A DUALITY EXACT SEQUENCE FOR LEGENDRIAN CONTACT HOMOLOGY

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Abstract. We establish a long exact sequence for Legendrian submanifolds $L \subset P \times \mathbb{R}$, where P is an exact symplectic manifold, which admit a Hamiltonian isotopy that displaces the projection of L off of itself. In this sequence, the singular homology H_* maps to linearized contact cohomology CH^* which maps to linearized contact homology CH_* which maps to singular homology. In particular, the sequence implies a duality between $\mathrm{Ker}(CH_* \to H_*)$ and $CH^*/\,\mathrm{Im}(H_*)$. Furthermore, this duality is compatible with Poincaré duality in L in the following sense: the Poincaré dual of a singular class which is the image of $a \in CH_*$ maps to a class $\alpha \in CH^*$ such that $\alpha(a) = 1$.

The exact sequence generalizes the duality for Legendrian knots in $\mathbb{R}^3$ [24] and leads to a refinement of the Arnold Conjecture for double points of an exact Lagrangian admitting a Legendrian lift with linearizable contact homology, first proved in [6].

1. Introduction

Legendrian contact homology, originally formulated in [2, 8], is a Floer-type invariant of Legendrian submanifolds that lies within Eliashberg, Givental, and Hofer's Symplectic Field Theory framework [9]. If $(P, d\theta)$ is an exact symplectic manifold with finite geometry at infinity, then Legendrian contact homology associates to a Legendrian submanifold $L \subset (P \times \mathbb{R}, dz - \theta)$ the "stable tame isomorphism" class of an associative differential graded algebra (DGA) $(\mathcal{A}(L), \partial)$. The algebra is freely generated by the Reeb chords of L — that is, integral curves of the Reeb field ∂_z that begin and end on L — and is graded using a Maslov index modulo twice the Maslov number of L. The differential comes from counting holomorphic curves in the symplectization of $(P \times \mathbb{R}, L)$; in the present case, this reduces to a count of holomorphic curves in P with boundary on the projection of L; see [4, 7] and below.

In general, it is difficult to extract information directly from the Legendrian contact homology DGA. An important computational technique is Chekanov's linearization $(Q(L), \partial_1)$ of $(\mathcal{A}(L), \partial)$. The linearization is defined when $\mathcal{A}(L)$ admits an isomorphism that conjugates ∂ to a differential ∂' which respects the word length filtration on $\mathcal{A}(L)$. In this case, the linearized homology is the E_1-term in the corresponding spectral sequence for computing the homology of ∂'. The linearized contact homology may depend on the choice of conjugating isomorphism, but the *set* of isomorphism classes of linearized contact homologies is invariant under deformations of L. Legendrian contact homology and its linearized version have turned out to be quite effective tools for providing obstructions to Legendrian isotopies [2, 5, 10, 16, 17, 20]. These results indicate the so-called "hard" properties of Legendrian embeddings.

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The main result of this paper describes an important structural feature of linearized Legendrian contact homology. Say that a Legendrian submanifold $L \subset P \times \mathbb{R}$ is *horizontally displaceable* if the projection of L to P can be completely displaced off of itself by a Hamiltonian isotopy. This condition always holds if $P = \mathbb{R}^{2n}$ (or, more generally, if $P = T^*(M \times \mathbb{R})$) or if L is a "local" submanifold that lies inside a Darboux chart.

Theorem 1.1. *Let $L \subset P \times \mathbb{R}$ be a closed, horizontally displaceable Legendrian submanifold. If L is spin, then let Λ be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Z}_n$. Otherwise, let $\Lambda = \mathbb{Z}_2$. The linearized contact homology and cohomology and the singular homology of L with coefficients in Λ fit into a long exact sequence:*

$$\cdots \to H_{k+1}(L) \xrightarrow{\sigma_*} H^{n-k-1}(Q(L)) \to H_k(Q(L)) \xrightarrow{\rho_*} H_k(L) \to \cdots . \tag{1.1}$$

Furthermore, if Λ is a field and if $\beta = \rho_*(\alpha) \in H_k(L)$, then the Poincaré dual $\gamma \in H_{n-k}(L)$ of β satisfies $\langle \sigma_*(\gamma), \alpha\rangle = 1$, where $\langle , \rangle$ is the pairing between the homology and cohomology of $Q(L)$.

