1 and equal to $\pm v$ on σ_0 whichever agrees with v' on U'. We now redefine u to be 1 where $v = v''$ and -1 where $v = -v''$ and notice that u, v'' and ω satisfy Equation (2.19) on Σ_0.

To complete the proof we need to extend u over Σ_1. We do this using the following observation: given a function $f : [0,1] \longrightarrow \mathbb{R}$ that is positive at 0 and negative at 1 there exists a function $g : [0,1] \longrightarrow \mathbb{R}$ such that g equals 1 near 0 and -1 near 1 and satisfies

$$gf - \frac{dg}{dt} > 0. \tag{2.20}$$

To see that this is true set $g(t) = h(t)\exp(\int_0^t f(x)\,dx)$ for some function h with $h' < 0$. We see that this g will satisfy Equation (2.20) and a little though shows how to pick h so that g satisfies all the above requirements. This observation tells us how to extend u over one leaf of $F \cap \Sigma_1$ (where $f = (\text{div}_\omega v)|_{leaf}$ and g will be the extension of u). Notice that smoothly varying f will smoothly vary g, thus we can extend u over all of Σ_1. $\square$

We now proceed to study characteristic foliations on a surfaces in a contact 3-manifolds. One of the main tool for manipulating characteristic foliations is the elimination lemma (proved in various forms by Giroux, Eliashberg and Fuchs, see [**E5**]) which allows one to cancel singularities under the right conditions.

LEMMA 2.23 (Elimination Lemma). *Let Σ be a surface in a contact 3-manifold (M, ξ). Assume that p is an elliptic and q is a hyperbolic singular point in Σ_ξ, they both have the same sign and there is a leaf γ in the characteristic foliation Σ_ξ that connects p to q. Then there is a C^0-small isotopy $\phi : \Sigma \times [0,1] \longrightarrow M$ such that ϕ_0 is the inclusion map, ϕ_t is fixed on γ and outside any (arbitrarily small) pre-assigned neighborhood U of γ and $\Sigma' = \phi_1(\Sigma)$ has no singularities inside U.*

An excellent proof of this may be found in [**E5**] (see also [**A**]). The idea is to construct a model of the part of the leaf of Σ_ξ connecting p and q using Lemmas 2.22 and 2.21. Then, in this model, explicitly write down the desired isotopy. Notice that the elliptic–hyperbolic pair that are canceled in this lemma are contained in a smooth curve in the characteristic foliation and that after the cancellation this curve remains part of the characteristic foliation. The proof also indicates that one can clearly add an elliptic-hyperbolic pair of singularities to a characteristic foliation along any leaf.

It is interesting to note that one has great freedom to alter the characteristic foliation near an elliptic point. We measure this freedom by means of a monodromy map, which we define for the characteristic foliation on C^0-small perturbations of the disk $D = \{(r, \theta, 0) : r \leq \pi\}$ in $(\mathbb{R}^3, \alpha = dz + r^2\, d\theta)$. Although this is a very restricted case it will be sufficient for our purposes. To the elliptic point e in D_ξ we associate a map $m : S^1 \longrightarrow S^1$ as follows: given a direction $\vartheta \in S^1$ at e the map m sends ϑ to the θ coordinate of the point where the trajectory of D_ξ leaving e in the direction ϑ exists the disk. Looking at Figure 2.3 (a) it is easy to see that $m : S^1 \longrightarrow S^1$ is the identity map for D_ξ; however, up to C^0-perturbations, we have a great deal of freedom to alter m.

LEMMA 2.24. *A C^0-small perturbation of D near the elliptic point e realizes any orientation preserving diffeomorphism of S^1 as the monodromy map of e.*

PROOF. Consider the map

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : (x, y) \mapsto (\frac{1}{y}, \gamma(x, y), f(x, y)).$$

If $f \equiv 0$, then the closure of the image of $R = \{(x, y) : 0 \leq x \leq 2\pi, y \geq \pi^{-1}\}$ is the disk D in the $r\theta$-plane of $\mathbb{R}^3$. The lines $\{x = c\}$ are mapped by F to curves $\gamma(c, y)$ in D that foliate all of D with a single singular point at the origin. If this was the characteristic foliation of D then by the proper choice of γ we would be done. To achieve this let

$$f(x, y) = -\int \frac{1}{(y + m)^2} \gamma_y(x, y)\, dy,$$

where m is a constant to be determined later. Note without loss of generality we can assume that $\gamma(x, y) = x$ for y outside some compact interval. Then consider

$$F^*\alpha = (f_x + \frac{1}{(y + m)^2}\gamma_x)\, dx + (f_y + \frac{1}{(y + m)^2}\gamma_y)\, dy.$$

One may easily compute that $f_y + \frac{1}{(y+m)^2}\gamma_y = 0$. Moreover, by choosing m large enough one can show that $f_x + \frac{1}{(y+m)^2}\gamma_x > 0$. Thus the lines $\{x = c\}$ will be tangent to $\ker F^*\alpha$. Hence $\gamma(c, y)$ will be leaves in the characteristic foliation of D', the closure of the image of F. Finally for m sufficiently large $|f(x, y)| < \epsilon$ for any prechosen ϵ. Hence D' is a C^0-small perturbation of D. $\square$

An easy corollary of this lemma is the following useful result:

COROLLARY 2.25. *By a C^0-small perturbation of a surface near an elliptic point e we can assume that any two rays that abut e lie on a smooth curve through e.*

Later we will also need the following manipulation of the characteristic foliation.

LEMMA 2.26. *Let p be a hyperbolic point in the characteristic foliation of a surface Σ and γ and γ' the union of separatrices of p indicated in Figure 2.7. Then there is a C^0-small perturbation of Σ fixed off of an arbitrarily small neighborhood of γ and on γ and γ' that achieves the modification of Σ_ξ indicated in Figure 2.7*

The proof of Lemma 2.26 involves constructing a model for the contact structure near p and then explicitly writing down the isotopy. The details of this argument may be found in [**F**].

Now if Σ is a surface in a contact manifold (M, ξ) then we can define $e_\pm$ and $h_\pm$ to be the number of $\pm$ elliptic and hyperbolic points, respectively, in Σ_ξ. Furthermore, set $d_\pm = e_\pm - h_\pm$.

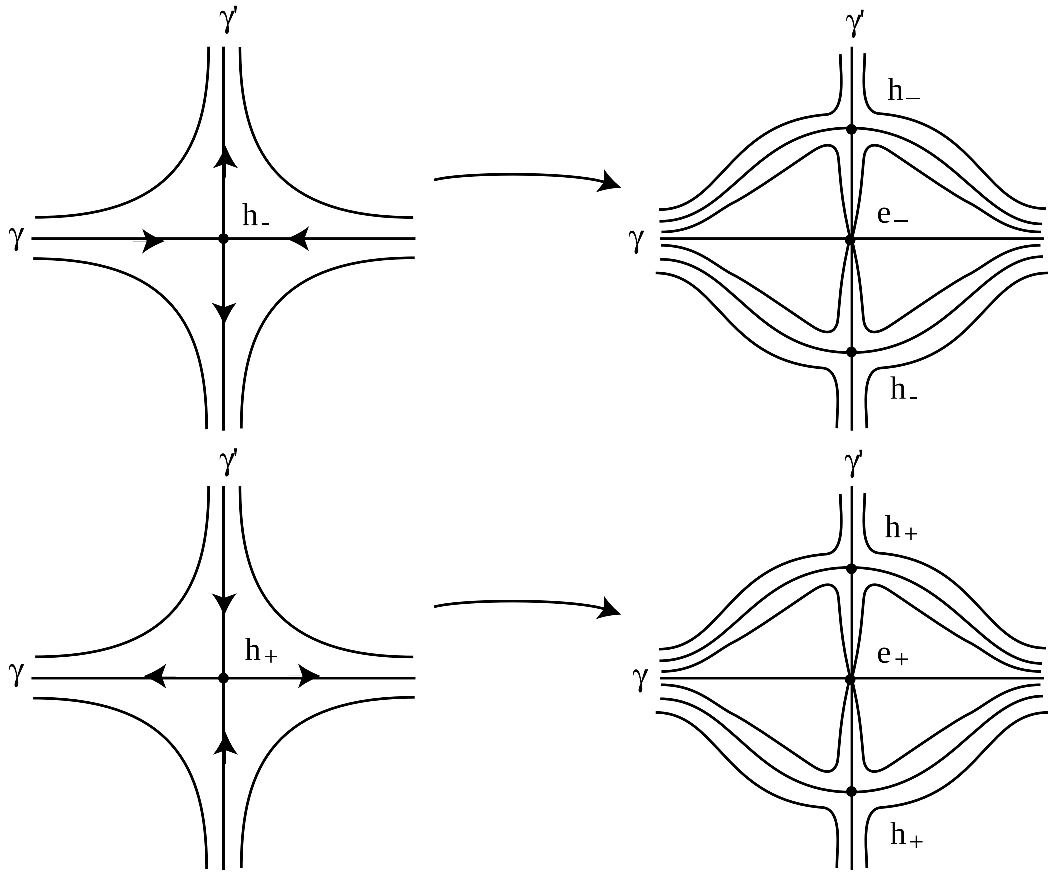


FIGURE 2.7. Alteration of Σ_ξ near a hyperbolic point.

PROPOSITION 2.27. *If $\Sigma \subset (M, \xi)$ is a closed surface, then*

$$d_\pm = \frac{1}{2}(\chi(\Sigma) \pm e(\xi)[\Sigma]),$$

where $e(\xi) \in H^2(M, \mathbb{Z})$ is the Euler class of ξ. If Σ has boundary γ a transversal knot, then

$$d_\pm = \frac{1}{2}(\chi(\Sigma) \mp l(\gamma)),$$

where $l(\gamma)$ is computed with respect to Σ.

A sketch of this proposition may be found in the proof of Theorem 5.4.

We now restrict our attention to tight contact manifolds for the rest of this section. On tight manifolds one can simplify the characteristic foliation on a surface quite a bit. The techniques we will discuss here were first developed by Eliashberg (see [**E4**] and [**E5**]).

THEOREM 2.28. *Let Σ be a closed surface in a tight contact 3-manifold. If Σ is a closed surface, then*

$$|e(\xi)[\Sigma]| \leq \begin{cases} 0 & \text{if } \Sigma = S^2, \\ -\chi(\Sigma) & \text{if } \Sigma \neq S^2. \end{cases}$$

If Σ has boundary γ a transversal knot to ξ, then

$$l(\gamma) \leq -\chi(\Sigma).$$

If Σ has boundary γ a Legendrian knot, then

$$\mathrm{tb}(\gamma) \leq -\chi(\Sigma) - |r(\gamma)|.$$

The third part of this theorem follows from the second part since for any Legendrian knot γ, Lemma 2.17 tells us that $l(T_\pm(\gamma)) = \mathrm{tb}(\gamma) \mp r(\gamma)$. The first two parts of this theorem will follow from Proposition 2.27 when we show that $d_- \leq 0$. We can show this by canceling all the negative elliptic points. To this end we develop some of Eliashberg's theory of Legendrian polygons and basins. A **Legendrian polygon** in Σ is a pair (D, f) where D is an oriented surface with piecewise smooth boundary and $f : D \longrightarrow \Sigma$ is an orientation preserving immersion such that f is injective on the interior of D, corners (vertices) of ∂D are mapped to singularities of Σ_ξ, smooth edges of ∂D are mapped to smooth leaves in Σ_ξ and f does not identify adjacent edges of ∂D. A Legendrian polygon is called simply connected if D is simply connected. By convention any elliptic point on ∂D will be thought of as a vertex. The only singularities of Σ_ξ the edges of ∂D can contain are hyperbolic, these will be called pseudoverticies.

Given an elliptic singularity e in Σ_ξ. We may look at the **basin** B_e of e, defined to be the union of all leaves of Σ_ξ that limit on e. It is quite clear that if $\overline{B_e}$ contains no limit cycles of Σ_ξ then it has the structure of a simply connected Legendrian polygon (if Σ has boundary then we must also assume that B_e is disjoint from the boundary). Moreover, the elliptic vertices of $\partial\overline{B_e}$ must all have sign opposite that of e (since a $+(-)$ elliptic point is a source (sink) of the flow).

THEOREM 2.29. *Let Σ be a surface in a tight contact manifold (M, ξ) (with at most one boundary component) with boundary transverse to ξ. After a C^0 small isotopy of Σ rel $\partial\Sigma$ we may assume that Σ_ξ has no positive hyperbolic or negative elliptic singularities.*

We will use this theorem later to keep track of characteristic foliations on surfaces but as mentioned above an immediate corollary is Theorem 2.28.

PROOF. We begin by trying to eliminate the negative elliptic points. Let $e \in \Sigma$ be such a point. Then $\overline{B_e}$ is a simply connected Legendrian polygon since we can ensure that $\overline{B_e}$ contains no limit cycles by adding a positive hyperbolic/elliptic pair of singular points along any repelling limit cycle (if Σ has boundary we orient Σ so that all the trajectories of Σ_ξ exit through the boundary, thus B_e will be disjoint from the boundary of Σ). Note using Lemma 2.26 we can ensure that no edges of the Legendrian polygon are identified. We may also assume that no vertices in the boundary of B_e are identified since if there was a pair (necessarily elliptic) we could introduce a positive elliptic/hyperbolic pair of points in B_e near them, the result of this will be to change B_e by separating the offending pair of points. All the elliptic vertices of $\partial\overline{B_e}$ are positive, thus there must be some hyperbolic points along the boundary. At least one of these hyperbolic points must be negative because if not then we would be able to use the the elimination lemma to cancel all the elliptic and hyperbolic singular points along $\partial\overline{B_e}$. Using Corollary 2.25 this would leave us with an embedded disk $\overline{B_e}$ that contained a limit cycle contradicting the tightness of ξ. We may now cancel e with one of the negative hyperbolic points along $\partial\overline{B_e}$. (If we had to use Lemma 2.26 then we might have added some negative elliptic points, but we can cancel all of them now.) Proceeding in this manner we may remove all the negative elliptic points.

Now given a positive hyperbolic point h consider a stable separatrix of h. This separatrix must have originated at a positive elliptic point since it could not have originated along the boundary of Σ (all the trajectories of Σ flow out of Σ) or at another hyperbolic point (by the genericity of Σ_ξ). Thus we may cancel h with this elliptic point along the separatrix. $\square$

We would like to end this section by noticing that given a surface "with corners" one may sometimes smooth the corners without changing the characteristic foliation. This was first observed by Makar-Liminov in [**ML**]. We begin by saying that Σ in a contact manifold (M, ξ) is a **surface with corners** if $\Sigma = \cup_{i=0}^n \Sigma_i$ where each of the Σ_i's is a smooth surface embedded in M satisfying the following: if $\Sigma_i \cap \Sigma_j \neq \emptyset$ $(i \neq j)$ then $\Sigma_i \cap \Sigma_j$ (called **corners** of Σ) is the union of components of the boundary of Σ_i (and Σ_j) and the intersection is transverse in M (i.e. $T_p\Sigma_i \cup T_p\Sigma_j$ spans T_pM for all $p \in \Sigma_i \cap \Sigma_j$), only two of the Σ_i's meat at a corner and $\Sigma_i \cap \Sigma_j$ is transverse to ξ. Notice that there is a well defined characteristic foliation on a surface with corners. Given a surface with corners we can clearly smooth out the corners. The following lemma tells us that we can smooth out the corners without changing the characteristic foliation.

Lemma 2.30. *Let Σ be a surface with corners in the contact manifold (M, ξ). Then there is a smooth embedded surface Σ' in M that agrees with Σ away from the corners and is C^0-close to Σ near the corners. Moreover, the characteristic foliation on Σ is homeomorphic to the characteristic foliation on Σ'.*

Idea of Proof. In a neighborhood N of a corner there are no singularities of the characteristic foliation of Σ. This neighborhood looks like a neighborhood N' of the z-axis in $(\mathbb{R}^3, dz + x\,dy)$ (mod $z \mapsto z+1$) with $N \cap \Sigma$ corresponding to the union of $\{(x,y,z) \in \mathbb{R}^3 : y = 0, x \geq 0\}$ and $\{(x,y,z) \in \mathbb{R}^3 : x = 0, y \geq 0\}$. We cannot use Lemma 2.20 to make this statement precise since Σ is not a smooth surface near the corner. We can however strengthen Lemma 2.20 (see for example the proof of Theorem 5.3) to get a contactomorphism from N to N'. We are left to see that we can smooth out the intersection of $\{(x,y,z) \in \mathbb{R}^3 : y = 0, x \geq 0\}$ and $\{(x,y,z) \in \mathbb{R}^3 : x = 0, y \geq 0\}$ in $\mathbb{R}^3$. One can explicitly do this. The details are left to the reader (or see [**ML**]). $\square$

4. A Little 4-Dimensional Topology

To describe 4-manifolds we will make extensive use of Kirby calculus. This is a way to describe and manipulate a handlebody decomposition of a 4-manifold using links in the 3-sphere. We will briefly describe these pictures; for a complete explanation the reader is referred to [**K1**] or [**GS**]. Given a Kirby picture (for example see Figure 2.8 or 2.9) one starts with a 4-ball and thinks of the picture as sitting in the boundary 3-sphere. A pair of 2-spheres in the picture (as in Figure 2.9) represents the attaching sphere of a 1-handle (a circle with a dot on it, as in Figure 2.10, also represents a 1-handle but for this see the above reference). A knot with a number by it in the picture represents the attaching sphere of a 2-handle (the number is the framing of the attaching sphere). If the manifold we are considering is closed then there is a unique way to add 3- and 4-handles, hence these do not appear in the Kirby pictures.

In trying to understand 4-manifolds it is important to first understand the basic building blocks. By this we mean those manifolds that cannot be decomposed into smaller manifolds under connected sum. A 4-manifold X is called **irreducible** if whenever $X = X_0 \# X_1$ then either X_0 or X_1 is homotopic to a 4-sphere. This definition of irreducible allows us to avoid the smooth 4-dimensional Poincaré conjecture. One of the many startling results obtained from the newly defined Seiberg-Witten invariants is the following result of Kotsckick's.

Theorem. *A closed, minimal simply connected symplectic 4-manifold is irreducible.*

One may find a proof of this beautiful result in [**Ko**]. The proof relies on the relation Taubes found between Seiberg-Witten invariants and pseudo-holomorphic curves in a symplectic manifold (see [**T**]). As mentioned in the introduction the main motivation for our work is to better understand to what extent the converse of this theorem holds. In the next two subsections we will consider two cut-and-paste operations that have been used to construct irreducible 4-manifolds.

4.1. Logarithmic Transformations in a Cusp Neighborhood. Consider a smooth 4-manifold X which contains an embedded 2-torus T^2 with trivial normal bundle: i.e., a neighborhood of T^2 in X is diffeomorphic to $T^2 \times D^2$. We form the (**generalized**) **logarithmic transform** (or log transform) of X along T^2 by gluing $T^2 \times D^2$ to $\overline{X \setminus (T^2 \times D^2)}$ by some diffeomorphism

$$\psi : \partial(T^2 \times D^2) \longrightarrow \partial(T^2 \times D^2).$$

We denote the result of performing a log transform on X by X_ψ. This construction arose from and received its name from algebraic geometry.

Lemma 2.31. *The diffeomorphism type of X_ψ is determined by*

$$\psi(\{pt\} \times \partial D^2).$$

Thus a log transform is uniquely specified by a primitive element in $H_1(T^3; \mathbb{Z})$.

Proof. Gluing $T^2 \times D^2$ to $\overline{X \setminus (T^2 \times D^2)}$ is the same as adding a 2-handle, two 3-handles and a 4-handle to $\overline{X \setminus (T^2 \times D^2)}$. Once the 2-handle is added there is only one way to add the remaining handles (this is a result from [**LP**]). The 2-handle is added along $\psi(\{\mathrm{pt}\} \times \partial D^2)$ with the framing determined by the product structure on $T^2 \times D^2$ and ψ. Now let $\psi' : \partial(T^2 \times D^2) \longrightarrow \partial(T^2 \times D^2)$ be another diffeomorphism such that $\psi'(\{\mathrm{pt}\} \times \partial D^2) = \psi(\{\mathrm{pt}\} \times \partial D^2)$. To complete the proof we need to show that ψ and ψ' induce the same framing on the 2-handle. Let $\phi = \psi' \circ \psi^{-1} : T^2 \times S^1 \longrightarrow T^2 \times S^1$ and notice that $\phi(\{\mathrm{pt}\} \times S^1) = \{\mathrm{pt}\} \times S^1$. If ϕ preserves the product framing on $T^2 \times D^2$ then ψ and ψ' induce the same framing on $\psi(\{\mathrm{pt}\} \times \partial D^2)$. To see that ϕ preserves the product framing, consider the $\mathbb{Z} \oplus \mathbb{Z}$ cover, $\mathbb{R}^2 \times S^1$, of $T^2 \times S^1$.

One may lift ϕ to a $\mathbb{Z} \oplus \mathbb{Z}$ equivariant map $\tilde{\phi} : \mathbb{R}^2 \times S^1 \longrightarrow \mathbb{R}^2 \times S^1$. Any two lifts of $\{\mathrm{pt}\} \times S^1$ determine the product framing of $\mathbb{R}^2 \times S^1$. Thus $\tilde{\phi}$, and hence ϕ, preserves the product framing. $\square$

The **multiplicity**, m, of a log transform is defined to be the degree of the map

$$\pi \circ \psi|_{\partial D^2} : \partial D^2 \longrightarrow \partial D^2,$$

where π is the projection of $\partial(T^2 \times D^2)$ to $S^1 = \partial D^2$. We may always assume that m is nonnegative since there is a diffeomorphism of $T^3 = \partial(T^2 \times D^2)$ that takes ∂D^2 to itself and reverses the orientation on it. From Lemma 2.31 we know that a log transform is determined by a primitive element α_ψ of

$$H_1(T^2 \times \partial D^2; \mathbb{Z}) \cong H_1(T^2; \mathbb{Z}) \oplus H_1(\partial D^2; \mathbb{Z}).$$

projecting α_ψ onto $H_1(\partial D^2; \mathbb{Z})$ yields m times a generator of $H_1(\partial D^2; \mathbb{Z})$. Now α_ψ projected onto $H_1(T^2; \mathbb{Z})$ is $l\alpha$ for some integer l and some α a primitive element in $H_1(T^2; \mathbb{Z})$. The element α is called the **direction** of the log transform. So, in general, a log transform is determined by m, α and l where m and l must be relatively prime (since α_ψ is a primitive element in $H_1(T^3; \mathbb{Z})$). We will write $X(m, \alpha, l)$ instead of X_ψ.

A **cusp neighborhood** N in a smooth 4-manifold X is the regular neighborhood of a PL embedded 2-sphere with a single nonlocally flat point which is a cone on a right handed trefoil knot and with self intersection number 0. This 2-sphere is called the **cusp**. Any such N is diffeomorphic to the 4-manifold in Figure 2.8. Note that the 4-manifold

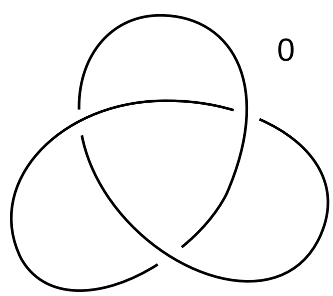


FIGURE 2.8. Cusp Neighborhood

shown in Figure 2.9 is diffeomorphic to a cusp neighborhood N. To see the diffeomorphism simply cancel the two 1-handles with the two -1 framed 2-handles in Figure 2.9; the result will be Figure 2.8. From Figure 2.9 it is not hard to see that ∂N is a T^2 bundle over S^1 with the monodromy map represented in the standard basis for T^2 by $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

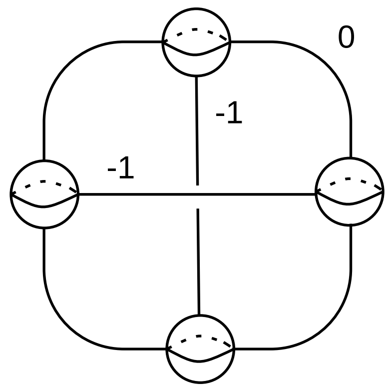


FIGURE 2.9. Another Picture of a Cusp Neighborhood

In fact, N is fibered by tori that degenerate to a singular fiber at the center. This singular fiber in the center is precisely the cusp 2-sphere that defines N as a cusp neighborhood. For more details on this and other facts about elliptic fibrations used below the reader is referred to [**HKK**]. So given a cusp neighborhood we have lots of tori on which to perform log transforms. The following theorem (first proved by Gompf in [**G1**]) shows why we are interested in cusp neighborhoods.

THEOREM 2.32. *Let N be a cusp neighborhood. The result of performing a log transform on a fiber of N is determined up to diffeomorphism relative to the boundary by the multiplicity m.*

SKETCH OF PROOF. Consider two log transforms on N. First, we may assume that they are performed on the same fiber since there is a diffeomorphism of N fixed on ∂N taking any torus fiber to another. Next, one shows that that the gluing map may be put in a standard from that depends only on the multiplicity m. More specifically, if $\psi : \partial(T^2 \times D^2) \longrightarrow \partial\overline{(N \setminus T^2 \times D^2)}$ is the map used to perform one of the log transforms then ψ may be represented by a matrix M in $\mathrm{SL}(3, \mathbb{Z})$, with respect to the basis $\lambda_1, \lambda_2, \mu$ for $H_1(T^3, \mathbb{Z})$, where λ_1, λ_2 is a basis for the first homology of T^2 and μ is a meridian to T^2 in $T^2 \times D^2$. Using diffeomorphisms of $\partial\overline{(N \setminus T^2 \times D^2)}$ that extend over $\overline{(N \setminus T^2 \times D^2)}$ (coming from the cusp fiber) and diffeomorphisms of $\partial(T^2 \times D^2)$ that extend over $T^2 \times D^2$ one can show that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & m \end{pmatrix}.$$

Thus ψ only depends on m. For more details see [**G1**]. $\square$

In [**FS1**] Fintushel and Stern made an extensive study of when one may find a cusp neighborhood in a complex surface and distinguished many non diffeomorphic h-cobordant manifolds by relating (under suitable hypothesis) the Donaldson invariants of a manifold X and its log transform $X(p)$. A Kirby picture of a p-log transformed cusp neighborhood, $N(p)$, is shown in Figure 2.10.

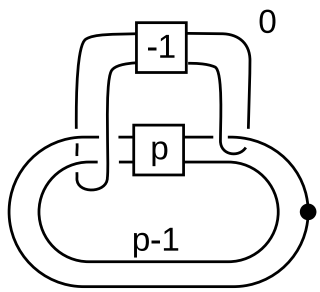


FIGURE 2.10. A p-log Transformed Cusp Neighborhood

4.2. Rational Blowdowns. Let X be a 4-manifold and suppose we can find spheres Σ_i, for $i = 0, \ldots, (p-2)$, embedded in X intersecting according to Figure 2.11, i.e. such that $\Sigma_0 \cdot \Sigma_0 = -(p+1)$, $\Sigma_i \cdot \Sigma_i = -2$, $\Sigma_{i-1} \cdot \Sigma_i = 1$, for $i = 1 \ldots (p-2)$, and all other intersections are 0. The Σ_i's have a neighborhood diffeomorphic to the

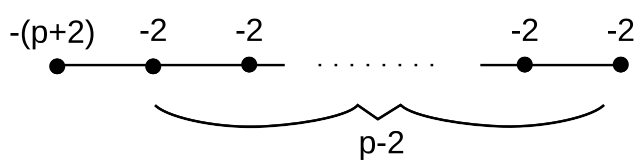


FIGURE 2.11. Graph of $C(p)$

4-manifold, $C(p)$, obtained by plumbing $(p-1)$ disk bundles over the 2-sphere together according to Figure 2.11 (a vertex represents a disk bundle with Euler number given by the label and an edge indicates the two vertices are plumbed together). An easy exercise in Kirby calculus shows that $\partial C(p)$ is the lens space $L(p^2, p-1)$. This lens space also happens to bound the 4-manifold $B(p)$ given in Figure 2.12. The **rational blowdown** of X along the Σ_i's, denoted X_p, is the result of gluing $B(p)$ to $\overline{X - C(p)}$. This is well-defined since any diffeomorphism of $L(p^2, p-1)$ extends over $B(p)$. To see this, recall in [**Bo**] Bonahon showed that $\pi_0(\text{Diff}\,(L(p^2, p-1))) = \mathbb{Z}_2$. $L(p^2, p-1)$ can be realized as the 2-fold cover of S^3 branched over a 2-bridge knot K.

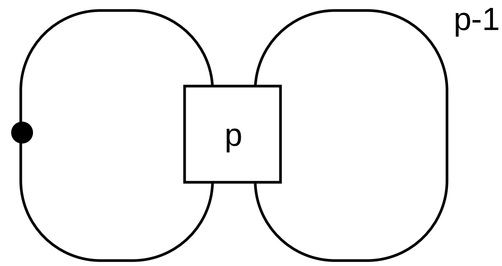


FIGURE 2.12. Rational Homology Ball

The generator g of $\pi_0(\text{Diff}\,(L(p^2, p-1)))$ is just the deck transform of this cover. Since K is slice we may consider the 2-fold cover of B^4 branched over the slicing disk. This manifold is none other than $B(p)$ (one may use the methods in [**AK**] to see this). Thus g extends over $B(p)$ as claimed. One of our main interests in rational blowdowns is the following proposition.

PROPOSITION 2.33. *Let X be a 4-manifold containing a cusp neighborhood N. The result of performing a p-log transform on a torus fiber of N, $X(p)$, may be obtained from $X\#_{(p-1)}\overline{\mathbb{C}P}^2$ by a rational blowdown.*

Before we begin the proof of this proposition we need to know that $C(p)$ is diffeomorphic to Figure 2.13. The verification of this is left as a good exercise in Kirby calculus for the reader.

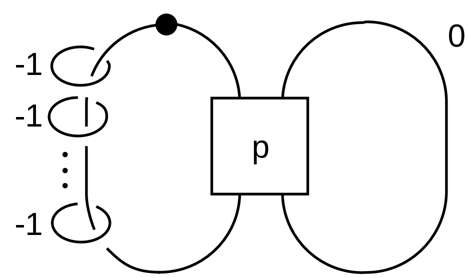


FIGURE 2.13. Another Picture of $C(p)$ ($p-1$ handles with -1 framing)

PROOF. In Figure 2.10, a p-log transformed cusp neighborhood, we see a copy of the $B(p)$. Figure 2.14 is obtained from Figure 2.10 by a rational blowup (we remove $B(p)$ and glue in $C(p)$). Pushing the 0 framed 2-handle that is linking the 1-handle p times over each of the -1 framed 2-handles and isotoping yields Figure 2.15. To get from Figure 2.15 to Figure 2.16 simply push the $-p+1$ framed 2-handle over the right most -1 framed 2-handle. To get to Figure 2.17 push the right

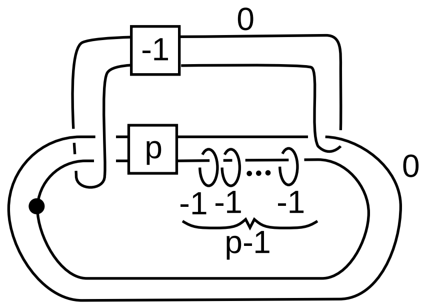


FIGURE 2.14

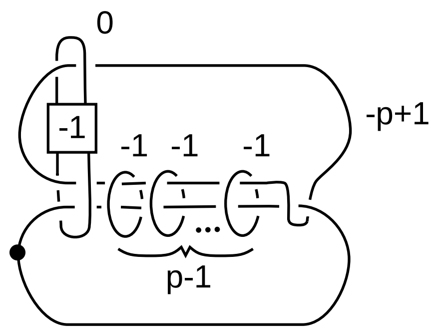


FIGURE 2.15

most -1 framed 2-handle over the -1 framed 2-handle next to it, then continue pushing the right most -1 framed 2-handle over its neighbor until there is only one -1 framed 2-handle left. Sliding the 0 framed handle over the -1 framed handle and canceling the 1-handle with the -1 framed 2-handle yields Figure 2.18. Now successively blowing down the -1 framed handle (we can do this $p-1$ times) yields the standard picture of a cusp neighborhood (Figure 2.8). $\square$

Another reason rational blowdowns are so interesting is that in [**FS2**] Fintushel and Stern used them to compute the Donaldson invariants of many manifolds and under suitable conditions related the Seiberg-Witten invariants of X and X_p.

4.3. Examples of Irreducible 4-Manifolds. In this section we will briefly discuss simply connected irreducible 4-manifolds. We begin with algebraic surfaces, which of course admit symplectic structures (since they are complex submanifolds of $\mathbb{C}P^n$, for some n); and then construct may irreducible manifolds from these by normal sums, log transforms and rational blowdowns. Some of these manifolds are

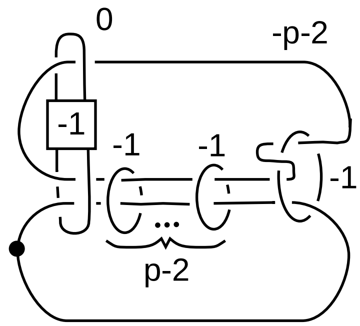


FIGURE 2.16

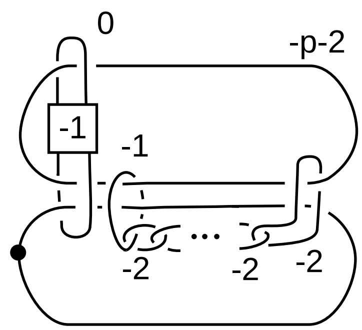


FIGURE 2.17

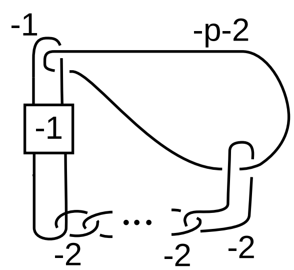


FIGURE 2.18

known to admit symplectic structures due to the work of Gompf on normal sums. Whether or not the remaining manifolds admit symplectic structures is the subject of Chapter 6.

Simply connected algebraic surfaces break into three classes: rational surfaces, elliptic surfaces and surfaces of general type. Throughout this section we will consider algebraic surfaces up to diffeomorphism, hence the subtleties involving complex structures shall be ignored. Surfaces of general type are just the algebraic surfaces that do not fall into one of the other groups. Thus we will not say anything specific about these surfaces, though we will see several examples of them below. Rational surfaces are simply those that are birationally equivalent to $\mathbb{C}P^2$. In other words, an algebraic surface X is rational if one can get from $\mathbb{C}P^2$ to X by a sequence of blowups and blowdowns. A complete list of rational surfaces is $\mathbb{C}P^2$, $S^2 \times S^2$ and blowups of these surfaces.

An Elliptic surface is a compact complex surface X and a holomorphic map $p: X \longrightarrow \Sigma$ onto a complex curve that has only finitely many critical values, and away from the critical values $p^{-1}(x)$ is an elliptic curve. Since we are interested in simply connected manifolds we must have $\Sigma = \mathbb{C}P^1$. The simplest example of an elliptic surface is $E(1) = \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P}^2$. The elliptic fibration can be seen by taking a generic pencil of cubic curves in $\mathbb{C}P^2$ and blowing up at the nine points in the base locus. Since a generic cubic curve is a torus this exhibits $E(1)$ as an elliptic surface (for more on this see [**G1**]). The facts pertinent to our discussion are that each of the exceptional spheres in $E(1)$ provide a section of the fibration with normal Euler number -1 and we can assume that $E(1)$ has exactly six singular fibers each of which is a cusp (recall that we are ignoring the complex structure on $E(1)$). We may now construct $E(2)$, also known as the K3 surface, by forming the normal connected sum of two copies of $E(1)$ along two regular fibers (the gluing map should be $(\mathrm{id}_{T^2} \times c): T^2 \times \partial D^2 \longrightarrow T^2 \times \partial D^2$ where the map $c: \partial D^2 \longrightarrow \partial D^2$ is complex conjugation). We may inductively define $E(n)$ to be the normal connected sum of $E(n-1)$ with $E(1)$. The manifold $E(n)$ contains nine sections with normal Euler number $-n$ and $6n$ cusp fibers. We may construct more elliptic surfaces $E(n; p, q)$ by performing log transforms of multiplicity p and q, where $(p, q) = 1$, on $E(n)$. A complete list of (simply connected) elliptic surfaces is given by $E(n; p, q)$, where n is a natural number and p and q are relatively prime natural numbers.