A version of Theorem 1.1 was described for Legendrian 1-knots in $J^1(\mathbb{R})$ in [24], where it was proven that off of a "fundamental class" of degree 1, the linearized Legendrian contact homology obeys a "Poincaré duality" between homology groups in degrees k and $-k$. Theorem 1.1 can also be interpreted as a Poincaré duality theorem for Legendrian contact homology, up to a fixed error term which depends only on the topology of the Legendrian submanifold. Any class in $H_k(Q(L))$ in the kernel of ρ_* has dual class in $H^{n-k-1}(Q(L))$ determined up to the image of σ_*. In particular, there is a "duality" between $\ker(\rho_*)$ and $\mathrm{coker}(\sigma_*)$. We call the quotient $H_k(Q(L))/\ker(\rho_*)$ the *manifold classes of L*. The last statement of Theorem 1.1 says that if β is in the image of ρ_*, then its Poincaré dual γ is not in the kernel of σ_*, and hence γ is not in the image of ρ_* by exactness. The non-degeneracy of the Poincaré pairing over a field shows that the converse of the last statement follows as well, and thus there is a bijective correspondence between pairs of Poincaré dual classes in $H_*(L)$ and manifold classes in $H_*(Q(L))$. We prove in Theorem 5.5 that, over $\mathbb{Z}_2$, the fundamental class $[L] \in H_n(L)$ is always in the image of ρ_*. In other words, in the pair of Poincaré dual classes ([point], $[L]$), it is $[L]$ which is hit by ρ_*. For Poincaré dual pairs of homology classes other than this one, it is impossible to say *a priori* which class is in the image of ρ_*, as Example 5.10 shows.

An important application of Legendrian contact homology was to the Arnold conjecture for Legendrian submanifolds, which states that the number of Reeb chords of a generic Legendrian submanifold $L \subset J^1(\mathbb{R}^n)$ is bounded from below by half the sum of Betti numbers of L. For Legendrian submanifolds with linearizable contact homology, this conjecture was proved in [6]. Theorem 1.1 gives the following refinement of that result:

Theorem 1.2. *Let $L \subset P \times \mathbb{R}$ be a horizontally displaceable Legendrian n-submanifold. Assume that the first Chern class of TP vanishes and that the Maslov number of L equals zero. Let Λ be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Z}_p$ if L is spin, and $\mathbb{Z}_2$ otherwise. Assume that the projection of L to $T^*(M \times \mathbb{R})$ is self-transverse and that the Legendrian contact homology of L admits a linearization. If c_m denotes the number of Reeb chords of L of grading m and if b_k denotes the k^{th} Betti number of L over Λ, then:*

$$c_m + c_{n-m} \geq b_m$$

for $0 \leq m \leq n$.

The key to the proof of Theorem 1.1 is to study the Legendrian contact homology of the "two-copy" link $2L$ consisting of L and another copy of L shifted high up in the z-direction. Holomorphic disks of $2L$ admit a description in terms of holomorphic disks of L together with negative gradient flow lines of a Morse function on L. This description of holomorphic disks allows for a particularly nice characterization of the linearized contact homology of $2L$ in terms of the linearized contact homology of L and its Morse homology. The observation that $2L$ is isotopic to a link with no Reeb chord connecting different components then yields the exact sequence.

The paper is organized as follows. In Section 2, we set notation and review the basic definitions and constructions in Legendrian contact homology. The algebraic framework for the statement and proof of Theorem 1.1 is established in Section 3 using the description of holomorphic disks with boundary on $2L$. The analysis necessary for this description is carried out in Section 6. In Section 4, the duality theorem is established. Section 5 contains the proof of Theorem 1.2, and discusses, via examples, the relationship between the manifold classes in linearized contact homology and the singular homology of the Legendrian submanifold. In Appendix A, we establish some results related to gradings in contact homology.

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2. Background Notions

This section sets terminology and gives a brief review of the basic ideas of Legendrian submanifold theory and Legendrian contact homology. See [4, 5, 6, 7] for details.

2.1. **The Lagrangian Projection.** We will use the projection $\Pi_P : P \times \mathbb{R} \to P$ throughout the paper; this is called the *Lagrangian projection.* If $L \subset P \times \mathbb{R}$ is a Legendrian submanifold then $\Pi_P(L) \subset P$ is a Lagrangian immersion with respect to the symplectic structure $d\theta$.

As mentioned in the introduction, the Reeb vector field of $dz - \theta$ is ∂_z where z is a coordinate along the $\mathbb{R}$-factor in $P \times \mathbb{R}$. Consequently, if L is a Legendrian submanifold then there is a bijective correspondence between Reeb chords of L and double points of $\Pi_P(L)$. We will use "Reeb chord" and "double point" interchangeably, depending on context.

2.2. **Legendrian Contact Homology.** The Legendrian contact homology of a Legendrian submanifold $L \subset P \times \mathbb{R}$ is defined to be the homology of a differential graded algebra (DGA) denoted by $(\mathcal{A}(L), \partial)$. In this paper, we will take the coefficients Λ of $\mathcal{A}(L)$ to be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Z}_n$ if L is spin and $\mathbb{Z}_2$ otherwise. In general, it is possible to define the DGA with coefficients in the group ring $\Lambda[H_1(L)]$.

2.2.1. *The Algebra.* Let L be an admissible[1] spin[2] Legendrian submanifold of $(P \times \mathbb{R}, dz - \theta)$ and label the double points of its Lagrangian projection with the set $\mathcal{Q} = \{q_1, \dots, q_N\}$. Above each q_i, there are two points q_i^+ and q_i^- in L, with q_i^+ having the larger z coordinate.