We can now construct our first examples of noncomplex irreducible 4-manifolds. These examples were first described in [**GM**]. Let $E'(2)$ be the the normal connected sum of two copies of $E(1)$ along two regular fibers, but instead of using the map described above in the construction of $E(2)$ we use the following map: let T^2 be a regular fiber in $E(1)$ near a cusp fiber, so we can fine a product structure on $T^2 = S^1 \times S^1$ so that $S^1 \times \{1\}$ and $\{1\} \times S^1$ bound embedded disks with (relative) normal Euler number -1 (i.e. vanishing cycles). Thus the boundary of a regular neighborhood of T^2 can be written as $T^3 = S^1 \times S^1 \times S^1$ where $S^1 \times \{1\} \times \{1\}$ and $\{1\} \times S^1 \times \{1\}$ are vanishing cycles and $\{1\} \times \{1\} \times S^1$ bounds a section. The map with which we form $E'(2)$ is now just a cyclic permutation of the S^1

factors. It was shown in [**GM**] that $E'(2)$ is diffeomorphic to $E(2)$. In order to get new manifolds we need to perform log transforms. Let us begin by examining $E'(2)$ more closely, see Figure 2.19. The manifold

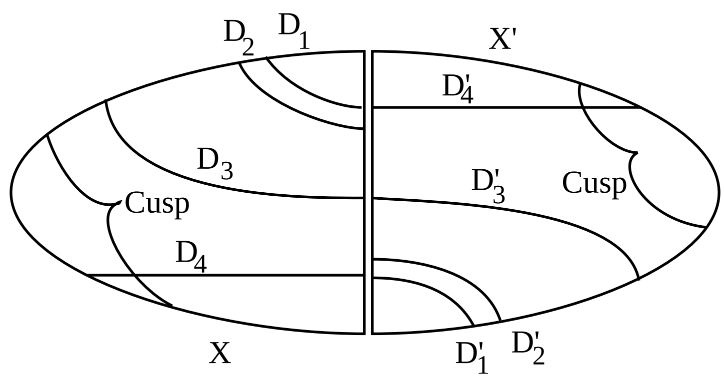


FIGURE 2.19. The manifold $E'(2)$

$E'(2) = X \cup X'$ where X and X' are two copies of $E(1)$ minus a neighborhood of a regular fiber T^2. By assuming that there are two cusp fibers near T^2 we can find two disjoint disks D_1 and D_2 in X that bound $S^1 \times \{1\} \times \{1\}$ and another disk D_3 that bounds $\{1\} \times S^1 \times \{1\}$. We also have a disk D_4 in X coming from a section of $E(1)$ that bounds $\{1\} \times \{1\} \times S^1$. Similarly X', after the cyclic permutation of the S^1 in the boundary, has disks D'_1 and D'_2 bounding $\{1\} \times \{1\} \times S^1$, a disk D'_3 bounding $\{1\} \times S^1 \times \{1\}$ and a disk D'_4 bounding $S^1 \times \{1\} \times \{1\}$. So inside $E'(2)$, we see that $D_4 \cup D'_1$ is a sphere that is dual to a cusp fiber in X. Let a neighborhood of this configuration be denoted N_1. We also have the sphere $D'_4 \cup D_1$ that is dual to a cusp fiber in X', denote a neighborhood of this N_2. Finally, the disks D_2 and D'_2 provide vanishing cycles for the torus $S^1 \times \{1\} \times S^1 \subset \partial X$ thus providing another cusp with dual sphere given by $D_3 \cup D'_3$, denote a neighborhood of this cusp and sphere by N_3. Then manifolds N_i are called nuclei and were first studied in [**G1**], where in it was show that given a simply connected manifold containing a nucleus N_i one can perform two log transforms on tori near the cusps of relatively prime multiplicities and the result will still be simply connected. Thus we can form the manifold $E'(2; p_1, q_1; p_2, q_2; p_3, q_3)$ by performing log transforms of relatively prime multiplicities p_i and q_i in N_i. Now let $K = E'(2; p_1, q_1; p_2, q_2; p_3, q_3)$, with all the pairs of p's and q's relatively prime and r is the number of even multiplicities of K, then

(1) K is a homotopy $E(2)$ if all the multiplicities are odd, otherwise it is homotopic to $3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P}^2$.

(2) the product $p_1q_1p_2q_2p_3q_3$ is a diffeomorphism invariant.
(3) if $r \leq 1$, the unordered triple (p_1q_1, p_2q_2, p_3q_3) is a diffeomorphism invariant.
(4) if $r = 2$, then the product p_iq_i which is odd is a diffeomorphism invariant.
(5) if no p_i or q_i vanishes and p_i or $q_i \neq 1$ for at least two values of i, then K is not a complex surface (or connected sum of complex surfaces).

These fact all appear in [**GM**]. Using the normal connected sum it is easy to see that K has a symplectic structure if $p_i = q_i = 1$ for some i (actually, this is easy to see if $p_3 = q_3 = 1$ but the general case follows form symmetries of the p_i's and q_i's). We will consider symplectic structures on the remaining K's in Chapter 6.

We can use elliptic surfaces to construct other noncomplex surfaces too. Most of the following constructions are due to Fintushel and Stern and first appeared in [**FS1**] or [**FS2**]. Recall the surface $E(4)$ contains nine disjoint sections of self intersection -4. Let $E_i(4)$ be the result of rationally blowing down i of these. It is clear that all the $E_i(4)$'s have symplectic structures since they can be realized by normal connected sums. Moreover, one can show that $E_i(4)$ is a complex surface for $i = 2, 3, 4$ or 9. However, $E_1(4)$ is not a complex surface and for $i = 5, 6, 7$ or 8 it is unknown if $E_i(4)$ is complex or not. We can now play the same game with $E(5)$. We start with the nine disjoint sections of self intersection -5 and would like to find dual 2-spheres with self intersection -2, thus identifying copies of $C(3)$ inside of $E(5)$. We can in fact find four such 2-spheres resulting in four disjoint $C(3)$ in $E(5)$. One may do this by realizing $E(5)$ as the 2-fold cover of $S^2 \times S^2$ branched over B which is the union of 4 copies of $\{pt\} \times S^2$ and 10 copies of $S^2 \times \{pt\}$ with double points resolved. This realized $E(5)$ as a genus 4 Lefschetz fibration with four singular fibers. A neighborhood of each of the singular fibers contains a copy of $C(3)$. Now let $E_i(5)$ be the result of blowing down i copies of $C(3)$. At this point it is unclear if all the $E_i(5)$'s have symplectic structures.

Next we claim that $E(n)$ contains two disjoint copies of $C(n-2)$. To see this notice that $E(n)$ can be constructed by forming the normal connected sum of two copies of $\mathbb{C}P^2 \# (4n+1)\overline{\mathbb{C}P}^2$ along a surface of genus $n-1$, this construction was first noticed by Stern (see [**Fu**]). Thus we realize $E(n)$ as a genus $n-1$ Lefschetz fibration with four singular fibers, two of which are shown in Figure 2.20. In these two singular fibers we clearly see copies of $C(n-2)$. Now let $G(n)$ be the result of rationally blowing down one the $C(n-2)$'s and let $H(n)$ be

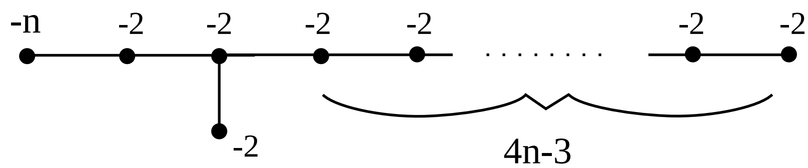


FIGURE 2.20. A singular fiber in $E(n)$.

the result of rationally blowing both $C(n-2)$'s. It is known that $G(n)$ is not homotopy equivalent to any complex surface and it is unclear if it supports a symplectic structure. We will consider this question in Chapter 6. The manifolds $H(n)$ are Horikawa surfaces, these are surfaces of general type (and thus admit symplectic structures).

CHAPTER 3

Convexity and Stein Structures in Dimension 4

Our main tool for cutting and pasting symplectic 4-manifolds will be convexity. There are several notions of convexity in symplectic geometry; an overview of some of these my be found in [**EG**]. We will only discuss a fairly strong notion of convexity here, called ω-convexity. After establishing the usefulness of ω-convexity below, the remaining sections of this chapter are devoted to showing the pieces that we want to cut-and-paste when doing a rational blowdown do in fact have ω-convex boundaries.

Let U be a domain in a symplectic manifold (X, ω) bounded by a hypersurface C. The hypersurface C is said to be **convex with respect to** ω or just ω**-convex** if there exists a vector field v defined in a neighborhood of C that is transverse to C, points out of U and is expanding (i.e. $L_v\,\omega = \omega$ where $L_v\omega$ stands for the Lie derivative of ω). We will sometimes abuse terminology and say that U has ω-convex boundary.

Given an ω-convex hypersurface C consider the 1-form $\alpha = i^*(\iota_v\omega)$, where v is the expanding vector field and $i : C \longrightarrow X$ is the inclusion map.

$$\begin{aligned} d\alpha &= di^*(\iota_v\omega) = i^*(d\iota_v\omega) = i^*(d\iota_v\omega + \iota_v d\omega) \\ &= i^*(L_v\omega) = i^*(\omega) \end{aligned}$$

So $d\alpha$ will be a symplectic form on any symplectic subbundle of $TC \cap TX$. One can easily check that $\ker\alpha$ is a such a symplectic subbundle (it is the symplectic complement of $\{v, \text{contact vector field of } \alpha\}$) and thus α is a contact 1-form on C.

Our interest in convexity is explained in the following theorem.

THEOREM 3.1. *Let U_i be a domain in the symplectic manifold (X_i, ω_i) with ω_i-convex boundary C_i, for $i = 0, 1$. If C_0 is contactomorphic to C_1, then there exists a symplectic structure on $(X_0 \setminus U_0) \cup_{C_0} U_1$.*

PROOF. Let $\alpha_i = \iota_{v_i}\omega_i$ be the contact structure induced on $C = C_i$ as the convex boundary of U_i (v_i is the expanding vector field). Form

$\mathrm{Symp}(C,\xi)$ where $\xi = \ker \alpha_0$. The form α_0 allows us to write

$$\mathrm{Symp}(C,\xi) = (0,\infty) \times C$$

where $\alpha_0(C) = \{1\} \times C$. By the proof of Proposition 2.13 we have a neighborhood N_0 of C in X_0 symplectomorphic to a neighborhood N_0' of $\alpha_0(C)$ in $\mathrm{Symp}(C,\xi)$. Let $\phi : C \longrightarrow C$ be the postulated contactomorphism between (C,α_0) and (C,α_1). By rescaling ω_1, if necessary, we have $f\alpha_0 = \phi^*\alpha_1$ where $f : C \longrightarrow \mathbb{R}$ is a positive function and $f(p) < 1$ for all $p \in C$. So we can think of $\alpha_1(C)$ in $\mathrm{Symp}(C,\xi)$ as the graph of f. Thus $\alpha_1(C)$ is disjoint from $\alpha_0(C)$ (in fact we may take $\alpha_1(C)$ to be disjoint from N_0' as well). Again the proof of Proposition 2.13 allows us to extend ϕ, thought of as a map from $\alpha_1(C) \subset \mathrm{Symp}(C,\xi)$ to $C \subset X_1$, to a symplectomorphism from a neighborhood N_1' of $\alpha_1(C)$ in $\mathrm{Symp}(C,\xi)$ to a neighborhood N_1 of C in X_1. Let $X_i^0 = X_i \setminus (U_i \setminus N_i)$ and T be the subset of $\mathrm{Symp}(C,\xi)$ bounded by the N_i', for $i = 0,1$. We may now use the symplectomorphisms constructed above to glue $N_i \subset X_i^0$ to $N_i' \subset T$, for $i = 0,1$, forming the manifold

$$Y = X_0^0 \cup_{N_0} T \cup_{N_1} (U_1 \cup N_1).$$

The manifold Y clearly has a symplectic form on it and is diffeomorphic to $(X_0 \setminus U_0) \cup_{C_0} U_1$ (since T just looks like a collar on $X_0 \setminus U_0$ and U_1 is identified to the other end of T by ψ). $\square$

This theorem clearly shows the usefulness in finding convex hypersurfaces in a manifold when trying to cut-and-paste symplectic structures. The remainder of this chapter will consist of trying to find convex hypersurfaces that will help us perform rational blowdowns symplectically.

1. Convex Structures on $C(p)$

In this section we would like to see under what conditions a set of $p-1$ spheres intersecting according to Figure 2.11 in a symplectic 4-manifold will have a neighborhood with convex boundary. The following lemma is a first step in understanding when we can find a such a neighborhood.

LEMMA 3.2. *Let Σ be a symplectic sphere in a symplectic 4-manifold (X,ω). If $\Sigma \cdot \Sigma$ is negative, then Σ has a neighborhood with ω-convex boundary.*

PROOF. Given such a Σ let $E = \nu(\Sigma)$ be its normal bundle in X. Choose a 2-form τ on the zero section $Z = \Sigma$ of E so that it gives the same orientation to Z that ω gives to Σ and represents an integral cohomology class in $H^2_{DR}(Z)$. Note that $c_1(E) = \kappa[\tau]$ for some real number $\kappa < 0$ (since $\Sigma \cdot \Sigma$ is negative).

On E we can define a function $r : E \longrightarrow \mathbb{R}$ which is the radius function on each fiber. We can also define a vector field $\frac{\partial}{\partial \theta}$ on $E \setminus Z$ that generates the standard S^1 action on $E \setminus Z$. There is a 1-form β on $E \setminus Z$ with $\beta(\frac{\partial}{\partial r}) = 0$, $\beta(\frac{\partial}{\partial \theta}) = 1$ and $d\beta = -2\pi\kappa p^*\tau$ (where $p : E \longrightarrow Z$ is projection). (We can find this 1-form by pulling back, to $E \setminus Z$, a connection 1-form on the unit circle bundle in E.) Set

$$\omega' = d(r^2 - \frac{1}{2\pi\kappa})\beta = 2rdr \wedge \beta + (r^2 - \frac{1}{2\pi\kappa})d\beta.$$

Notice that ω' is only defined on $E \setminus Z$ but we claim that ω' extends to a symplectic form on all of E. To see that ω' extends, notice that as r goes to 0, ω' goes to $p^*\tau + dx \wedge dy$ (where $dx \wedge dy$ is the standard volume form on the fiber) which is well defined on all of E. As ω' is closed, we are left to see that ω' is nondegenerate. For this we have

$$\omega' \wedge \omega' = 4r(r^2 - \frac{1}{2\pi\kappa})dr \wedge \beta \wedge d\beta$$

(note: $d\beta \wedge d\beta = \text{const.}(p^*\tau \wedge p^*\tau) = \text{const.}(p^*(\tau \wedge \tau)) = 0$ since Z is 2-dimensional). Now $\beta \wedge d\beta$ an volume form on the unit S^1 bundle P in E and $E \setminus Z$ is $(0, \infty) \times P$. So $dr \wedge \beta \wedge d\beta$ is a volume form on $E \setminus Z$ and hence is nondegenerate. On Z we saw above that $\omega' = p^*\tau + dx \wedge dy$, so $\omega' \wedge \omega' = p^*\tau \wedge dx \wedge dy$ is a volume from near Z as well.

Now let $v = \left(\frac{r^2 - \frac{1}{2\pi\kappa}}{2r}\right)\frac{\partial}{\partial r}$ and compute

$$\begin{aligned} L_v\omega' &= d\iota_v\omega' = d\iota_v[2rdr \wedge \beta + (r^2 - \frac{1}{2\pi\kappa})d\beta] \\ &= d[2r\frac{(r^2 - \frac{1}{2\pi\kappa})}{2r}\beta] = d[(r^2 - \frac{1}{2\pi\kappa})\beta] \\ &= \omega'. \end{aligned}$$

Thus v is an expanding vector field for ω' and it is transversely pointing out of any sufficiently small disk bundle in E. Thus any small disk bundle about Z in E has ω'-convex boundary. By scaling ω' if necessary (which will not affect convexity) we can arrange that ω' on Z agrees with ω on Σ. Thus by Theorem 2.5 there is an neighborhood U' of Z symplectomorphic to a neighborhood U of Σ. Inside U' there is a disk bundle with ω'-convex boundary thus there is also a neighborhood of Σ inside U with ω-convex boundary. $\square$

At this point one might hope that given two spheres, with negative self-intersection numbers, in a 4-manifold that have a single transverse intersection, we could find a neighborhood with ω-convex boundary.

This however is not the case in general, as the following example illustrates. Take two disk bundles over the sphere, each with normal Euler number -1, and plumb them together. After blowing down one of the -1 spheres we get a 4-manifold diffeomorphic to $S^2 \times D^2$. So if the original manifold had ω-convex boundary then so would $S^2 \times D^2$, but this contradicts a theorem of Eliashberg's [**E3**].

Before the next lemma we need to review Weinstein's construction of symplectic 4-manifolds with convex boundary (see [**W2**]). First we define a standard 2-handle as a subset of $\mathbb{R}^4$ with symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $f = x_1^2 + x_2^2 - \frac{1}{2}(y_1^2 + y_2^2)$ and $F = x_1^2 + x_2^2 - \frac{\epsilon}{6}(y_1^2 + y_2^2) - \frac{\epsilon}{2}$, where $\epsilon > 0$. Set $A = \{f = -1\}$ and $B = \{F = 0\}$. We define the standard 2-handle H to be the component of $\mathbb{R}^4 \setminus (A \cup B)$ that contains the origin. The attaching region is $A \cap H$ and the core of the handle is the intersection of the y_1y_2-plane with H. Now given a Legendrian knot L in the boundary of a symplectic 4-manifold with ω-convex boundary we can add H along L (the framing we add H with is determined by the canonical framing of L) to obtain a new symplectic manifold with ω-convex boundary.

LEMMA 3.3. *Let (X, ω) be a symplectic four manifold and Σ_i be embedded 2-spheres, for $i = 0, \ldots p-2$, that intersect according to Figure 2.11. Assume that Σ_0 is symplectically embedded and that the other Σ_i's are Lagrangian submanifolds. Then there exists a small regular neighborhood $C(p)$ of the Σ_i's that has ω-convex boundary.*

PROOF. Lemma 3.2 gives us a tubular neighborhood N_0 of Σ_0 with an ω-convex boundary. We would like to see how to extend N_0 to a neighborhood N_1 of $\Sigma_0 \cup \Sigma_1$ with ω-convex boundary. To do this we will try to add a 2-handle to N_0, as discussed above, inside X. The core of this 2-handle will be $\Sigma_1 \setminus (\Sigma_1 \cap N_0)$. Thus we need to see that $L_1 = \Sigma_1 \cap \partial N_0$ is a Legendrian knot inside ∂N_0 (with the contact structure induced by ω and v, the expanding vector field for N_0). By shrinking the tubular neighborhood N_0 if necessary, and recalling the symplectic structure constructed on N_0 in Lemma 3.2, we may assume that there is a Darboux coordinate chart U that contains $N_0 \cap \Sigma_1$ yielding the following local model: $\mathbb{R}^4$, $\omega' = dx \wedge dy + dz \wedge dw$, $U \cap \Sigma_0$ corresponds to the xy-plane, N_0 corresponds to a tubular neighborhood N of the xy-plane and $U \cap \Sigma_1$ corresponds to the graph of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ from the xy-plane to the zw-plane such that $f(0,0) = (0,0)$. We also claim that we can replace v by $\frac{\partial}{\partial R}$ in our local model, where $\frac{\partial}{\partial R}$ stands for the radial vector field on $\mathbb{R}^4$. To see this we will find an expanding vector field transverse to ∂N that interpolates between $\frac{\partial}{\partial R}$

and v and is equal to $\frac{\partial}{\partial R}$ near $\Sigma_1 \cap N$, and then replace v with this vector field. To this end let ρ be a the pull-back to $\mathbb{R}^4$ of a smooth function on $\mathbb{R}^2$ (the xy-plane) that is 1 in a neighborhood V of $(0,0)$ and 0 outside a larger neighborhood V', where V is chosen large enough so that $V \times \mathbb{R}^2$ contains $\Sigma_1 \cap N$. Set $\alpha = \iota_v \omega'$ and $\alpha' = \iota_{\frac{\partial}{\partial R}} \omega'$. Since $d(\alpha - \alpha') = \omega' - \omega' = 0$ and we are in $\mathbb{R}^4$ we can find a function g such that $dg = \alpha - \alpha'$. Now let $\beta = \alpha - d(\rho g)$ and notice that $d\beta = \omega'$. Thus the vector field w corresponding (under ω') to β will be expanding:

$$L_w \omega' = d\iota_w \omega' = d\beta = \omega'.$$

Also note that on $V \times \mathbb{R}^2$, $w = \frac{\partial}{\partial R}$ and outside of $V' \times \mathbb{R}^2$, $w = v$. Finally, note that w is transverse to ∂N since the vector field $v - \rho(v - \frac{\partial}{\partial R})$ clearly is, the vector field corresponding to $(d\rho)g$ is tangent to ∂N and w is the sum of these two vector fields.