Let $Q(L)$ be the free Λ-module generated by the set of double points $\mathcal{Q}$, and let $\mathcal{A}(L)$ be the free unital tensor algebra of $Q(L)$. This algebra should be considered to be a *based* algebra, i.e. the generating set $\mathcal{Q}$ is part of the data. For simplicity, we will frequently suppress the L

[1] This is a genericity condition, together with a requirement that the submanifold behaves nicely near a double point. See [4, 7].

[2] This condition is unnecessary if we work with $\mathbb{Z}_2$ coefficients.

in the notation for $Q(L)$ and $\mathcal{A}(L)$, and we will write elements of $\mathcal{A}$ as sums of words in the elements of $\mathcal{Q}$.

2.2.2. *The Grading.* Before defining the grading, we need to make some preliminary definitions and choices. Let $c_1(P)$ denote the first Chern class of TP equipped with an almost complex structure compatible with the symplectic form $d\theta$. Define the *greatest divisor* $g(P)$ as follows: if $c_1(P) = 0$, then $g(P) = 0$; otherwise, if $c_1(P) \neq 0$, then let $g(P)$ be the largest positive integer such that $c_1(P) = g(P)a$ and such that $g'a \neq 0$ if $0 < g' < g(P)$, for some $a \in H^2(P;\mathbb{Z})$.

The set of complex trivializations of TP over a closed curve γ in P is a principal homogeneous space over $\mathbb{Z}$. In particular, given two trivializations Z and Z' of TP along γ, there is a well-defined distance $d(Z, Z') \geq 0$ which is the absolute value of the class in $\pi_1\big(GL(k,\mathbb{C})\big) \cong \mathbb{Z}$ determined by Z' if Z is considered as the reference framing.

Definition 2.1. Let $g \in \mathbb{Z}$, $g \geq 0$. A $\mathbb{Z}_g$*-framing* of TP along a closed curve $\gamma \subset P$ is an equivalence class of complex trivializations of TP along γ, where two trivializations Z_0 and Z_1 belong to the same equivalence class if $d(Z_0, Z_1)$ is divisible by g.

In Appendix A, we show how choices of sections in certain frame bundles of TP over the 3-skeleton of some fixed triangulation of P induce a *loop* $\mathbb{Z}_g$*-framing of* P where $g = g(P)$, i.e., a $\mathbb{Z}_g$-framing of $TP|_\gamma$ for any loop γ in P. We also show that such loop $\mathbb{Z}_g$-framings are unique up to an action of $H^1(M;\mathbb{Z}_g)$. If $g = 0$, then $\mathbb{Z}_g$-framings are ordinary framings and the framings are unique up to action of $H^1(M;\mathbb{Z})$. In the special case $P = T^*M$, it is straightforward to see that $c_1(P) = 0$ and that there is a canonical loop framing; see Remark A.4. In what follows, we assume that a loop $\mathbb{Z}_g$-framing for P has been fixed.

If γ is a loop in L, then the tangent planes of L give a loop of Lagrangian subspaces of TP along γ. Using a trivialization Z representing the $\mathbb{Z}_g$-framing of $TP|_\gamma$ (see Lemma A.3), we get a Maslov index

$$\mu(\gamma, Z) \in \mathbb{Z}$$

of the loop of tangent planes along γ. Since a change of complex trivialization by one unit changes the Maslov index by 2 units, this gives a homomorphism

$$H_1(L;\mathbb{Z}) \to \mathbb{Z}_{2g}.$$

A generator $m(L)$ of the image of this homomorphism is called a *Maslov number of* L.

We are now ready to define the grading. Assume that $L \subset P \times \mathbb{R}$ is connected. For each of the double points in $\mathcal{Q}$, choose a *capping path* γ_i in L that runs from q_i^+ to q_i^-. The Lagrangian projections of tangent planes to L along γ_i gives a bundle of Lagrangian subspaces over γ_i in TP. Pick a trivialization Z of TP over γ_i representing its $\mathbb{Z}_g$-framing. This gives a path Γ_i of Lagrangian subspaces in $\mathbb{C}^n \approx \mathbb{R}^{2n}$. Let $\hat{\Gamma}_i$ be the result of closing the path Γ_i to a loop using a positive rotation (see [5]) along the complex angle between the endpoints of Γ_i. The *Conley Zehnder index* $\nu(\gamma_i, Z)$ is the Maslov index of $\hat{\Gamma}_i$. Let

$$|q_i|_Z = \nu(\gamma_i, Z) - 1 \in \mathbb{Z}.$$

The grading $|q_i|$ of q_i is

$$|q_i| = \pi(|q_i|_Z \mod 2g) \in \mathbb{Z}_{2g}/m(L), \tag{2.1}$$

where $\pi\colon \mathbb{Z}_{2g} \to \mathbb{Z}_{2g}/m(L)$ is the projection. Note that this grading is independent of the choice of capping path and of the choice of representative framing Z. It does however depend on the choice of loop $\mathbb{Z}_g$-framing.