Now let us examine our local model. If the function f representing Σ_1 is linear then we can represent it by a 2×2 matrix (using the basis for $\mathbb{R}^4$ above) and the graph is a Lagrangian surface if and only if the determinant is -1. One may now easily check that in this case $\Sigma_1 \cap \partial N$ is Legendrian. Now given an arbitrary function f whose graph, Γ_f, is Σ_1 we claim that we can perturb Σ_1 near the origin so that it is the graph, Γ_A, of the linear map given by $A = df_{(0,0)}$. Then by shrinking N if necessary we can conclude that $\Sigma_1 \cap \partial N$ is Legendrian. To prove the above claim we will look at Γ_f as the graph of a closed 1-form σ on $T^*\Gamma_A$ (we may do this by using Theorem 2.7 to identify a neighborhood of Γ_A in $\mathbb{R}^4$ and a neighborhood of the zero section in $T^*\Gamma_A$). Now we just need to see that we can perturb σ into a closed 1-form σ' that vanishes near the origin. To do this let ρ be a function on $\Gamma_A \cong \mathbb{R}^2$ that is 0 near $(0,0)$ and 1 further out. Since σ is a closed form on $\mathbb{R}^2$ there is a function g such that $dg = \sigma$. Now set $\sigma' = d(\rho g)$. Clearly σ' is closed, 0 near the origin and equal to σ away from the origin.

Now abstractly form the symplectic manifold Y, using Weinstein's techniques, by gluing a 2-handle to a copy of N_0 along L_1 so that Y also has convex boundary. Thus N_0 inside Y is symplectomorphic to N_0 inside X. Using a relative version of Theorem 2.7 (see [**Gr3**]) we can extend this symplectomorphism over a neighborhood of the core of the 2-handle since the core is a Lagrangian disk and so is $\Sigma_1 \setminus (\Sigma \cap N_0)$. Examining Weinstein's construction we see that we can find a new 2-handle H inside the one we added that is contained entirely in the neighborhood where the symplectomorphism is defined. Thus $N_1 = N_0 \cup H$ is a neighborhood of $\Sigma_0 \cup \Sigma_1$ with ω-convex boundary inside X.

We may continue to add handles as above to N_1 to get $C(p)$ if we can see that $\Sigma_2 \cap \partial N_1$ is a Legendrian knot in N_1 (the argument for the rest of the Σ_i's is exactly the same). To see this we once again look at a local model. A neighborhood of $\Sigma_1 \cap \Sigma_2$ is symplectomorphic to a neighborhood of the origin in $\mathbb{R}^4 = T^*\mathbb{R}^2$ where Σ_1 goes to $\mathbb{R}^2$ and Σ_2 goes to a fiber in $T^*\mathbb{R}^2$ (this is easy to do since Σ_1 and Σ_2 are Lagrangian and have transverse intersection). Arguing as above we can assume that our expanding vector field is $\frac{\partial}{\partial r}$, the radial vector field in the fiber of $T^*\mathbb{R}^2$. Now one can easily check in this model that $\Sigma_2 \cap \partial N_1$ is Legendrian. $\Box$

We would now like to see what can be said about $C(p)$ when all the spheres Σ_i are symplectically embedded. For this we will need the following lemma.

LEMMA 3.4. *If Σ is a symplectically embedded sphere in a symplectic 4-manifold X with $\Sigma \cdot \Sigma = -2$, then the symplectic structure on the ambient 4-manifold may be changed near Σ to make Σ a Lagrangian sphere. In addition, any symplectic surface in X that has a (positive) transverse intersection with Σ will remain symplectic in the new symplectic structure.*

PROOF. First consider the symplectic 4-manifold $(Y = S^2 \times S^2, \omega = \omega_S \oplus \omega_S)$. Let $S_0 \subset Y$ be the graph of the antipodal map on S^2 and $S_1 \subset Y$ be the graph of the identity map on S^2. One may readily check that $S_0 \cdot S_0 = -2$ and $S_1 \cdot S_1 = 2$. Moreover, S_0 is a Lagrangian sphere and S_1 is a symplectic sphere. Finally, we may observe that $S_0 \cap S_1 = \emptyset$ and that each of the S_i's intersects the symplectic sphere $\{x\} \times S^2$ once, for all $x \in S^2$.

Now given Σ as in the lemma we may perform a normal sum of Σ in X and S_1 in Y (see Theorem 2.6). Topologically X is unchanged (since $Y \setminus S_1$ is a disk bundle over S_0 which is isomorphic to the normal disk bundle to Σ in X) but Σ has been replaced by S_0, a Lagrangian sphere, and the symplectic structure on X has been altered near this sphere. Any symplectic surface with a positive transverse intersection with Σ may be summed with some $\{x\} \times S^2$ in Y when we perform the normal sum, thus remaining symplectic in the new symplectic structure. $\Box$

THEOREM 3.5. *Let (X, ω) be a symplectic four manifold and Σ_i be symplectically embedded 2-spheres, for $i = 0, \dots p-2$, that intersect according to Figure 2.11. Then if $p = 2$ or 3 there exists (after changing ω near the spheres) a small regular neighborhood $C(p)$ of the Σ_i's that has ω-convex boundary.*

PROOF. If $p = 2$, then we are done by Lemma 3.2. If $p = 3$, then there are just two spheres involved. We may use the previous lemma to make Σ_1 into a Lagrangian sphere while preserving the fact that Σ_0 is symplectic. Now Lemma 3.3 gives us the neighborhood $C(p)$ with ω-convex boundary. □

It is reasonable to think that $C(p)$ has an ω-convex boundary for $p > 3$. If this were true then all the theorems in Chapter 6 would have much greater applicability. We hope to return to this question in a future paper.

2. Stein Structures and the Convexity of $B(p)$

In this section we will show that the rational homology ball $B(p)$ also has an ω-convex boundary. To do this we show that $B(p)$ admits a Stein structure and that all Stein manifolds have an ω-convex "boundary." A **Stein manifold** is a proper nonsingular complex analytic subvariety of $\mathbb{C}^n$. Given a function $\psi : X \longrightarrow \mathbb{R}$ on a Stein manifold X we define the 2-form $\omega_\psi = -d(J^*(d\psi))$ where $J^* : T^*X \longrightarrow T^*X$ is the adjoint operator to the complex structure J on X. We call ψ a **plurisubharmonic function** on X if the symmetric from $g_\psi(\cdot, \cdot) = \omega_\psi(\cdot, J\cdot)$ is positive definite. Note that this implies that ω_ψ is a symplectic structure on X; and, moreover, $h_\psi = g_\psi + i\omega_\psi$ is a Hermitian metric on X. Hence we see that X is a Kähler manifold. It is easy to see that any Stein manifold admits a proper exhausting plurisubharmonic function. For example the restriction of the radial distance function on $\mathbb{C}^n$ to X will be such a function. Grauert (in [**Gra**]) proved a complex manifold X is a Stein manifold if and only if X admits an exhausting plurisubharmonic function. Thus we know that any Stein manifold admits a symplectic structure. It can in fact be shown that this symplectic structure is essentially unique. In [**EG**] it was shown that given any two plurisubharmonic functions ψ and ϕ on a Stein manifold X, (X, ω_ψ) is symplectomorphic to (X, ω_ϕ).

Our interest in Stein manifolds is indicated in the next lemma.

LEMMA 3.6. *The gradient vector field ∇_ψ of a plurisubharmonic function ψ on a Stein manifold X is an expanding vector field for ω_ψ (the gradient is taken with respect to g_ψ).*

Thus the nonsingular level sets of ψ inherit a contact structure from ω_ψ.

Proof. First by definition we have $\iota_{\nabla_\psi} g_\psi = d\psi$. So

$$\begin{aligned}\iota_{\nabla_\psi}\omega_\psi(\cdot,\cdot) &= \omega_\psi(\nabla_\psi,\cdot) = -g_\psi(\nabla_\psi, J\cdot)\\ &= -J^* g_\psi(\nabla_\psi,\cdot) = -J^* d\psi.\end{aligned}$$

Thus

$$\begin{aligned}L_{\nabla_\psi}\omega_\psi &= d\iota_{\nabla_\psi}\omega_\psi + \iota_{\nabla_\psi} d\omega_\psi\\ &= d\iota_{\nabla_\psi}\omega_\psi = -dJ^* d\psi = \omega_\psi.\end{aligned}$$

Hence ∇_ψ is an expanding vector field for ω_ψ. □

We now show that the manifold $B(p)$ has a Stein structure on it. For this we will need the following theorem.

Theorem 3.7. *An oriented 4-manifold is a Stein manifold if and only if it has a handle decomposition with all handles of index less than or equal to 2 and each 2-handle is attached to a Legendrian circle γ with the framing on γ equal to* $\mathrm{tb}(\gamma) - 1$.

This theorem is implicit in Eliashberg's paper [**E2**]. For a complete discussion of this theorem see the paper [**G3**] of Gompf. We have the following immediate corollary.

Corollary 3.8. *The (interior of the) rational homology ball $B(p)$ can be given the structure of a Stein manifold. Thus $B(p)$ is a symplectic 4-manifold with convex boundary.*

Proof. Figure 3.1 shows a handle decomposition of $B(p)$. To see

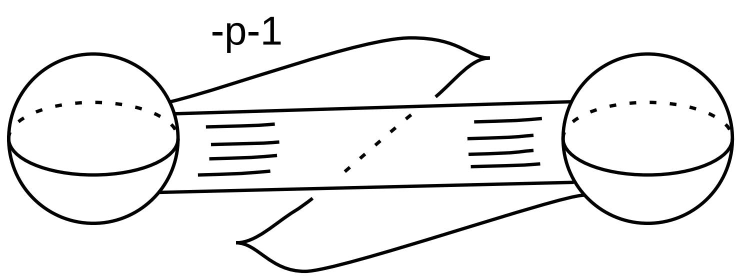


Figure 3.1. Stein Structure on $B(p)$

that this is really $B(p)$ take the standard picture for $B(p)$, Figure 2.12, and slide the 2-handle over the 1-handle. One may easily compute using Formula (2.17) (we did not discuss this formula in the presence of 1-handles but it is still valid, see [**G3**]) that the Thurston-Bennequin invariant of the attaching sphere for the 2-handle in Figure 3.1 is $-p$. Thus Theorem 3.7 tells us that $B(p)$ is indeed Stein.

□

CHAPTER 4

Homotopy Classes of 2-Plane Fields on 3-Manifolds

In Chapter 5 we will show that the two contact structures induced on the lens spaces $L(p^2, p-1)$ as the boundary of $B(p)$ and $C(p)$ are contactomorphic. With this goal in mind, in this chapter we show that they are homotopic as 2-plane fields. In the first section we discuss the general theory of 2-plane fields on 3-manifolds and define a complete set of invariants for them. These invariants were first defined in Gompf's paper [**G3**]. In the second section we will compute these invariants for for the two contact structures on $L(p^2, p-1)$ thus showing they are homotopic.

1. General Theory

Throughout this section let M be an oriented 3-manifold and let ξ be a transversely oriented 2-plane field on M. The most obvious invariant of ξ is the Euler class $e(\xi)$ of ξ thought of as an oriented two dimensional vector bundle. If M bounds an almost complex 4-manifold X such that ξ is the field of complex lines in TM then we get another interpretation of $e(\xi)$. Since TX is a complex bundle over X, $c_1(X)$ is a well defined two dimensional cohomology class. We claim that $c_1(X)$ restricted to $\partial X = M$ is $e(\xi)$. To see this first notice that $TX|_M = \xi \oplus L$ where L is the trivial complex line bundle over M. Now we have

$$e(\xi) = c_1(\xi) = c_1(\xi) + c_1(L) = c_1(\xi \oplus L) = c_1(TX|_M).$$

Now place a Riemannian metric on M. Since ξ is transversely oriented we may choose a unit vector field v on M such that v_m is perpendicular to ξ_m for all $m \in M$. Note that given v we may find ξ so knowing v is equivalent to knowing ξ. It is well known that every oriented 3-manifold has a trivial tangent bundle (see [**K1**]). By fixing a trivialization $TM \cong M \times \mathbb{R}^3$ we may regard v as a section of the unit sphere bundle $M \times S^2 \subset M \times \mathbb{R}^3$ and thus as a map $f_\xi : M \longrightarrow S^2$. If we homotope ξ then v will also be homopoted as will f. Thus a homotopy class of 2-plane fields gives us a homotopy class of maps from M to S^2

(this correspondence of course depends on the trivialization of TM). Now let $\Phi(M)$ be the set of homotopy classes of 2-plane fields on M. From the above discussion we have

$$\Phi(M) = [M, S^2]. \tag{4.1}$$

We may now use the Thom–Pontrjagin construction (see [**P**] and [**M**]) to better understand the set $\Phi(M)$, but first we will need a few definitions. A **framed submanifold** L of M is a submanifold with a fixed trivialization $\mathcal{F}$ of its normal bundle. Two framed submanifolds $(L_j, \mathcal{F}_j)$, $j = 0, 1$, are **framed cobordant** if there exists a framed submanifold $(N, \mathcal{N})$ of $M \times [0,1]$ such that $N \cap M \times \{j\}$ is L_j and $\mathcal{F}_j$ is induced from $\mathcal{N}$.

THEOREM 4.1. *For a fixed trivialization of TM we have*

$$\Phi(M) = \coprod \Phi_x(M),$$

where the disjoint union is taken over $x \in H^2(M, \mathbb{Z})$ and

$$\Phi_x(M) = \{\xi \in \Phi : PD[f_\xi^{-1}(pt.)] = x\}.$$

Moreover, $e(\xi) = 2x$ and $\Phi_x(M)$ is isomorphic to $\mathbb{Z}_{d(2x)}$ where $d(y)$ is the divisibility of y in $H^2(M, \mathbb{Z})$ modulo torsion.

PROOF. Given a map $f : M \longrightarrow S^2$ we may assume that f is smooth and thus we get a link L_f in M as the preimage of a regular value of f and a framing $\mathcal{F}_f$ on L_f pulled back from S^2. One can check that the framed cobordism class of $(L_f, \mathcal{F}_f)$, up to framed cobordism, is independent of f, up to homotopy, and the regular value of f we pick. Thus, we have set up a map from $[M, S^2]$ to framed cobordism classes of framed links in M. This is part of the Thom–Pontrjagin construction. The other part gives an inverse to the this map (see [**M**]). Thus we have a one-to-one correspondence. So using Equation (4.1) we have a one-to-one correspondence between 2-plane fields and framed cobordism classes of framed links given by sending ξ to $(L_{f_\xi}, \mathcal{F}_{f_\xi})$. Note $e(\xi) = c_1(\xi) = f_\xi^* c_1(TS^2)$ since $\xi = f_\xi^* TS^2$. Thus $2L_{f_\xi} = PD(e(\xi))$ because $PD(c_1(TS^2)) = 2[p]$ where $[p]$ is the 0-homology class of a the regular value p and $L_{f_\xi} = f_\xi^{-1}(p)$. Now given $x \in H_1(M; \mathbb{Z})$ set

$$\Phi_x(M) = \{\xi \in \Phi(M) : [L_{f_\xi}] = x\}.$$

Clearly this is the same as the $\Phi_x(M)$ defined in the statement of the theorem.

We now want to show that $\Phi_x(M)$ is isomorphic to $\mathbb{Z}_d$ where d is the divisibility of $2x$ in $H_1(M; \mathbb{Z})$ modulo torsion. To this end choose a knot K representing x and choose a framing $\mathcal{F}$ on K. Now we have a map $h : \mathbb{Z} \longrightarrow \Phi_x(M)$ given by sending n to K with framing $\mathcal{F}_n$, where $\mathcal{F}_n$ is the framing on K given by adding n right handed twists to $\mathcal{F}$. The map h is clearly a surjection. Now suppose that $h(n)$ is framed cobordant to $h(m)$. Thus we have a framed surface in $M \times [0, 1]$ or by gluing $M \times \{0\}$ to $M \times \{1\}$ we get a closed surface T in $M \times S^1$ with self-intersection $n - m$. Let $C = x \times [S^1] \in H_2(M \times S^1; \mathbb{Z})$ and note

$$n - m = [T] \cdot [T] = (([T] - C) + C) \cdot (([T] - C) + C) = 2([T] - C) \cdot C$$

where the last equality follows from $([T]-C) \in H_2(M; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \otimes H_0(S^1; \mathbb{Z})$ which is true since $([T] - C) \cap [M \times \{0\}] = 0$ (if part of $([T] - C)$ lay in $H_1(M; \mathbb{Z}) \otimes H_1(S^1; \mathbb{Z})$ then it would have nontrivial intersection with $M \times \{0\}$). Now $n - m = ([T] - C) \cdot (2x)$ in $H_*(M; \mathbb{Z})$ again since $([T] - C)$ lives in $H_2(M; \mathbb{Z})$. Thus $n - m$ is divisible by d. Conversely, let $\alpha \in H_2(M; \mathbb{Z})$ be a homology class with the property that $c_1(\xi)(\alpha) = d$ (one can easily find such a class by Poincaré duality since we are working modulo torsion). We can now find a closed surface T representing $\alpha + C$ in $H_2(M \times S^1; \mathbb{Z})$ so that when we cut $M \times S^1$ open to get $M \times [0, 1]$, T will become a surface Σ that cobounds a copy of K in $M \times \{0\}$ and a copy in $M \times \{1\}$. Notice

$$(\alpha + C) \cdot (\alpha + C) = 2\alpha \cdot C = \alpha \cdot (2x) = c_1(\xi)(\alpha),$$

where $\alpha \cdot (2x)$ is computed in M. Thus we have constructed a framed cobordism $(\Sigma, \mathcal{N})$ (the framing on Σ comes from T) from $(K, \mathcal{F}_n)$ to $(K, \mathcal{F}_{n+d})$. So the map $h : \mathbb{Z} \longrightarrow \Phi_x(M)$ is onto and has kernel $d\mathbb{Z}$. $\square$

So far our discussion of $\Phi(M)$ has relied on fixing a trivialization of TM. In practice it is quite hard to manipulate trivializations. For example, given a 3-manifold described in two different ways it will be quite difficult, in general, to compare trivializations in the different presentations of the manifold. This problem led Gompf in [**G3**] to define two invariants of 2-plane fields that can be computed and compared without keeping track of the trivializations and completely determine in which homotopy class a plane field lies. As Theorem 4.1 indicates there is a 2-dimensional invariant that refines the Euler class and a 3-dimensional invariant (an integer mod d). We will only discuss these invariants when $e(\xi)$ is a torsion element since this is all we will need in our applications and is easier than the general case. For the general definitions see [**G3**].

We begin with the 3-dimensional invariant. Let X we an almost complex 4-manifold that M bounds so that ξ is the field of complex lines in TM (we can always find such a manifold, see for example [**G3**]).

We now define our 3-dimensional invariant to be the rational number

$$\theta(\xi) = c_1^2(X) - 2\chi(X) - 3\sigma(X)$$

where $c_1^2(X)$ is defined as follows: we would like to say that $c_1^2(X)$ is just the 4-dimensional cohomology class $c_1^2(X)$ evaluated on the fundamental homology class of X but since X is not a closed manifold it does not have an absolute 4-dimensional fundamental homology class. By Poincaré duality we can represent $c_1(X)$ by an element $c' \in H_2(X, \partial X; \mathbb{Z})$. Consider the following piece of the long exact sequence for $(X, \partial X)$

$$\ldots \longrightarrow H_2(X) \xrightarrow{j} H_2(X, \partial X) \xrightarrow{b} H_1(\partial X) \longrightarrow \ldots$$

Clearly $b(c') \in H_1(\partial X)$ is the Poincaré dual to the Euler class $e(\xi)$ (which is the restriction to $\partial X = M$ of $c_1(X)$). If in the above sequence we use rational coefficients then $b(c') = 0$ since $e(\xi)$ is a torsion class over $\mathbb{Z}$ and therefore zero over $\mathbb{Q}$. Thus we may find a class $c \in H_2(X; \mathbb{Q})$ such that $j(c) = c'$. Absolute homology classes have a well defined intersection pairing so we define $c_1^2(X)$ to be $c \cdot c$. Note that c is not necessarily unique but any other homology class that maps to c' will have the same self intersection number. We claim that $\theta(\xi)$ only depends on M and the homotopy class of ξ. To see this let ξ and ξ' be two 2-plane fields on M in the same homotopy class. Now choose almost complex 4-manifolds X and X' that bound (M, ξ) and $(\overline{M}, \xi')$ respectively, where $\overline{M}$ is M with the opposite orientation. One may easily check that the manifold Y formed by gluing X and X' together along their boundary will have an almost complex structure. Since Y is a closed almost complex manifold equation (2.8) tells us that

$$c_1^2(Y) - 2\chi(Y) - 3\sigma(Y) = 0. \tag{4.2}$$

We also clearly have

$$\chi(Y) = \chi(X) + \chi(X') \quad \text{and}$$
$$\sigma(Y) = \sigma(X) + \sigma(X').$$

Now consider $c_1(Y)$. We claim that $PD(c_1(Y)) = c + c'$ where c (respectively c') is an absolute 2-homology class (over $\mathbb{Q}$) in X (respectively X') that is the pull back of the relative 2-homology class $PD(c_1(X))$ (respectively $PD(c_1(X'))$). This is quite easy to see by considering the restrictions of $c_1(Y)$ to M, X and X'. So finally we have

$$c_1^2(Y) = c_1^2(X) + c_1^2(X'),$$

and thus equation (4.2) says

$$\theta(\xi) = -\theta'$$

where $\theta' = \theta(\xi')$, computed as a 2-plane field on $\overline{M}$. Now fixing our choice of X' we see that if we compute $\theta(\xi)$ using any almost complex 4-manifold X that bounds (M, ξ) we will get $-\theta'$. Thus $\theta(\xi)$ is independent of the choice of X. Moreover, note the above argument shows that $\theta(\xi)$ flips sign if we reverse the orientation on M.

We now turn our attention to the 2-dimensional invariant of a 2-plane field ξ. Let $\mathrm{Spin}(M)$ be the group of spin structures on M and $G = \{x \in H_1(M; \mathbb{Z}) : 2x = PD(c_1(\xi))\}$. For the reader unfamiliar with spin structures see [**GS**] for a discussion of all the facts needed below. The invariant we will define is a map

$$\Gamma(\xi, \cdot) : \mathrm{Spin}(M) \longrightarrow G.$$

This invariant will clearly refine the Euler class since $2\Gamma(\xi, s) = e(\xi)$ for any $s \in \mathrm{Spin}(M)$. We will give two definitions of this invariant. From the first it is fairly easy to see that it is well–defined. Given a spin structure s on M we can find a trivialization τ on TM that induces s. By Theorem 4.1, τ assigns to ξ a map $f_\xi : M \longrightarrow S^2$. We now choose a regular value, p, of f_ξ and set

$$\Gamma(\xi, s) = [f^{-1}(p)].$$

Another "more intrinsic" (and more amiable to computation) way to define $\Gamma(\xi, \cdot)$ is as follows: let v be a vector field in ξ with zero locus (counted with multiplicity) 2γ where γ is a smooth curve in M. The vector field v gives a trivialization of ξ on $M \setminus \gamma$ and hence a trivialization of TM on $M \setminus \gamma$. This trivialization induces a spin structure on $M \setminus \gamma$, and finally, since v vanishes to order 2 on γ we can extend this spin structure over γ to obtain a spin structure s' on all of M. We can now define

$$\Gamma(\xi, s') = [\gamma].$$

To see that this is the same as the first definition choose a vector field w on S^2 with on zero of order 2 at p. Then $v = f_\xi^*(w)$ (notice that f_ξ gives an isomorphism from ξ_m to $T_{f_\xi(m)}S^2$ and thus $f_\xi^*(w)$ is well–defined) is a vector field in ξ that vanishes to order 2 along the curve $f^{-1}(p)$. Moreover, it is not hard to see that the spin structure that v induces on M is in fact s. Hence, our two definitions are the same if the second one is well–defined.

To see that the second definition is well–defined let v_0 and v_1 be two vector fields as described in the definition. We can now form the difference class $\Delta(v_0, v_1)$ of the nonzero vector fields v_0 and v_1 in $\xi|_{M \setminus (\gamma_0 \cup \gamma_1)}$ where γ_i is the zero set of v_i. Note this is a cohomology class in $H^1(M \setminus (\gamma_0 \cup \gamma_1); \mathbb{Z})$. (Difference classes of vector fields in ξ are defined analogously to difference classes of spin structures, see [**GS**].) Moreover, we have

$$\partial PD(\Delta(v_0, v_1)) = 2([\gamma_0] - [\gamma_1]). \tag{4.3}$$

Since both v_0 and v_1 vanish to order two we can uniquely extend the mod 2 reduction of $\Delta(v_0, v_1)$ to a cohomology class Δ in $H^1(M; \mathbb{Z}_2)$. It is not hard to see that Δ is the difference class $\Delta(s_0, s_1)$ for the spin structures s_i associated to v_i. Now let β be the Bockstein homomorphism induced by the coefficient sequence $\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2$. Then we clearly have (using Equation (4.3))

$$\beta(PD(\Delta(s_0, s_1))) = [\gamma_0] - [\gamma_1]. \tag{4.4}$$

Thus if v_0 and v_1 define the same spin structures then $[\gamma_0] = [\gamma_1]$. Hence $\Gamma(\xi, \cdot)$ is well–defined. In addition, we have seen that $\Gamma(\xi, \cdot)$ is $H^1(M; \mathbb{Z}_2)$-equivariant where $H^1(M; \mathbb{Z}_2)$ acts on the set of spin structures as usual and on G by the Bockstein homomorphism. Note that this implies that $\Gamma(\xi, \cdot)$ is determined by its value on one spin structure.

We are now ready to state the following:

THEOREM 4.2. *Let ξ_0 and ξ_1 be two 2-plane fields on a closed oriented 3-manifold. If $e(\xi_0)$ is a torsion class, then ξ_0 and ξ_1 are homotopic if and only if, for some choice of spin structure s, $\Gamma(\xi_0, s) = \Gamma(\xi_1, s)$ and $\theta(\xi_0) = \theta(\xi_1)$.*

We refer the reader to [**G3**] for a proof of (a stronger version of) this result.

2. Our Examples

We shall now consider the contact structures induced on $L(p^2, p-1)$ as the convex boundary of $B(p)$ and $C(p)$. Let ξ_0 denote the contact structure induced from $B(p)$ and ξ_1 denote the one induced from $C(p)$. We begin by computing the Euler class for these two examples. To do this we compute the first Chern class of the almost complex 4-manifolds they bound. To compute $c_1(B(p))$ we use the following lemma, whose proof follows easily from the proof of Theorem 3.7 (see [**G3**]).

LEMMA 4.3. *If X is a Stein 4-manifold obtained by adding handles to D^4 as in Theorem 3.7, then $c_1(X)$ is represented by the cocycle*

$$c = \sum r(\gamma_i) f_{h_i},$$

where the sum is over the knots γ_i to which the 2-handles h_i are attached and f_{h_i} is the cochain that is 1 on core of h_i and 0 elsewhere.

The handle decomposition in Figure 3.1 of $B(p)$ provides a convenient basis for the cellular chain groups of $B(p)$. Namely,

$$\begin{aligned} C_1(B(p);\mathbb{Z}) &= \langle \alpha \rangle & C_2(B(p);\mathbb{Z}) &= \langle h \rangle \\ C_1(\partial B(p);\mathbb{Z}) &= \langle a \rangle & C_2(B(p),\partial B(p);\mathbb{Z}) &= \langle A \rangle \end{aligned}$$

where α, h and A correspond to the core of the 1-handle, core of the 2-handle and co-core of the 2-handle, respectively, and a corresponds to the meridian of the surgery unlink in the standard picture of $L(p^2, p-1) = \partial B(p)$. Now let γ be the curve to which the 2-handle was added. Using formula (2.18) we can easily compute $r(\gamma) = -1$ (when γ is given the appropriate orientation). So $c_1(B(p))$ is represented by

$$c(B(p)) = -f_h. \tag{4.5}$$

The Poincaré dual to $c(B(p))$ is represented by the relative cycle $-A \in C_2(B(p), \partial B(p);\mathbb{Z})$. So the Poincaré dual of the restriction of $c(B(p))$ to $\partial B(p)$ is $-pa$, since $\partial A = pa$ as can easily be seen by Kirby calculus or from the long exact homology sequence of the pair $(B(p), \partial B(p))$. Thus we have

$$e(\xi_0) = PD(-pa). \tag{4.6}$$

The handle body decomposition of $C(p)$ indicated in Figure 2.11 gives the following cellular chain groups

$$\begin{aligned} &C_2(C(p);\mathbb{Z}) = \langle h_0, \dots, h_{p-2} \rangle \quad C_2(C(p), \partial C(p);\mathbb{Z}) = \langle B_0, \dots B_{p-2} \rangle \\ &C_1(\partial C(p);\mathbb{Z}) = \langle a \rangle \end{aligned}$$

where h_i, b_i correspond to the core, respectively co-core, of the 2-handles and a is as above. For our applications we can always assume that the homology classes corresponding to the h_i's are represented by symplectic 2-spheres Σ_i in $C(p)$. Choosing an almost complex structure compatible with the symplectic structure on $C(p)$ we have the adjunction formula

$$c_1(C(p))(\Sigma_i) = \chi(\Sigma_i) - \Sigma_i \cdot \Sigma_i. \tag{4.7}$$

Thus

$$c_1(C(p))(\Sigma_i) = \begin{cases} -p & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \tag{4.8}$$

So $c_1(C(p))$ is represented by the cochain

$$c = -pf_{h_0}. \tag{4.9}$$

The Poincaré dual to $c_1(C(p))$ is represented by $-pB_0$. Hence the Poincaré dual to the restriction of $c_1(C(p))$ to $\partial C(p)$ is given by $-pa$ since $\partial B_0 = a$ as can easily be seen from Kirby calculus. Thus

$$e(\xi_1) = PD(-pa). \tag{4.10}$$

We now refine our above computations by computing $\Gamma(\xi_i, \cdot)$. If p is odd then $\mathrm{Spin}(L(p^2, p-1))$ and $G = \{x \in H_1(M; \mathbb{Z}) : 2x = PD(c_1(\xi))\}$ contain only one element making it easy to compute Γ. We have

$$\Gamma(\xi_i, s) = -\frac{1}{2}(pa), \tag{4.11}$$

where s is the unique element in $\mathrm{Spin}(L(p^2, p-1))$. If p is even then the situation is a little more complicated and we will proceed as follows: Let s denote the spin structure induced on $L(p^2, p-1)$ as the boundary of $C(p)$ (notice that $C(p)$ has a unique spin structure on it). The spin structure s is characterized by the fact that it will extend across a 2-handle added to a with framing 0. To compute $\Gamma(\xi_1, s)$ we must find a vector field in ξ_1 that will induce s. We saw above that $PD(c_1(C(p))|_{\partial C(p)}) = PDe(\xi_1) = -pa$. Thus we may find a vector field in ξ_1 that vanishes along $-a$ with multiplicity p, or more to the point a vector field v in ξ_1 that vanishes along $-\frac{p}{2}a$ with multiplicity 2. This vector field clearly induces the spin structure s. Thus

$$\Gamma(\xi_1, s) = -\frac{p}{2}a.$$

Now to compute $\Gamma(\xi_0, s)$ we need to know that s is the spin structure on $\partial B(p)$ that will extend across a 2-handle attached to a, in Figure 4.1, with framing 0 (it is a simple exercise in Kirby calculus to see that a in Figure 4.1 is the same as the a above; it is important to note, however, that the 0-framings also correspond). Our next task is to find a vector

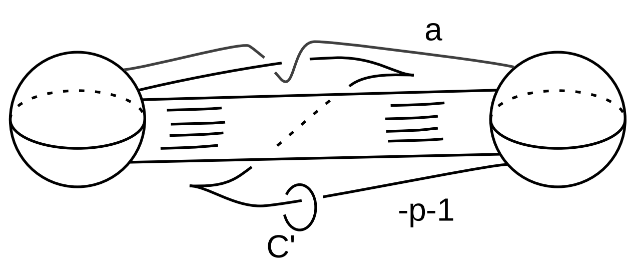


FIGURE 4.1. a and C' in $\partial B(p)$

field v that lies in ξ_0, induces s and has a zero locus of multiplicity 2. We begin with the vector field v' defined as follows: let $v' = \frac{\partial}{\partial x}$ the region of $\mathbb{R}^3 \subset S^3$ to which we are adding handles. As shown in [**G3**], it makes sense to say $v' = \frac{\partial}{\partial x}$ on all of $S^1 \times S^2 = \partial(\text{0-handle} \cup \text{1-handle})$. Finally we can extend v' over $\partial B(p)$ to a vector field in ξ_0 with zero set equal to C',in Figure 4.1 (since C' is the boundary of the co-core of the 2-handle). It is clear that v' is not the vector field we want since its zero set has multiplicity 1. Next we will construct a difference class between v' and the vector field we can use in our computation. To this end consider the surface F_0 indicated in Figure 4.2. Notice that the framing F_0 induces on k is $\text{tb}(k) = \text{blackboard} - 1$. So if we set $F = F_0 \cup$core of 2-handle the $\partial F = -C'$, since the 2-handle is attached with framing $\text{tb}(k) - 1$. Furthermore, the curve C, with $[C] = -\frac{p}{2}a$, lies on $F_0 \subset F$ (see Figure 4.2). It is now easy to see that $F' = F \setminus C$

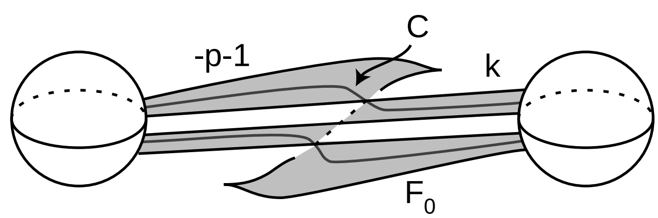


FIGURE 4.2. The surface F_0 in $\partial B(p)$

is an integral homology class in $\partial B(p) \setminus \{C \cup C'\}$ and $\partial F' = 2C - C'$. The class $PD[F']$ is the difference class for v' and some vector field v on $\partial B(p)$ with zero locus $2C$. Now v will induce a spin structure, s_v, on $\partial B(p)$ and

$$\Gamma(\xi_0, s_v) = [C] = -\frac{p}{2}a.$$

Finally we need to see that s_v is the same as s. This will be true if s_v extends over a 2-handle added to a with framing 0. In order to verify this, note that $a \cap F' = 1$ and thus the spin structure on a neighborhood of a induced by v is not the one that is induced from v'. Moreover, the spin structure induced from v' does not extend over a 2-handle added to a with framing 0 since on the core of the 2-handle the the spin structure is the one induced by the tangent vector field to its boundary circle which of course does not extend (its degree is odd). Thus we finally have

$$\Gamma(\xi_0, s) = [C] = -\frac{p}{2}a.$$

We would now like to compute the three dimensional invariants of ξ_0 and ξ_1. To do this we will need to calculate $c_1^2(B(p))$ and $c_1^2(C(p))$. We begin with $c_1^2(B(p))$. Remember, to do this calculation we must pull $PD(c_1(B(p))) \in H_2(B(p), \partial B(p); \mathbb{Q})$ back to an absolute homology class and then calculate the self-intersection number of this class. For $B(p)$ this is quite simple since $H_2(B(p);\mathbb{Q}) = 0$. Thus $c_1^2(B(p)) = 0$ and we get

$$\theta(\xi_1) = c_1^2(B(p)) - 2\chi(B(p)) - 3\sigma(B(p)) = 0 - 2(1) - 3(0) = -2. \tag{4.12}$$

Now to compute $c_1^2(C(p))$ we will need to consider the following part of the long exact sequence for $(C(p), \partial C(p))$

$$\begin{array}{ccc} H_2(C(p);\mathbb{Q}) & \xrightarrow{j} & H_2(C(p), \partial C(p);\mathbb{Q}) \\ \| & & \| \\ \langle [h_0] \ldots, [h_{p-2}] \rangle & \xrightarrow{M} & \langle [B_0], \ldots, [B_{p-2}] \rangle \end{array}$$

where M is the matrix

$$\begin{pmatrix} -(p+2) & 1 & 0 & 0 & & & \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & & & \\ & & & & \ddots & & \\ & & & & 1 & -2 & 1 \\ & & & & 0 & 1 & -2 \end{pmatrix}$$

representing j in the basis given on the second row. One may easily check that

$$c = \frac{1}{p} \sum_{i=1}^{p-1} (p-i)[h_{i-1}] \tag{4.13}$$

is the unique element of $H_2(C(p);\mathbb{Q})$ that M maps to $PD(c_1(C(p))) = -p[B_0]$. So we get

$$\begin{aligned} c_1^2(C(p)) = c \cdot c &= \frac{1}{p^2} (\sum_{i=1}^{p-1} (p-i)[h_{i-1}])^2 \\ &= \frac{1}{p^2} \left[-(p-1)^2(p+2) - 2\sum_{i=2}^{p-1}(p-i)^2 + 2\sum_{i=1}^{p-1}(p-i)(p-i-1) \right] \\ &= 1 - p. \end{aligned}$$

Thus

$$\begin{aligned} \theta(\xi_1) &= c_1^2(C(p)) - 2\chi(C(p)) - 3\sigma(C(p)) \\ &= (1-p) - 2(p) - 3(-(p-1)) = -2. \end{aligned} \tag{4.14}$$

The above calculations and Theorem 4.2 give us the main result of this chapter.