The grading of a word in $\mathcal{A}$ is the sum of the gradings of its constituent letters. This grading will be extended to the two-copy $2L$ in Section 3.1.

2.2.3. *The Differential.* The differential on $\mathcal{A}$ is a degree -1 endomorphism that comes from counting rigid holomorphic disks in P with boundary on $\Pi_P(L)$. More precisely, a *marked disk* D_{m+1} is the unit disk in $\mathbb{C}$ together with $m+1$ marked points $\{x_0, x_1, \ldots, x_m\}$ in counter-clockwise order along its boundary. Let $\partial\widehat{D}_{m+1}$ be the boundary of D_{m+1} with the marked points removed. Over a double point q of $\Pi_P(L)$ lie two points q^+ and q^- in L; let W^+ be a neighborhood of q^+ in L, and similarly for W^-. Given a continuous map $u : (D_{m+1}, \partial D_{m+1}) \to (P, \Pi_P(L))$ with $u(x_j) = q$, say that u has a *positive puncture* (resp. *negative puncture*) at q if, as the boundary of D_{m+1} near x_j is traversed counter-clockwise, its image under u lies in $\Pi_P(W^+)$ (resp. $\Pi_P(W^-)$) before x_j and in $\Pi_P(W^-)$ (resp. $\Pi_P(W^+)$) after.

Now we can define the moduli spaces of holomorphic disks with boundary on L. Fix an almost complex structure J on P compatible with $d\theta$. For $a, b_1, \ldots, b_k \in \mathcal{Q}$ and $A \in H_1(L)$, define the moduli space $\mathcal{M}_A(a; b_1, \ldots, b_k)$, to be the set of maps $u : D_{m+1} \to P$ that are J-holomorphic, i.e. that satisfy $du + J \circ du \circ i = 0$, and so that:

- The boundary of the punctured disk is mapped to $\Pi_P(L)$,
- The map u has a positive puncture at x_0 with $u(x_0) = a$,
- The map u has negative punctures at x_j, $j > 0$, with $u(x_j) = b_j$, and
- The restriction $u|_{\partial\widehat{D}_{m+1}}$ admits a continuous lift into L which, together with the capping paths γ_a and $-\gamma_{b_j}$, gives a loop representing A.

These disks should be taken modulo holomorphic reparametrization. For admissible Legendrian submanifolds, it is possible to prove that the moduli spaces are smooth manifolds. More precisely, note that

$$\mathcal{M}_A(a; b_1, \ldots, b_k) = \cup_\alpha \mathcal{M}_A^\alpha(a; b_1, \ldots, b_k),$$

where α ranges over the homotopy classes represented by the disk maps.

Consider a map of the boundary of a punctured disk corresponding to $A \in H_1(L)$ and fix a homotopy class of disk maps α with this boundary condition. Fix a trivialization TP along the boundary of the disk map which extends over the disk and fix representatives $Z_a, Z_{b_1}, \ldots, Z_{b_k}$ of the $\mathbb{Z}_g$-framings of the capping paths of $a, b_1, \ldots, b_k$ which agree with this trivialization at the double points. This gives a trivialization of TP along the closed-up loop representing A. Consider the loop of Lagrangian planes tangent to $\Pi_P(L)$ along this closed-up loop and let $\mu(A, \alpha)$ denote its Maslov index measured using the trivialization just described. The following transversality result is a consequence of Proposition 2.3 in [7].

Lemma 2.2. *For an open dense set of almost complex structures J, which are standard in a neighborhood of L, $\mathcal{M}_A^\alpha(a; b_1, \ldots, b_k)$ is a transversely cut out manifold of dimension d where*

$$d = |a|_{Z_a} - \sum_j |b_j|_{Z_{b_j}} + \mu(A, \alpha) - 1.$$

Proof. Since the area of any non-trivial J-holomorphic disk is bounded from below and since there are only finitely many Reeb chords, it follows that there are only finitely many Reeb chord collections $(a; b_1, \dots, b_k)$ such that $\mathcal{M}_A(a; b_1, \dots, b_k)$ may be non-empty. Gromov compactness then implies that for each fixed Reeb chord collection, there are at most finitely many homology classes A and homotopy classes α such that $\mathcal{M}^\alpha_A(a; b_1, \dots, b_k)$ may be non-empty. The lemma then follows from Proposition 2.3 in [7]. $\square$

We will also need to consider moduli spaces of rigid holomorphic disks with exactly two positive punctures a_1, a_2 and an arbitrary number of negative punctures. There are new transversality issues in this case, as the setup allows for multiple covers. In order to deal with such issues, we study closely related holomorphic disks. More precisely, we take two copies L_0 and L_1 of L and let $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots c_l)$ denote the moduli space of holomorphic disks which are maps $u : S \to P$, where S is a strip $\mathbb{R} \times [0, 1]$ with boundary punctures $x_1, \dots, x_k$ along $\mathbb{R} \times \{0\}$ and $y_1, \dots, y_l$ along $\mathbb{R} \times \{1\}$, and where u maps the punctures at $\pm\infty$ to a_1 and a_2, respectively, the puncture x_j to b_j, the puncture y_j to c_j, has positive punctures are $\pm\infty$ and negative punctures elsewhere and takes the boundary components in $\mathbb{R} \times \{0\}$ to L_0 and those in $\mathbb{R} \times \{1\}$ to L_1, and where A encodes the homology class of the boundary data as before. In complete analogy with the one punctured case, we have

$$\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots c_l) = \cup_\alpha \mathcal{M}^\alpha_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots c_l),$$

where α is the homotopy class of the disk map. Again, we choose a trivialization TP for the boundary of the disk which extends over the disk and is compatible trivializations Z_c representing the $\mathbb{Z}_g$-framings of the capping paths of the Reeb chords of the disk. Using the same notation as in Lemma 2.2, we have the following result:

Lemma 2.3. *For generic perturbation of L_1, if*

$$d = |a_1|_{Z_{a_1}} + |a_2|_{Z_{a_2}} - \sum_{j=1}^{k} |b_j|_{Z_{b_j}} - \sum_{j=1}^{l} |c_j|_{Z_{c_j}} - n + 1 + \mu(A, \alpha) \leq 0$$

then $\mathcal{M}^\alpha_A(a_1, a_2; b_1, \dots, b_k, c_1, \dots, c_l)$ is a transversely cut out manifold of dimension d.

Proof. This follows from the proof of Proposition 2.3 (2) in [7]: we may achieve transversality for low dimensional moduli spaces by perturbing the Lagrangian boundary condition near one of the positive punctures. $\square$

Remark 2.4. It is essential in the proof of Lemma 2.3 that the perturbations near one of the positive punctures are independent from those at the other. This is not true for multiply covered disks if the boundary of the disk lies on a single immersed Lagrangian L, but using two Lagrangians resolves this problem.

Remark 2.5. Note that Lemma 2.3 gives one way of resolving the transversality problems arising from multiple covered disks. In order to compute the number of disks which would arise on the two-copy version one would have to study the obstruction bundles over certain multiply covered disks on L.

When discussing moduli spaces below, we fix an almost complex structure J so that Lemma 2.2 holds and a perturbation of L_1 so that Lemma 2.3 holds.

If L is spin, the moduli spaces can be consistently oriented (see [6]). If $\mathcal{M}_A(a; b_1, \dots, b_k)$ is zero dimensional, then by Gromov compactness it is compact and the algebraic count of points

$\#\mathcal{M}_A(a; b_1, \dots, b_k)$ makes sense. This count allows us to define the differential on generators as follows:

$$\partial a = \sum_{\dim(\mathcal{M}_A(a;b_1,\dots,b_k))=0} \big(\#\mathcal{M}_A(a; b_1, \dots, b_k)\big) b_1 \cdots b_k. \tag{2.2}$$

We then extend the differential to $\mathcal{A}(L)$ via linearity and the Leibniz rule.

The following lemma combines the definition of ∂ with Stokes' Theorem.

Lemma 2.6. *Let $\ell(q_i)$ be the length of the Reeb chord lying above q_i. If $u \in \mathcal{M}(a; b_1, \dots, b_k)$, then:*

$$\ell(a) - \sum_k \ell(b_j) \geq C \operatorname{Area}(u) > 0,$$

for some constant $C > 0$.

The central result of the theory is the following theorem:

Theorem 2.7 ([4, 5, 6, 7])**.** *The differential ∂ satisfies $\partial^2 = 0$ and the "stable tame isomorphism class" (and hence the homology) of the DGA $(\mathcal{A}, \partial)$ is invariant under Legendrian isotopy.*

See [2] for the definition of "stable tame isomorphism"; we will need only some straightforward consequences of the definition, not its precise formulation, in this paper.

2.3. **Linearization.** Legendrian contact homology is not an easy invariant to use, as it is a non-commutative algebra given by generators and relations. One important tool in extracting useful information from Legendrian contact homology is Chekanov's *linearized contact homology* [2]. To define it, break the differential ∂ into components $\partial = \sum_{r=0}^{\infty} \partial_r$, where the image of ∂_0 lies in the ground ring and ∂_r maps Q to $Q^{\otimes r}$. If $\partial_0 = 0$ — that is, if there are no constant terms in the differential ∂ — then the equation $\partial^2 = 0$ implies that $\partial_1^2 = 0$ as well, and hence that (Q, ∂_1) is a chain complex in its own right.