THEOREM 4.4. *The two contact structures* ξ_0 *and* ξ_1 *on* $L(p^2, p-1)$ *are homotopic as 2-plane fields.*

CHAPTER 5

Tight Contact Structures on Lens Spaces

It is still not clear what one expects to find when studying tight contact structures on lens spaces. We begin by considering the existence question. Given a lens space $L(p,q)$, with $p > 0$ and $q > 0$ relatively prime and $q < p$, one may easily construct a tight contact structure on it using Gompf's Stein calculus [**G3**]. To do this let $r_0, r_2, \ldots, r_n$ be a continued fractions expansion of $\frac{-p}{q}$ and notice that both Kirby diagrams in Figure 5.1 represent the lens space $L(p,q)$. Now it is clear,

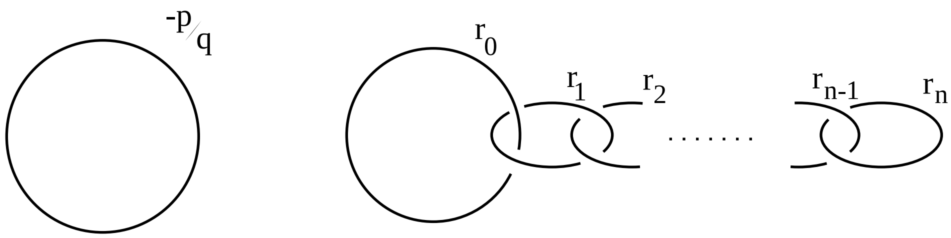


FIGURE 5.1. Two Kirby diagrams of $L(p,q)$

since $r_i < -1$ for all i, that the Kirby diagram on the right may be made into a Stein diagram thus realizing $L(p,q)$ as the boundary of a Stein manifold. In general this is all that we can say about tight contact structures on $L(p,q)$. We can say considerably more about $L(p,1)$ (see [**E5**]). If p is odd, then we can realize all elements of $H^2(L(p,1);\mathbb{Z})$ as the Euler class of a tight contact structure except possibly the zero class. If p is even, then we can realize every element of $H^2(L(p,1);\mathbb{Z})$ as the half Euler class (i.e. Γ-invariant) of a tight contact structure except possibly the class $\frac{p}{2}$, see Figure 5.2. This leads to the following conjecture:

CONJECTURE. *Each element in $H^2(L(p,1);\mathbb{Z})$ is realized by a unique tight contact structure except the zero class if p is odd or the class $\frac{p}{2}$ if p is even.*

As mentioned in Section 2.3 there is not a lot known about the uniqueness of tight contact structures on 3-manifolds. In particular, Eliashberg has shown (see [**E4**]) that $S^1 \times S^2$, S^3 and $\mathbb{R}P^3$ have unique

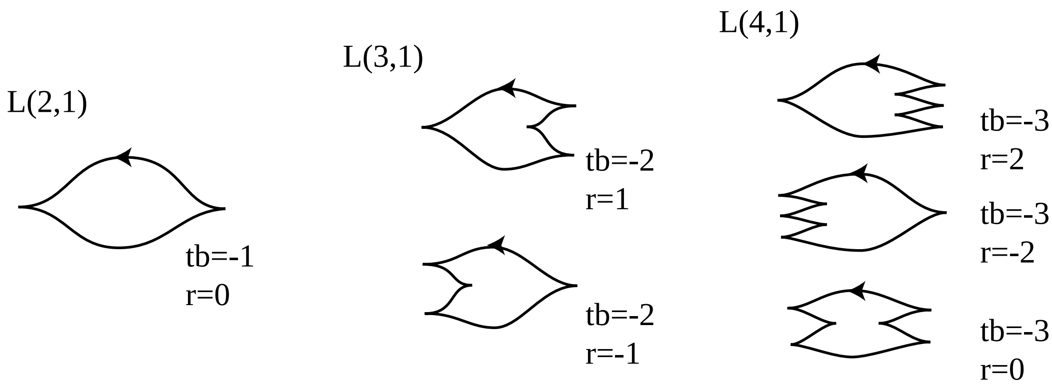


FIGURE 5.2. Tight contact structures on $L(p, 1)$

tight structures. Thus the uniqueness question has been answered for $L(p, q)$ only when $p = 0, 1$ or 2. In this chapter we will reprove Eliashberg's $p = 2$ result and then extend it to show that there is only one tight contact structure realizing the Euler class "$e = q + 1$" when p is odd and a similar but more technical result when p is even (see Theorem 5.2). A corollary of this result is the following theorem that will be needed in Chapter 6.

THEOREM 5.1. *If the spheres used to from $C(p)$ are all symplectic or Lagrangian, then the two contact structures ξ_0 and ξ_1 induced on $L(p^2, p-1)$ as the boundary of $B(p)$ and $C(p)$, respectively, are contactomorphic.*

1. Characteristic Foliations on Generalized Projective Planes

In this section we discuss a procedure for showing two tight contact structures on a given lens space are contactomorphic. For this we use two decompositions of a lens space. First, every a lens space $L = L(p, q)$ (we will normalize p and q so that $p \geq 0$ and $q < |p|$) has a genus one Heegaard splitting. More specifically,

$$L = V_0 \cup_M V_1,$$

where each V_i is a solid torus and $M : \partial V_0 \longrightarrow \partial V_1$ is a diffeomorphism represented in a standard basis for $T^2 = \partial V_i$ by

$$M = \begin{pmatrix} q & p' \\ p & r \end{pmatrix},$$

where $rq - pp' = -1$. (A standard basis is given by μ, the boundary of a meridinal disk, and λ, a longitude for T^2 given by the product structure on V_i and oriented so that $\mu \cap \lambda = 1$.) Secondly, L has a simple CW decomposition. To see this let Γ be a core curve in V_0 and Δ a meridional disk in V_1. Now $\gamma = \partial\Delta$ is a meridional curve on ∂V_1 and is mapped by M to a (p,q)-curve on ∂V_0. We can find an annulus A in V_0 with its interior embedded in the interior of $V_0 \setminus \Gamma$, one boundary component on $\gamma \subset \partial V_0$ and the other boundary component wrapping p times around Γ. We will call $D = \Delta \cup A$ a **generalized projective plane** in L. Given a point x on Γ, a CW decomposition of L is given by

$$\{x\} \cup \Gamma \cup D \cup B,$$

where B is a 3-ball. We can now state the main result of this chapter.

THEOREM 5.2. *Let $L = L(p,q)$ be a lens space and ξ_i, $i = 0,1$, be two tight contact structures on L. If*

(1) *p is odd and $e(\xi_i)(D) = (q+1) \mod p$, for $i = 0,1$, or*
(2) *p is even and $\Gamma(\xi_i, s) \cdot D = \frac{1}{2}(q+1) \mod p$, for $i = 0,1$, where s is the spin structure on L that does not extends over a 2-handle attached to Γ with framing 0,*

then ξ_0 and ξ_1 are contactomorphic.

The proof of this theorem involves most of the rest of this chapter. We begin by showing that the "characteristic foliation" of a tight contact structure on D determines that structure on L. We then turn our attention to simplifying this foliation. Before we embark on this journey let us first pause to show that Theorem 5.1 follows from this theorem.

PROOF OF THEOREM 5.1. Theorem 5.2 tells us that we need to compute $e(\xi_i)(D)$ (when p is odd). In the standard picture of $L(p^2, p-1)$ as surgery on an unknot we will take our disk Δ to be the meridional disk in the surgery torus. Considering $C(p)$ (the case of $B(p)$ is just the same), the Poincaré dual to $e(\xi)$ will be represented by $-p$ times a meridional curve a (see Figure 5.3). To see how this curve intersects D we will push it into the surgery torus. One may readily compute that a goes to $p-1$ times a longitude and hence (being careful about orientations) we have

$$D \cap (\text{a push off of } a) = p - 1.$$

Thus we have

$$e(\xi_i)(D) = -p(p-1) + mp^2 = -p^2 + p + mp^2 = p + n'p^2 = q + 1 + n'p^2,$$

where $n' = m - 1$.

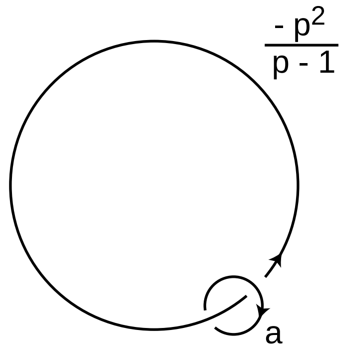


FIGURE 5.3. The lens space $L(p^2, p-1)$

Now if p is even we need to compute $\Gamma(\xi_i, s) \cdot D$. In Section 4.2 we computed

$$\Gamma(\xi_i, s') = -\frac{1}{2}(pa),$$

where s' is the spin structure on L that extends across a 2-handle attached to Γ with framing 0. So to compute $\Gamma(\xi_i, s)$ we first compute

$$\beta(PD(\Delta(s, s'))) = \frac{p^2}{2}a,$$

where β is the Bockstein homomorphism discussed in Chapter 4. Now we can compute

$$\begin{aligned}\Gamma(\xi_i, s) \cdot D &= \Gamma(\xi_i, s') \cdot D + \beta(PD(\Delta(s, s'))) \cdot D \\ &= -\frac{1}{2}p(p-1) + \frac{p^2}{2}(p-1) + mp^2 = \frac{1}{2}p + m'p^2 \\ &= \frac{1}{2}(q+1) + m'p^2.\end{aligned}$$

□

We now begin to standardizing our contact structure on L. Given an oriented contact structure ξ on L we may ambiently isotope L so that Γ is transverse to ξ. We may now write down a standard model for V_0 that will be used many times in this chapter. Let U be a tubular neighborhood of the z-axis in $\mathbb{R}^3$ modulo the action $(r, \theta, z) \mapsto (r, \theta + \frac{2\pi q}{p}, z+1)$. Now by shrinking V_0 if necessary we can find a diffeomorphism from U to V_0 taking the z-axis to Γ and $S = \{(r, \theta, z) : \theta = \frac{2k\pi}{p} \text{ for } k = 0, \ldots, p-1\}$ to A. We will see that this diffeomorphism may be isotoped into a contactomorphism without changing the properties described above. Even though D is not an embedded surface, ξ will clearly induce a singular foliation, D_ξ, on it and we can choose the orientation on D so that the flow D_ξ is directed toward Γ. If ξ is tight D_ξ essentially determines ξ.

THEOREM 5.3. *Let L_0 and L_1 be two copies of $L(p,q)$. Let ξ_i be a tight oriented contact structure on L_i and D_i the generalized projective plane in L_i, $i = 0, 1$. Assume that the 1-skeleton Γ_i of D_i is transverse to ξ_i. If a diffeomorphism $f : L_0 \longrightarrow L_1$ may be isotoped so that it takes $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$ then it may be isotoped into a contactomorphism.*

PROOF. We would like to begin by isotoping f to a contactomorphism near Γ. To this end we note that f takes $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$ and thus Γ_0 to Γ_1; moreover, if $p > 2$, the fact that f takes $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$ tells us that f takes $(\xi_0)_m$ to $(\xi_1)_{f(m)}$ for $m \in \Gamma_0$. Thus using Theorem 2.12 we may isotope f into a contactomorphism near Γ. Unfortunately, without exercising care, this isotopy might change the fact that f takes D_0 to D_1, much less $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$. To see that this does not happen consider the model U for V_0 described above and let ψ be the diffeomorphism from U to V_0. Now set $\alpha = \psi^*\alpha_0$ and $\alpha' = \psi^* f^*\alpha_1$. We have

$$\alpha = a\,dr + b\,d\theta + c\,dz$$

and

$$\alpha' = a'\,dr + b'\,d\theta + c'\,dz,$$

where a, b, c, a', b' and c' are functions on U. We may also take c and c' to be nonzero since Γ is transverse to ξ_i. If we restrict both these forms to S we get

$$\alpha|_S = a\,dr + c\,dz$$

and

$$\alpha'|_S = a'\,dr + c'\,dz.$$

Since α and α' have the same kernel on S it is easy to see that $a = hc$ and $a' = hc'$ for some function h (note for this we need to know that the kernel always has a radial component, but by taking V_0 small enough we can assume this). One may now check that on S, $\alpha' = \frac{c'}{c}\alpha$. So back in L_0 we have that $\alpha_0|_A$ is some nonzero multiple of $f^*\alpha_1|_A$. Moreover, since $\frac{c'}{c}$ is a well-defined function near Γ we may clearly extend it over L_0 and thus by rescaling α_1 we may assume that $\alpha_0|_A = \alpha_1|_A$. Thus referring back to the proof of Lemma 2.20 we see that the vector field used to define the above isotopy points along the characteristic foliation on A. Hence after the above isotopy, f is a contactomorphism when restricted to a neighborhood N of Γ and still takes $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$.

Now we may use Lemma 2.20 applied to the disk $D_0 \setminus (D_0 \cap N')$, where N' is a neighborhood of Γ_0 contained in N, to further isotope f to a contactomorphism in a neighborhood U of all of D_0. Finally, consider the 3-ball $B = L_0 \setminus (L_0 \cap U')$, where U' is a neighborhood of D_0 contained in U. Theorem 2.14 says we may now isotope $f|_B$ rel ∂B to a contactomorphism on B, thus obtaining a contactomorphism from (L_0, ξ_0) to (L_1, ξ_1). $\square$

This theorem shows the importance of understanding the characteristic foliation D_ξ. To do this we will always assume that the 1-skeleton Γ of D is transverse to the contact structure ξ. Notice that V_0 is a tubular neighborhood of Γ. By shrinking V_0, if necessary, we may assume that $\gamma = D \cap \partial V_0 = \partial\Delta$ is a transverse curve and that no singularities of D_ξ lie in $A = D \cap V_0$.

Theorem 5.4. *Assume we are in the situation described in the preceding paragraph. Then*

$$e(\xi)(D) \equiv (-l + q) \mod p, \tag{5.1}$$

where $l = l(\gamma)$.

Proof. We begin by recalling that Proposition 2.27 tells us that

$$-l = d_+ - d_-$$

where $d_\pm$ is the number of $\pm$-elliptic points minus the number of $\pm$-hyperbolic points in Δ_ξ. One way to see this is to let w be the vector field directing Δ_ξ and v the nonzero vector field used to compute $l(\gamma)$. So if γ' is the knot formed by pushing γ along v, then $l(\gamma) = I(\gamma', \Delta)$; and, if γ'' is the knot formed by pushing γ along w, then $I(\gamma'', \Delta) = 0$. Let

$$\phi : \Delta \longrightarrow S^1 : x \mapsto (\text{angle between } v_x \text{ and } w_x).$$

Then it is a standard fact that

$$\begin{aligned} I(\gamma'', \Delta) - l(\gamma) &= \text{total variation of } \phi \text{ around } \gamma = \text{degree}(\phi|_\gamma) \\ &= \sum_{\text{sing. pts. } p_i} -\text{degree}(\phi|_{\overline{\partial(\Delta - B(p_i))} - \gamma}) \\ &= \sum_{\text{sing. pts. } p_i} \text{degree}(\phi_{\partial B(p_i)}) \end{aligned}$$

where $B(p_i)$ is a small ball about the singular point p_i. One can now easily check that

$$\text{degree}(\phi_{\partial B(p_i)}) = \begin{cases} +1 & p_i \text{ is } + \text{ elliptic or } - \text{ hyperbolic,} \\ -1 & p_i \text{ is } - \text{ elliptic or } + \text{ hyperbolic.} \end{cases}$$

Now to compute $e(\xi)(D)$ we will choose a vector field in ξ that is nonzero along the 1-skeleton of D and extend it to a vector field in ξ on all of D. Then $e(\xi)(D)$ will be the signed count of the zeros of this vector field. We will find this vector field by extending w across A. To do this we just have to look in V_0, we can use our model U for V_0 (see page 73). In this model A will be a p-pronged star shaped graph in the xy-plane crossed with the z-axis. We only have w defined along $A \cap \partial V_0$ and there it is pointing radially inward. Now define w along the z-axis to be the vector field that points along the negative y-axis at $(0,0,0)$ and twists by $\frac{2\pi q}{p}$ as we traverse the z-axis from $(0,0,0)$ to $(0,0,1)$. This will give a nonzero vector field along Γ in V_0. Now extend w across A arbitrarily. Thus we have

$$\begin{aligned} e(\xi)(D) &= \sum_{\text{x a zero of } w} \text{index}(x) \\ &= \sum_{\text{x a zero of } w|_\Delta} \text{index}(x) + \sum_{\text{x a zero of } w|_A} \text{index}(x). \end{aligned}$$

At a positive elliptic point x of Δ_ξ (hence a zero of w), ξ_x and $T_x\Delta$ have the same orientation and $w \in T\Delta \cap \xi$ near x; thus index(x) will be the same thought of as a vector field in $T\Delta$ or one in ξ. So, index$(x) = +1$. Similarly index$(x) = +1$ if x is a negative hyperbolic point in Δ_ξ and index$(x) = -1$ if x is a negative elliptic point or a positive hyperbolic point of Δ_ξ. Thus

$$\sum_{\text{x a zero of } w|_\Delta} \text{index}(x) = -l.$$

To compute the second sum we need to consider our model of V_0 again. Let $A' = S^1 \times [0,1]$ be an annulus and $f : A' \longrightarrow V_0$ a map from A' to A sending $S^1 \times \{0\}$ to Γ and $S^1 \times \{1\}$ to γ. Now $f^*\xi$ is $A' \times \mathbb{R}^2$. If we choose a trivialization of $f^*\xi$ so that f^*w twists 0 times as we traverse $S^1 \times \{1\}$ then f^*w will twist q times about $S^1 \times \{0\}$. Thus we may conclude that

$$\sum_{\text{x a zero of } w|_A} \text{index}(x) = q.$$

Thus finishing the proof of Equation (5.1). $\square$

This theorem is sufficient for our purposes when p is odd but for p even we need the following refinement.

THEOREM 5.5. *Let p be even and s be the spin structure on L described in Theorem 5.2. Then*

$$\Gamma(\xi, s) \cdot D \equiv \frac{1}{2}(-l + q) \mod p. \tag{5.2}$$

PROOF. We start by canceling the negative elliptic and positive hyperbolic points in D_ξ. Now the strategy of the proof is to start with the vector field w used in the proof of Theorem 5.4 and form another vector field $v \in \xi$ by coalescing the zeros of w into zeros of multiplicity 2. While constructing v from w we will be careful not to change w in a neighborhood of Γ thus the spin structure induced in a neighborhood of Γ by w and v will be the same. It is not hard to see that this structure will not extend across a 2-handle attached to Γ with framing 0. Thus the spin structure induced by v will be s and we can use v to compute $\Gamma(\xi, s)$. Assuming we can construct such a v then it should be clear that

$$\Gamma(\xi, s) \cdot D \equiv \frac{1}{2}(-l + q) \mod p.$$

We now begin to construct v. On V_0 notice that we have a map from $A \times \mathbb{R}^2 \longrightarrow TL|_{V_0}$ that sends A to A and $\{\text{pt.}\} \times \mathbb{R}^2$ to $\xi_{\{\text{image of pt.}\}}$. We can thus pull w back to a section of $(\partial A) \times \mathbb{R}^2$. It is quite easy to see (see the proof of Theorem 5.4 for the definition of w on A) that we can extend this section over A to a vector field with exactly $\frac{q-1}{2}$ zeros of multiplicity 2 and one zero of multiplicity 1. Now let v on $A \subset L$ be in image of this section. Thus v is a vector field in ξ that has $\frac{q-1}{2}$ multiplicity 2 zeros and a multiplicity 1 zero at x. Notice that we can assume that w and v agree in a neighborhood of Γ. Moreover, we can assume that x is connected by a leaf in the foliation to an elliptic point e_0 in Δ. The remaining elliptic points $e_1, \ldots, e_n$ pair up with hyperbolic points $h_1, \ldots, h_n$, i.e. e_i is connected to h_i by a stable sepratrix on h_i.

We must now see how to coalesce e_i with h_i for $i = 1 \ldots n$ and x with e_0. Let N_i be a neighborhood, in Δ, of the stable sepratrix of h_i that connects it to e_i. We will see how to alter w in N_i so that it has a single zero of multiplicity two in N_i and is unchanged near the boundary. Thus v will be w on the complement of the N_i's (and A) and this new vector field on N_i. The neighborhood N_i can be broken into three pieces: a neighborhood E_i of e_i, a neighborhood H_i of h_i and $N_i' = N_i \setminus (E_i \cup H_i)$ (see Figure 5.4). In E_i the contact planes

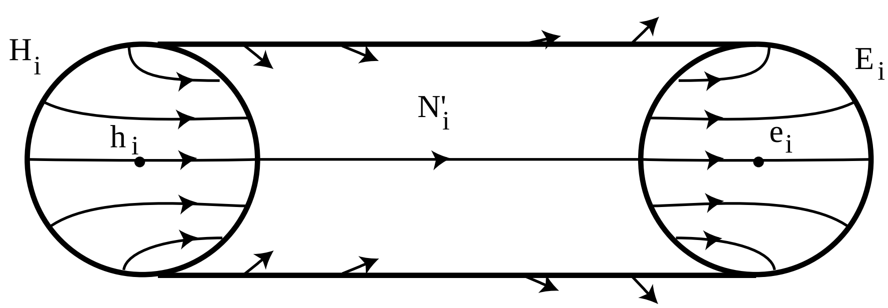


FIGURE 5.4. The neighborhood N_i

are almost tangent to the tangent planes of E_i. Thus we can choose a nonzero vector field tangent to E_i as shown in Figure 5.4 and then tilt them to lie in the contact planes, call the resulting vector field v. We can do the same in H_i. Now over N_i' the contact planes form a trivial bundle. In addition, we have a trivialization given to us by the vector field w (since it is nonzero in N_i'). We already have v defined on part of the boundary of N_i', on the rest of the boundary set v equal to w. As we traverse the boundary of N_i', v will spin around twice as measured with respects to w. Thus we can extend v over N_i' so as to have precisely one zero of multiplicity two. Finally, we can alter w in a neighborhood of the leaf connecting e_0 to x just as we did in the N_i's. This completes the construction of v on D. We can easily extend the to a vector field v in a neighborhood of D with the requited properties. Thus we are left to extend v over a 3-ball ($= L(p,q) \setminus$ (neighborhood of D)). This is an easy exercise we leave to the reader (be careful that the zeros continue to have multiplicity two). □

This theorem tells us the type of singularities we can expect to encounter in our proof of Theorem 5.2.

Corollary 5.6. *For a contact structures ξ on $L(p,q)$ satisfying (1) or (2) of Theorem 5.2 we have*

$$l = -1 + 2np.$$

Proof. Theorem 5.4 tells us that

$$l = -e(\xi)(D) + q + mp,$$

for some m. If our contact manifold satisfies (1) of Theorem 5.2 then we clearly have

$$l = -(q+1) + p + mp = -1 + mp.$$

Now since p is odd if m is not even then $-1 + mp$ is even. This contradicts Proposition 2.27 which implies that l must be odd. Thus m is even and we may write

$$l = -1 + 2np,$$

as required. Now if our manifold satisfies (2) of Theorem 5.2 then Theorem 5.5 tells us

$$\begin{aligned} l &= -2\Gamma(\xi, s) \cdot D + q + 2np \\ &= -2(\frac{1}{2}(q+1)) + q + 2np = -1 + 2np. \end{aligned}$$

□

This corollary indicates how we might try to prove Theorem 5.2.

Corollary 5.7. *Let ξ_0 and ξ_1 be two contact structures on $L = L(p,q)$ as in Theorem 5.2. If we can isotope D_0 in (L,ξ_0) and D_1 in (L,ξ_1) so that $l=-1$ in both ξ_0 and ξ_1, then ξ_0 is contactomorphic to ξ_1.*

Proof. Since $l=-1$ in both cases we can use Theorem 2.29 to make the characteristic foliation on D_i quite simple, consisting of a single positive elliptic point in Δ_i with leaves leaving this point and flowing to Γ_i. Given any map from D_0 to D_1 it is quite clear that it can be isotoped to take $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$. This may of course be realized by an ambient isotopy of L. Thus Theorem 5.3 tells us (L,ξ_0) and (L,ξ_1) are contactomorphic. □

Thus we are left to try to simplify the characteristic foliation Δ_ξ.

2. Simplifying the Characteristic Foliation

In this section we will see how to simplify the foliation on a generalized projective plane in a lens space $L(p,q)$ with a tight oriented contact structure ξ. As a warm up let us prove Corollary 2.15: there is a unique tight contact structure on $\mathbb{R}P^3$. As noted in Chapter 2 this is an unpublished result of Eliashberg's; we assume his proof is similar.

proof of Corollary 2.15. We may construct $\mathbb{R}P^3$ by gluing the 3-ball B^3 to $\mathbb{R}P^2$. Later we will show that for any tight contact structure ξ on $\mathbb{R}P^3$ we can arrange (after an isotopy) that the characteristic foliation $\mathbb{R}P^2_\xi$ has one singular point (necessarily elliptic) and no limit cycles. Assuming this for the moment, let ξ and ξ' be two tight contact structures on L and L', two copies of $\mathbb{R}P^3$. Isotope $\mathbb{R}P^2$ in L and L' so that the characteristic foliation on $\mathbb{R}P^2$ is as described above. Clearly, one may construct a diffeomorphism from L to L' which takes $\mathbb{R}P^2$ to $\mathbb{R}P^2$ and $\mathbb{R}P^2_\xi$ to $\mathbb{R}P^2_{\xi'}$. Lemma 2.20 will allow us to extend our diffeomorphism to a contactomorphism from a neighborhood U of $\mathbb{R}P^2$ in L to a neighborhood U' of $\mathbb{R}P^2$ in L'. Notice $\overline{L\setminus U}$ and $\overline{L'\setminus U'}$ are both 3-balls, B and B' respectively and the diffeomorphism from L to L' takes B to B'. Since the diffeomorphism is a contactomorphism from U to U' we have $(\partial B)_\xi$ mapped to $(\partial B')_{\xi'}$. Thus by Theorem 2.14 we may isotope our diffeomorphism to a contactomorphism on all on L, finishing the proof.

We now show that we can isotope $\mathbb{R}P^2$ in $(\mathbb{R}P^3,\xi)$ as claimed above. We can consider $\mathbb{R}P^3$ as $L(2,1)$ so we decompose $\mathbb{R}P^3$ as in the previous section. That is: let D denote a projective plane in $\mathbb{R}P^3$ and assume that the 1-skeleton Γ of D is transverse to ξ. We can find a Heegaard splitting $\mathbb{R}P^3 = V_0\cup V_1$ with Γ the core curve in V_0 and $\Delta = D\cap V_1$

the meridional curve in V_1. Finally, take V_0 sufficiently small that $\gamma = \partial\Delta$ is a transverse curve. Isotope Δ so that it contains only positive elliptic and negative hyperbolic points. Theorem 5.4 tells us that $l(\gamma) = -1 + 2n$ where n is necessarily a nonpositive integer.

First we claim that we can isotope D so that $n = 0$ or -1. To do this we notice that Δ may be isotoped so that Δ_ξ appears as in Figure 5.5 (this will be proved below in more generality, see Theorem 5.9). By

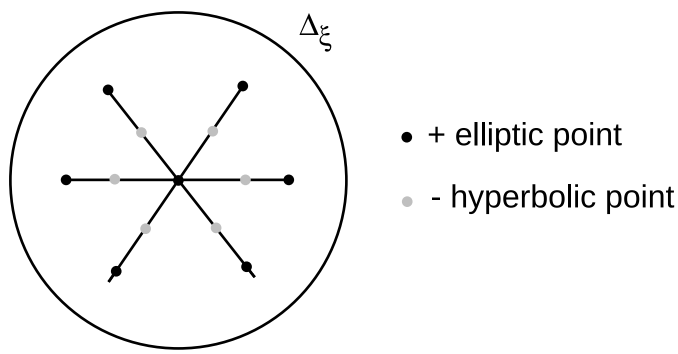


FIGURE 5.5. Singularities on Δ

this we mean that the singularities in Δ_ξ form a star with one positive elliptic point in the center and $|n|$ edges each with one negative hyperbolic point in the interior and one positive elliptic point at the end. By following the flow of D_ξ the boundary of Δ is mapped two-to-one onto Γ. Choosing a point x in Γ we can use this map to break $\gamma = \partial\Delta$ into two arcs B_0 and B_1. Clearly, if $n < -2$ then there must be a hyperbolic point h with both of its unstable separatrices exiting Δ through, say B_0. This is also true for $n = -2$; to see this label the hyperbolic points h_0 and h_1. Traversing $\gamma = \partial\Delta$ in a counterclockwise direction we encounter both the end points of the unstable separatrices of, say h_0, and then both of the ones from h_1. Label these end points coming from h_0, h_0^f or h_0^s according to the order in which they are encountered (f for first and s for second) and similarly for h_1. We can choose the point x in Γ so that one of its preimage points lies behind the point h_0^f and h_1^s does not lie between it and h_0^f. Now it should be clear that if h_0^f and h_0^s do not lie within one of the B_i's then h_1^f and h_1^s must. Note that the unstable separatrix of h separates an elliptic point e from the rest of Δ. Now isotope part of Γ in D so that it lies on a subarc of B_0 that contains the ends of the unstable separatrix of h (see Figure 5.6). Then push a subarc B' of $\Gamma \cap B_0$ across the unstable separatrices of h in Δ as indicated in Figure 5.7. We have found a new transverse curve

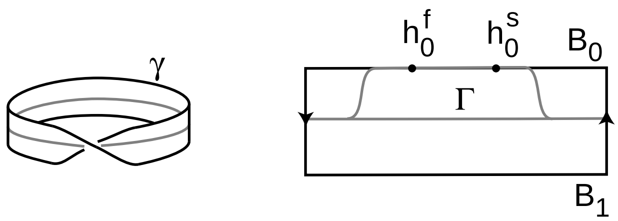


FIGURE 5.6. Isotopy of Γ

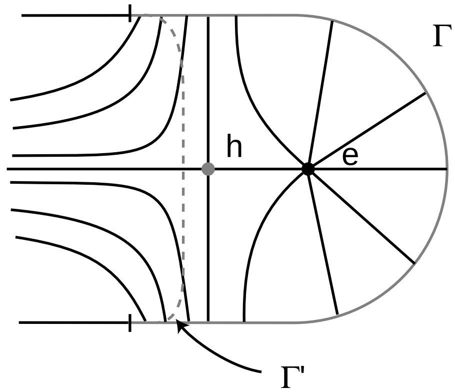


FIGURE 5.7. Isotopy of Γ in Δ

Γ' in D. Using Γ' we can see a new decomposition of $\mathbb{R}P^3 = V_0' \cup V_1'$ where Γ' is the core of V_0' and $D \cap V_1' = \Delta'$. We now need to compare Δ_ξ' and Δ_ξ. Notice $\Delta \cap \Gamma'$ separates Δ into two parts Δ_0 and Δ_1 where Δ_0 contains h and e. Moreover, Δ' is just $\Delta_1 \cup_{B'} \Delta_0$ (see Figure 5.8). Remember that we have not actually moved D so D_ξ looks the "same,"

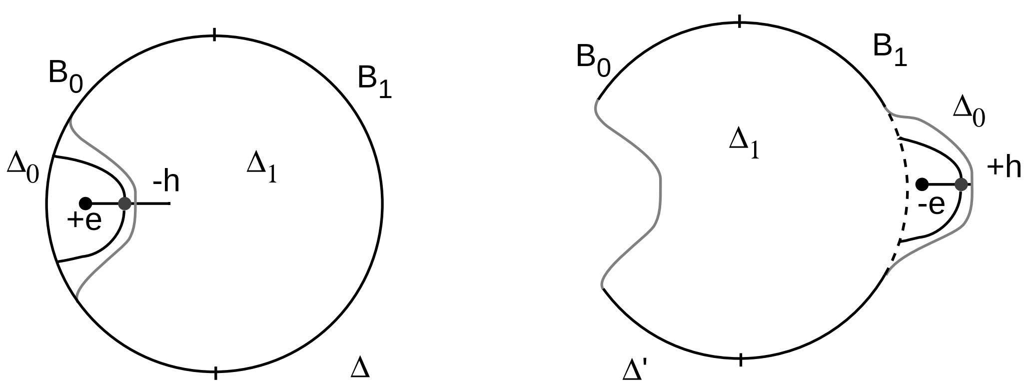


FIGURE 5.8. Δ and Δ'

i.e. has the same singularities (note D in nonorientable so there are no signs on the singularities of D_ξ). But the orientation Δ_0 inherits as a subset of Δ' is opposite the orientation it inherits as a subset of Δ. Thus Δ' has $-n$ positive elliptic points, 1 negative elliptic point, $-n-1$ negative hyperbolic points and 1 positive hyperbolic point (h and e changed sign since the orientation on Δ_0 changed). We may now cancel h with a positive elliptic point and e with a negative hyperbolic point (see the proof of Theorem 2.29 for this), thus by Proposition 2.27

$$l = -(-n-1) - (-n-2) = -1 + 2(n+2).$$

Hence, repeating the above process as necessary, we can arrange that $n = 0$ or -1 as claimed above.

If $n = 0$, then we are done since D_ξ is as claimed above: one elliptic point and no closed orbits. So we are left to deal with the $n = -1$ case. We claim that this cannot occur in a tight contact structure on $\mathbb{R}P^3$. To see this suppose we have a tight contact structure with $n = -1$. So there is precisely one negative hyperbolic point h in Δ_ξ. If the unstable separatrices of h both exit Δ through, say, B_0 then we may proceed as in the previous paragraph to find a Γ' and Δ' so that $\gamma' = \partial\Delta'$ is an unknot with $l > 0$, contradicting the tightness of ξ (see Theorem 2.28). Thus we may assume that each B_i has an unstable separatrix exiting through it. If we follow one of the unstable separatrices of h through Γ and back onto Δ we see that it will have to end at an elliptic point e (it cannot end back at h since we are assuming that D_ξ is generic). It is easy to see that the other unstable separatrix of h when it returns to Δ will also end at e (see Figure 5.9). By perturbing Δ a little

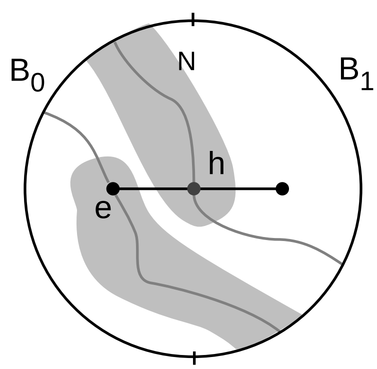


FIGURE 5.9. An impossible characteristic foliation

in a neighborhood of e we may assume that the unstable separatrices of h hit e on opposite sides (as in Corollary 2.25). Now let N be a neighborhood of one of the unstable separatrix of h (see Figure 5.9).

Orienting N so that e is a positive elliptic point will make h a positive hyperbolic point of N_ξ since if the orientation of N agrees with that of Δ at e it must disagree with it at h. Thus we may cancel e with h in N and both the unstable separatrix of h will remain part of the characteristic foliation of D, i.e. they will form a closed orbit in D_ξ. This closed orbit bounds a disk in D contradicting the tightness of ξ. Thus n is not equal to -1. $\square$

Now let us see how we can generalize the above proof. As in the previous section let D denote a generalized projective plane in $L = L(p, q)$. Further assume that the 1-skeleton Γ of D is transverse to ξ. We can also find a Heegaard splitting $L = V_0 \cup V_1$ with Γ the core curve in V_0 and $\Delta = D \cap V_1$ the meridional curve in V_1. Finally, take V_0 sufficiently small that $\gamma = \partial\Delta$ is a transverse curve and the characteristic foliation on $A = D \cap V_0$ contains no singularities.

THEOREM 5.8. *In the situation described above if $l(\gamma) < -2p$, then we may isotope D to D' so that D' satisfies all the properties listed above for D and $l(\gamma') = l(\gamma) + 2p$.*

First we need to understand what Δ_ξ looks like. As always we can use Theorem 2.29 to insure that Δ_ξ has only positive elliptic and negative hyperbolic points. By Proposition 2.27 we know that there are k positive elliptic points and $k-1$ hyperbolic points, for some positive integer k. We can actually standardize Δ_ξ even more:

LEMMA 5.9. *We may isotope Δ so that the singularities of Δ_ξ form a star, that is the union of the singularities and stable separatrices forms a graph with the vertices elliptic points, one hyperbolic point in the interior of each edge, one k-valent vertex and $k-1$ univalent vertices (see Figure 5.5).*

PROOF. We begin by defining the graph of singularities in Δ_ξ to be the union of singular points and stable separatrices in Δ_ξ. We will show that by a C^0-small isotopy of Δ, supported away from the boundary, that we can affect the graph of singularities in Δ_ξ as shown in Figure 5.10. Since the graph of singularities in Δ_ξ must be a tree (if there was a closed loop in the graph one of the unstable separatrices of a hyperbolic point on the loop would have no place to go) a sequence of these isotopies will clearly yield the conclusion of the lemma.

Now assume that part of the graph of singularities in Δ_ξ is as shown on the left hand side of Figure 5.10. Let D be a subdisk of Δ that contains all the singularities of Δ_ξ and is disjoint from the boundary of Δ. Lemma 2.22 tells us that Δ is a convex disk and thus, by Lemma 2.21,

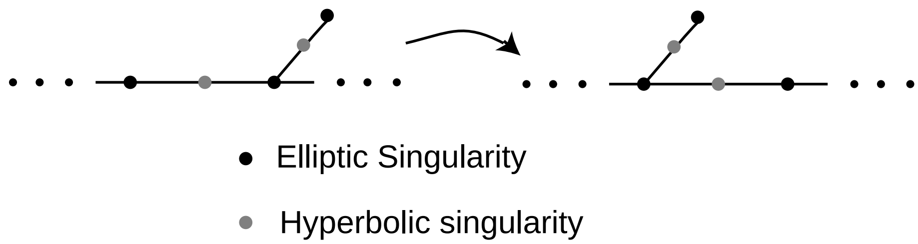


FIGURE 5.10. Change in Graph of Singularities.

has a neighborhood contactomorphic to $\Delta \times (-\delta, \delta)$ with a vertically invariant (tight) contact structure on it. Note that this implies that the characteristic foliation on $D \times \{t\}$ is diffeomorphic to D_ξ for all $t \in (-\delta, \delta)$. We may now form the ball $B = D \times [-\epsilon, \epsilon]$, for some ϵ, and sphere $S = \partial B$ (to be precise we should smooth the corners on S as discussed at the end of Section 2.3, but since this will not affect the characteristic foliation, see Lemma 2.30, we will ignore this point). To complete the proof we will find a disk D'' in B with which we can replace D in Δ, so that the new Δ will have characteristic foliation as indicated on the right side of Figure 5.10. To this end, let D' be a disk in $(\mathbb{R}^3, dz + x\,dy)$ with the characteristic foliation indicated in Figure 5.11 (it is not hard to construct such a disk). By Lemma 2.22,

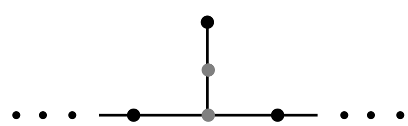
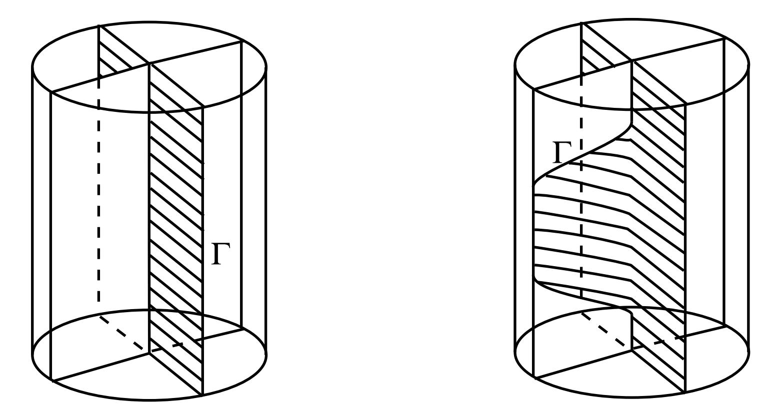


FIGURE 5.11. Characteristic Foliation on D'.