Even if the DGA does not satisfy $\partial_0 = 0$, it may be tame isomorphic to one that does; in this case, we call the DGA *good*. The existence of such an isomorphism is equivalent to the existence of an *augmentation* of the DGA $(\mathcal{A}, \partial)$, i.e. a graded algebra map $\varepsilon : \mathcal{A} \to \mathbb{Z}$ such that $\varepsilon(1) = 1$ and $\varepsilon \circ \partial = 0$. To see why this is true, define an isomorphism $\Phi^\varepsilon : \mathcal{A} \to \mathcal{A}$ on the generators of $\mathcal{A}$ by:

$$\Phi^\varepsilon(q_i) = q_i + \varepsilon(q_i). \tag{2.3}$$

This map conjugates $(\mathcal{A}, \partial)$ to a good DGA $(\mathcal{A}, \partial^\varepsilon)$. It is easy to check that ∂^ε may be computed from ∂ by replacing q_i with $q_i + \varepsilon(q_i)$ in the expression for the differential.

For any given augmentation ε, it is straightforward to compute the homology $H_*(Q, \partial_1^\varepsilon)$. The set $\mathcal{H}(\mathcal{A}, \partial)$ of all such "linearized" homologies taken with respect to all augmentations of $(\mathcal{A}, \partial)$ is an invariant of the stable tame isomorphism class of $(\mathcal{A}, \partial)$ [2]. It is this set of linearized homologies for which we will prove duality.

2.4. **Examples.** Before proceeding to the proof of duality, let us look at a few basic examples in $J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}$. In these examples, we use the *front projection* $\Pi_F : J^1(\mathbb{R}^n) \to J^0(\mathbb{R}^n) = \mathbb{R}^{n+1}$. In the front projection, a Reeb chord occurs when two tangent planes to $\Pi_F(L)$ at some point in $\mathbb{R}^n$ are parallel.

In terms of the front projection, the Conley-Zehnder index of a capping path γ_i is calculated as follows. Let $x_i \in N$ be the projection of the double point q_i to N. Around $q_i^\pm$, the front

FIGURE 1. Front projection of the Flying Saucer in dimension 3, on the left, and 5, on the right.

projection $\Pi_F(L)$ is the graph of the functions $h_i^{\pm}$ with domain in a neighborhood of x_i. Define the difference function for q_i to be $h_i = h_i^+ - h_i^-$. Since the tangent planes to $\Pi_F(L)$ are parallel at $q_i^{\pm}$, the difference function h_i has a critical point at x_i and, if L is admissible, the Hessian $d^2 h_i$ is nondegenerate at x_i. For the proof of the following lemma, see [5].

Lemma 2.8. *If γ_i is a generic capping path for q_i, then*

$$\nu(\gamma_i) = D(\gamma_i) - U(\gamma_i) + \mathrm{Index}_{x_i}(h_i), \tag{2.4}$$

where $D(\gamma_i)$ (resp. $U(\gamma_i)$) is the number of cusps that γ_i traverses in the downward (resp. upward) z-direction.

Example 2.9 (The flying saucer). The front projection of the simplest Legendrian submanifold of $\mathbb{R}^{2n+1}$ is shown in Figure 1. This Legendrian L is an n-sphere and $\Pi_P(L)$ has exactly one double point c. The grading on this double point is $|c| = n$. Thus, $\mathcal{A}(L)$ is the $\mathbb{Z}$-algebra generated by c. Moreover, for $n \geq 2$, one easily sees that $\partial c = 0$ for degree reasons. Thus, the differential is good and the linearized contact homology is:

$$H_k(Q(L)) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Example 2.10. Consider the two flying saucers in Figure 2. Connect them by a curve $c(s)$ parametrized by $s \in [-1, 1]$ that runs from the cusp edge of one flying saucer to the cusp edge of the other. Take a small neighborhood of c whose cross-sections are round balls whose radii decrease from $s = -1$ to $s = 0$, and increase from $s = 0$ to $s = 1$. Finally, introduce cusps along the sides of the neighborhood and join it smoothly to the two flying saucers. If n is even, the resulting Legendrian sphere L has the same classical invariants as the flying saucer. Note that this is Example 4.12 of [5].

The connecting tube can be chosen such that there are six Reeb chords $q_1, \dots, q_6$ that come from the two flying saucers and exactly one more Reeb chord q at the center of the connecting tube. The degrees of these chords are:

$$\begin{aligned} |q_1| &= |q_2| = |q_5| = n \\ |q_3| &= |q_6| = |q| = n - 1 \\ |q_4| &= 0. \end{aligned}$$

For $n > 2$, the fact that there are no chords of degree 1 implies that $\mathcal{A}(L)$ must be a good algebra. Further, it is clear that $H_0(Q(L)) \cong \mathbb{Z}$. We will calculate the remainder of the linearized cohomology using Theorem 1.1 in Section 5.2.