D' is a convex surface and thus there is a vertically invariant (tight) contact structure ξ' on $D' \times (-\delta, \delta)$ so that $(D' \times \{t\})_{\xi'}$ is diffeomorphic to the characteristic foliation on D' in $\mathbb{R}^3$, for all $t \in (-\delta, \delta)$. Let B' be the ball $D' \times [-\epsilon', \epsilon']$, for some ϵ', and S' be the sphere $\partial B'$. Now by choosing ϵ and ϵ' correctly and isotoping S' to remove the separatrix connections between hyperbolic points we can arrange that S and S' have identical characteristic foliations. Let ψ we a diffeomorphism from B to B' taking the characteristic foliation on S to the one on S'. We can, moreover, assume that ψ takes $(\partial D) \times [-\epsilon, \epsilon]$ to $(\partial D') \times [-\epsilon', \epsilon']$ in such a way that $(\partial D) \times \{t\}$ is mapped to $(\partial D') \times \{f(t)\}$, where $f : [-\epsilon, \epsilon] \longrightarrow [-\epsilon', \epsilon']$ is smooth and $f(0) = 0$. Now Theorem 2.14 tells us that ψ may be isotoped rel boundary into a contactomorphism from B to B'. Set $\Delta' = (\Delta \setminus D) \cup \psi^{-1}(D' \times \{0\})$ (and smooth corners). Notice that Δ' is a C^0-small isotopy of Δ fixed near the boundary and the characteristic foliation on Δ' is indicated in Figure 5.11. We can clearly isotope Δ' further so that the graph of singularities of Δ'_ξ is as shown on the right hand side of Figure 5.10. Thus completing the proof of our claim. □

Proof of Theorem 5.8. We first use Lemma 5.9 to arrange the singularities of Δ_ξ into a star. Since we are assuming that $l < -2p$ we know there are at least p edges (hyperbolic points) in the star. By following the flow of the characteristic foliation on D there is a p-to-1 mapping from $\gamma = \partial\Delta$ to Γ. The inverse image of a point x in Γ will break γ up into p arcs, $B_0, B_1, \ldots, B_{p-1}$. We can also use the flow on A_ξ and x to break A into p rectangles $A_0, A_1, \ldots, A_{p-1}$. As in the proof of Corollary 2.15 we can assume, by properly choosing the point in Γ, that there is a hyperbolic singularity h in Δ_ξ whose unstable separatrices have their end points contained in one of the B_i's, say B_0, and separate exactly one elliptic point e from the rest of D.

We would now like to perform the same isotopy of Γ that we did in the proof of Corollary 2.15. But in the present situation we will have to isotope D as well, so the characteristic foliation on D will necessarily change. To see how it will change we consider the isotopy one step at a time. The first part of the isotopy is to ambiently isotope a piece of Γ in A to a subarc of B_0 containing the end points of the unstable separatrix of h. To see the effect of this isotopy on the rest of A (hence D since this isotopy can be supported arbitrarily close to V_0) we will consider our standard model U for V_0 (see page 73). In this model, remember, A will be the image of the "suspended cross" shown in Figure 5.12 (a) when $p = 4$. In this model we can also take the point x in Γ to be



(a) V_0 before isotopy (b) V_0 after isotopy

Figure 5.12. Model for V_0

$(0,0,0)$. The ambient isotopy of Γ takes A to A', where A' is shown in Figure 5.12 (b). This isotopy clearly moves Γ through transverse knots and A' has no singularities (since the tangencies of A' are "bounded away from the horizontal" and the 2-planes in ξ are arbitrarily close to horizontal planes). Thus our isotopy, so far, has not significantly affected D_ξ.

The next step in our isotopy is to push part of $\Gamma \cap B_0$ across the unstable separatrices of h. Note this is not an isotopy through transverse knots, as the one above, but at the end of the isotopy Γ will again be a transverse knot. To see what happens to the rest of D we will consider the case when $p = 3$, the case for larger p being analogous. Near $\Gamma \cap B_0$ we have Δ on one side of Γ and A_1 and A_2 fanning out behind Γ (see Figure 5.13 (a)). We can assume that A_1, say, meets $\Gamma \cap B_0$ so that

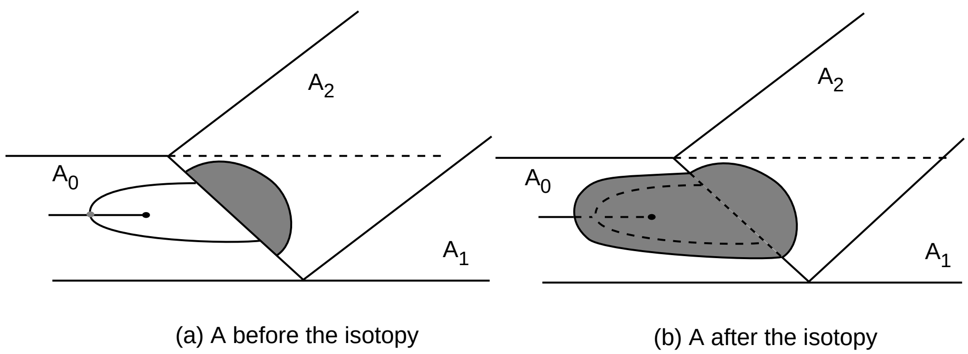


FIGURE 5.13. A_1 and A_2 near $\Gamma \cap B_0$

$A_1 \cup \Delta$ form a smooth surface and A_2 is above A_1. When we push part of $\Gamma \cap B_0$ across the unstable separatrices of h we will transfer h and e from Δ to A_1 as we did in the proof of Corollary 2.15. Now we will have to drag part of A_2 along with us through the isotopy (the gray part in Figure 5.13), but notice that we can drag it so that it is arbitrarily close to A_1 (see Figure 5.13 (b)). Thus the characteristic foliation on the part of A_2 we dragged along is topologically equivalent to the foliation just transferred onto A_1, since Peixoto's Theorem (see [**AP**]) says that the foliation Δ_ξ is structurally stable (all the fixed points are hyperbolic (in the sense of dynamics), there are no periodic orbits and no saddle points are connected by a leaf). In particular, this means that we have added an elliptic and a hyperbolic singularity to A_2. After isotoping Γ we have a new transverse curve Γ' and a new generalized projective plane D'. Using Γ' we also get a new Heegaard decomposition $L = V_0' \cup V_1'$ where V_0' is a small tubular neighborhood of Γ'. Let $\Delta' = D' \cap V_2'$ and $\gamma' = \partial\Delta'$.

We now claim that $l(\gamma') = l(\gamma) + 2p$. To see this let Δ_0 be the component of $\Delta \setminus \Gamma'$ that contains h and e. We can see that Δ' will essentially look like Δ with Δ_0 removed in one place and $p-1$ copies of it glued on along subarcs of $B_1, \ldots, B_{p-1}$ (see Figure 5.14). Now just

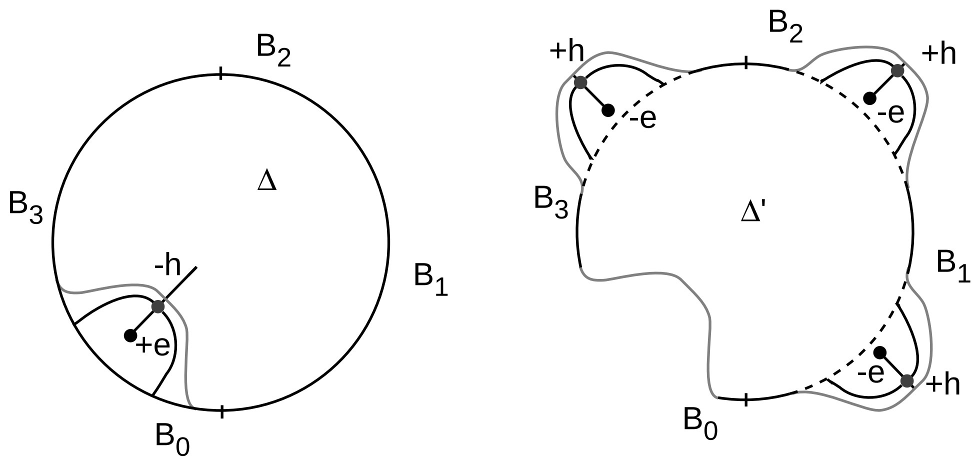


FIGURE 5.14. Δ and Δ' when $p = 4$.

as in the proof of Corollary 2.15 the orientations on the copies of D_0 in Δ' will be opposite that of D_0 in Δ. Thus Δ'_ξ has one less positive elliptic and one less negative hyperbolic point than Δ_ξ but has $p-1$ more negative elliptic and positive hyperbolic points. Thus when we cancel the negative elliptic and positive hyperbolic points from Δ'_ξ we will have p fewer positive elliptic and negative hyperbolic point than Δ_ξ had. Hence our claim follows from Proposition 2.27. □

We are now ready to finish the proof of our main result.

PROOF OF THEOREM 5.2. Corollary 5.6 tells us that for the contact structures we are interested in $l(\gamma) = -1 + 2np^2$ where n is a nonpositive integer. Theorem 5.8 insures that we can isotope D so that $l(\gamma) = -1$. Thus Corollary 5.7 finishes the proof. □

CHAPTER 6

Symplectic Constructions in Dimension 4

In this chapter we will prove our main results about symplectic manifolds and examine their consequences. We begin, in Section 1, by determining when a rational blowdown and log-transform may be done symplectically. In the last section we show that may irreducible 4-manifolds are symplectic and discuss the relation between irreducible 4-manifolds and symplectic 4-manifolds.

1. Rational Blowdowns and Symplectic Manifolds

We begin this section with the following result:

THEOREM 6.1. *Let (X, ω) be a symplectic 4-manifold. Suppose there exists embedded spheres Σ_i in X, for $i = 0, \ldots, (p-2)$, such that $\Sigma_0 \cdot \Sigma_0 = -(p+1)$, $\Sigma_i \cdot \Sigma_i = -2$, $\Sigma_{i-1} \cdot \Sigma_i = 1$, for $i = 1 \ldots (p-2)$, and all other intersections are 0 (see Figure 6.1). Let $C(p)$ be a tubular neighborhood of the Σ_i's. If $C(p)$ may be chosen to have ω-convex boundary and the Σ_i's are either Lagrangian or symplectic spheres, then the rational blowdown X_p has a symplectic structure on it.*

From the proof below it is be clear that we may take the symplectic structure on $X_p \setminus B(p)$ to be the same as the one on $X \setminus C(p)$.

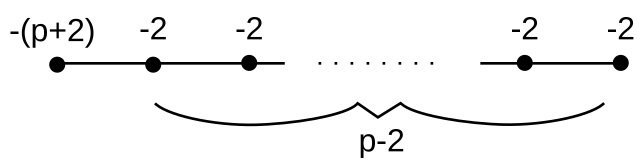


FIGURE 6.1. Graph of $C(p)$

PROOF. By assumption we know that $C(p)$ in X has ω-convex boundary. Corollary 3.8, moreover, says that $B(p)$ has ω'-convex boundary, where ω' is the symplectic structure $B(p)$ inherits from its Stein structure. Thus Theorem 3.1 tells us that X_p will have a symplectic structure if the contact structures induced on $L(p^2, p-1)$ as the convex boundary of $C(p)$ and $B(p)$ are the same. But this is the content of Theorem 5.1 $\square$

In specific examples it can be difficult to directly verify the hypothesis that $C(p)$ have ω-convex boundary. Lemma 3.3 and Theorem 3.5 help us find such $C(p)$. For the convenience of the reader we restate these results as:

THEOREM 6.2. *Let (X,ω) be a symplectic 4-manifold. Suppose there exists embedded spheres Σ_i in X, for $i = 0,\ldots,(p-2)$, such that $\Sigma_0 \cdot \Sigma_0 = -(p+1)$, $\Sigma_i \cdot \Sigma_i = -2$, $\Sigma_{i-1} \cdot \Sigma_i = 1$, for $i = 1\ldots(p-2)$, and all other intersections are 0 (see Figure 6.1). If Σ_0 is a symplectically embedded sphere and either*

i) *Σ_i is Lagrangian for $i = 1,\ldots,(p-2)$, or*
ii) *$p = 3$ and Σ_1 is symplectically embedded,*

then we may take $C(p)$ to have an ω-convex boundary.

An immediate corollary of Theorems 6.1 and 6.2 is the following:

COROLLARY 6.3. *Let (X,ω) be a symplectic 4-manifold containing spheres $\Sigma_0,\ldots,\Sigma_{p-2}$ along which X may be rationally blowdown. If Σ_0 is symplectically embedded and the rest of the spheres are Lagrangian or if $p = 2$ or 3 and all the spheres are symplectically embedded, then X_p has a symplectic structure.*

We would like to use this corollary to explore symplectic log transforms. In order to relate rational blowdowns to log transforms we need to be an a cusp neighborhood (see Proposition 2.33); moreover, if we wish the log transform to be a symplectic operation we will need some relation between the cusp and the symplectic structure. Recall a cusp has a neighborhood that has an elliptic fibration over a disk with one singular fiber, the cusp (see Figure 2.9). We may perturb the fibration to obtain an elliptic fibration over the disk with two fishtail fibers, F_1 and F_2. A **fishtail fiber** is an immersed 2-sphere with one transverse double point and self-intersection number 0. On F_1, say, there is a curve that bounds a disk D (the vanishing cycle for F_2) and inside a neighborhood N of $F_1 \cup D$ is a cusp. In fact, we can think of N as an elliptic fibration over the disk with a single cusp C as the singular fiber or with two fish tails, F_1 and another, as the singular fibers. Thus N is a cusp neighborhood. If N is in a symplectic manifold (X,ω) then we will call N a **symplectic cusp neighborhood** if F_1 is symplectic. We will also say a cusp C is a **symplectic cusp** if it has a neighborhood in which C may be perturbed into two fishtail fibers both of which are symplectic. Clearly if (X,ω) contains a symplectic cusp then it contains a symplectic cusp neighborhood.

THEOREM 6.4. *Let (X, ω) be a symplectic 4-manifold that contains a symplectic cusp neighborhood N. Then a $p = 2$ or 3 log transform on a torus fiber in N yields a symplectic manifold $X(p)$. Moreover, if C is a symplectic cusp, then performing multiplicity 2 and 3 log transforms on torus fibers in a neighborhood of C yields a symplectic manifold $X(2, 3)$.*

PROOF. Let F be the symplectic fishtail in N. If we blow up X at the double point of F, then the proper transform of F is an embedded symplectic 2-sphere with self-intersection number -4, see Figure 6.2. If

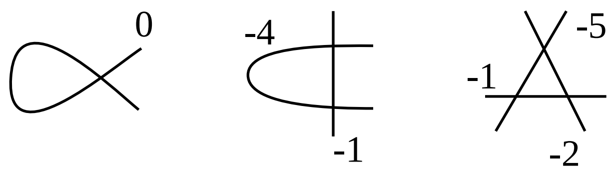


FIGURE 6.2. The fiber F and its blow ups.

we blow up again (at the intersection of F and the exceptional divisor) we will obtain two embedded symplectic 2-spheres realizing the configuration in Figure 6.1, see Figure 6.2. Thus, by Theorem 6.2, we can realize $C(2)$ in $X\#\overline{\mathbb{C}P}^2$ or $C(3)$ in $X\#2\overline{\mathbb{C}P}^2$ with ω-convex boundary. So the rational blowdown will be a symplectic manifold that is diffeomorphic to to the p log-transform $X(p)$, by Proposition 2.33.

If we have a symplectic cusp in X then there are two symplectic fishtail fibers. After blowing up and rationally blowing down once, as above, we will have a symplectic manifold $X(3)$, say, that still contains a symplectic fishtail fiber; moreover, it will still be in the neighborhood of a cusp. Thus when we perform the second blow up and rational blowdown we will get a symplectic manifold diffeomorphic to $X(2, 3)$. $\square$

2. Examples of Symplectic 4-Manifolds

In this section we would like to return to the question of symplectic structures on the irreducible 4-manifolds constructed in Section 4.3. We begin with the Gompf–Mrowka examples.

THEOREM 6.5. *If p_i, q_i are relatively prime numbers for $i = 1, \ldots, 3$ and p_3 and q_3 are in the set $\{1, 2, 3\}$, then the manifold $E'(2; p_1, q_1; p_2, q_2; p_3, q_3)$ supports a symplectic structure.*

PROOF. As mentioned in Section 4.3 the manifold $E'(2; p_1, q_1; p_2, q_2; p_3, q_3)$ has a symplectic structure on it if $p_3 = q_3 = 1$. Thus we need to see that the cusp neighborhood N_3 on which the p_3 and q_3 log-transforms are performed is in fact a symplectic cusp neighborhood. There are several ways to do this. The simplest is to notice that N_3 is a cusp neighborhood in a (complex) elliptic fibration of $K'(2)$ (see [**GM**]), thus is a symplectic cusp. $\square$

Next recall that $E(n)$ contains two disjoint copies of $C(n-2)$. It is clear from the construction in Section 4.3 that the spheres in $C(n-2)$ are actually symplectic spheres. Moreover $E(4)$ contains nine copies of $C(2)$ containing symplectic spheres and $E(5)$ contains four copies of $C(3)$ in which all of the spheres are symplectic. Thus $E_i(4)$ and $E_j(5)$ all have symplectic structures for $i = 1, \dots, 9$ and $j = 1, \dots, 4$.

We would like to end by saying that these results should apply to many other 4-manifolds. In particular, we think that they will apply to most of the manifolds studied in [**FS1**]. In a future paper we will consider when symplectic cusps may be found in these and other 4-manifolds.

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