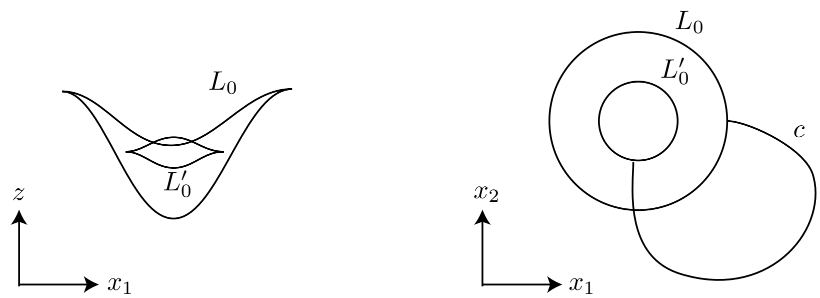

FIGURE 2. On the left hand side, the x_1z-slice of part of L_1 is shown. To see this portion in $\mathbb{R}^3$, rotate the figure about its center axis. On the right hand side we indicate the arc c connecting the two copies of L_0.

Remark 2.11. In contrast to dimension three, where DGAs with $\partial_0 = 0$ almost never arise directly from the geometry of a Legendrian knot, this is not infrequently the case in higher dimensions. In dimension three, however, the theory of the existence of augmentations has been well-studied (see [11, 12, 19, 22, 23]), while less work has been done in higher dimensions. One exception is Ng's use of augmentation ideals in his study of Knot Contact Homology [18], which is based on the technology of Legendrian contact homology.

3. Algebraic Framework

In this section we present the algebraic setup for the linearized contact homology of the two-copy $2L$ of a Legendrian submanifold $L \subset P$. The chain complex $Q(2L)$ encodes the chain complexes for the linearized contact homology, the linearized contact cohomology, and the Morse homology of a function f on L. Similar setups for the two-copy of a Legendrian submanifold were used in [6] for the proof of double-point estimates of exact Lagrangians as well as in [24] for the proof of duality for Legendrian contact homology in three dimensions.

3.1. **The Two-Copy Algebra.**

The *two-copy* $2L$ of a Legendrian submanifold $L \subset P \times \mathbb{R}$ is the Legendrian link composed of L and another copy of L, denoted $\tilde{L}$, shifted a large distance s in the z-direction. The Reeb chords of this link are degenerate: at every point of L which is not a Reeb chord endpoint, there starts a Reeb chord ending on $\tilde{L}$, while at every Reeb chord endpoint of L, there starts two such chords. This degenerate situation is similar to a Morse-Bott degeneration in Morse theory. We use the following perturbation of $\tilde{L}$ in order to get back into a generic situation. Let f be a C^1-small Morse-Smale function on $\tilde{L}$. A neighborhood of $\tilde{L} \subset P \times \mathbb{R}$ can be identified with the 1-jet space $J^1(L)$, and we perform a Legendrian isotopy pushing along the 1-jet extension of f. For simplicity, we will call also this perturbed Legendrian submanifold $\tilde{L}$. Note that the z-coordinate of a point p in the new $\tilde{L}$ differs by $f(p)$ from the corresponding point in the old $\tilde{L}$.

The generators of $\mathcal{A}(2L)$ come from two sources. First, for every double point q of $\Pi_P(L)$, there are four double points of $\Pi_P(2L)$: two copies of the original double point of $\Pi_P(L)$, one denoted q^0 in $\Pi_P(L)$ and one denoted $\tilde{q}^0$ in $\Pi_P(\tilde{L})$, and two intersections between $\Pi_P(L)$ and $\Pi_P(\tilde{L})$. Second, each critical point of f gives a double point of $\Pi_P(2L)$. Thus, we can split the generators of the two-copy DGA into four types:

Pure generators: Double points q^0 of $\Pi_P(L)$ and their nearby counterparts $\tilde{q}^0$ for $\Pi_P(\tilde{L})$,

Mixed q generators: Intersections q^1 between $\Pi_P(L)$ and $\Pi_P(\tilde{L})$ so that $(q^1)^-$ lies near $(q^0)^-$ and $(q^1)^+$ lies near $(\tilde{q}^0)^+$,

Mixed p generators: Intersections p^1 between $\Pi_P(L)$ and $\Pi_P(\tilde{L})$ so that $(p^1)^-$ lies near $(q^0)^+$ and $(p^1)^+$ lies near $(\tilde{q}^0)^-$, and

Mixed Morse generators: Double points c^1 corresponding to critical points of f.

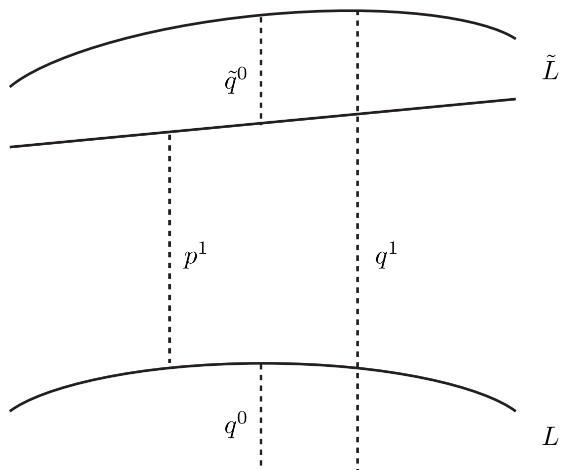

FIGURE 3. Three of the four types of generators of the two-copy algebra, as seen from a cross-section of the front projection of $2L$.

The first three types of generators are pictured schematically in Figure 3. For a small perturbation f, we have the following relationships between the lengths ℓ of each type of chord:

$$\begin{aligned} \ell(q^1) &\approx s + \ell(q^0), \\ \ell(c^1) &\approx s, \\ \ell(p^1) &\approx s - \ell(q^0). \end{aligned} \tag{3.1}$$

To define the grading on $\mathcal{A}(2L)$, we follow the relative grading of [2, 17]. First, choose a basepoint $p \in L$ away from the Reeb chords of $2L$ and let $\tilde{p}$ be its z-translate in $\tilde{L}$. Fix an identification of $T_p(L)$ and $T_{\tilde{p}}\tilde{L}$. Next, choose capping paths γ_i for the Reeb chords of L that run through the marked point p and denote by γ_i^+ and γ_i^- the paths p splits γ_i into, where γ_i^+ (γ_i^-) contains the starting point (ending point) of γ_i. Finally, choose capping paths for the double points on $\tilde{L}$ to be z-translates of these (with small deformations near the endpoints when necessary). Given suitable trivializations of TL ($T\tilde{L}$) along γ_i ($\tilde{\gamma}_i$), we can then define loops $\hat{\Gamma}_i^\pm$ of Lagrangian planes in $\mathbb{R}^{2n}$ by concatenating Γ_i^+ and Γ_i^- and closing up as before.

It is obvious that the q^0 ($\tilde{q}^0$) generators inherit their gradings from $\mathcal{A}(L)$. Using the concatenation property of the Conley-Zehnder index [21], it is easy to see that

$$|q_i^1| = |q_i^0|. \tag{3.2}$$

It is easy to see that the loops $\hat{\Gamma}_i^\pm$ for the p^1 generators are simply the reverses of the loops for the q^1 generators. Since the Morse index of the double point p^1 is n minus the index at q^1, by

Lemma 2.8 we obtain $\nu(p_i^1) = n - \nu(q_i^1)$. It follows that:

$$|p_i^1| = -|q_i^1| + n - 2. \tag{3.3}$$

For the gradings of the c^1 generators, note that the difference function between neighborhoods of $c_i^{\pm}$ is given by the sum of f and a constant function. Choosing γ^+ to be the reverse of a small perturbation of the translate of γ^-, we can use the proof of Lemma 2.8 in [5] to get:

$$|c_i^1| = \mathrm{Index}_{c_i}(f) - 1. \tag{3.4}$$

In order to distinguish the differential ∂ on $\mathcal{A}(L)$ from the differential on $\mathcal{A}(2L)$, we will denote the latter by $\widehat{\partial}$. Furthermore, provided the Morse function f is sufficiently small, the algebras $\mathcal{A}(L)$ and $\mathcal{A}(\tilde{L})$ are canonically isomorphic DGAs. We will use this identification without further comment below.

3.2. **The Linearized Complex.** Assume that the contact homology of $L \subset P \times \mathbb{R}$ is linearizable and let $\varepsilon\colon \mathcal{A}(L) \to \Lambda$ be an augmentation. Define the algebra map $\widehat{\varepsilon} : \mathcal{A}(2L) \to \Lambda$ on the generators by:

$$\widehat{\varepsilon}(x) = \begin{cases} \varepsilon(x) & x \text{ is a } q^0 \text{ or } \tilde{q}^0 \text{ generator,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then $\widehat{\varepsilon}$ is an augmentation of $(\mathcal{A}(2L), \widehat{\partial})$. To see this, first note that Lemma 2.6 and equation (3.1) imply that the differential of any q^0 or $\tilde{q}^0$ generator agrees with the original differential ∂. In particular,

$$\widehat{\varepsilon} \circ \widehat{\partial}(q^0) = \widehat{\varepsilon} \circ \widehat{\partial}(\tilde{q}^0) = 0.$$

Second, note that every term in the differential of a mixed Reeb chord must contain at least one more mixed chord. This implies:

$$\widehat{\varepsilon} \circ \widehat{\partial}(q^1) = \widehat{\varepsilon} \circ \widehat{\partial}(p^1) = \widehat{\varepsilon} \circ \widehat{\partial}(c^1) = 0,$$

and the augmentation property follows.

The techniques just used generalize to the following lemma.

Lemma 3.1. *If a holomorphic disk with boundary on $2L$ has one positive puncture and its positive puncture maps to a mixed Reeb chord then it has exactly one negative puncture mapping to a mixed Reeb chord. If a holomorphic disk with boundary on $2L$ has one positive puncture and its positive puncture maps to a pure Reeb chord then all its negative punctures map to pure Reeb chords as well.*

Proof. Immediate from Lemma 2.6. □

Let us continue to explore the consequences of Lemma 2.6. Consider the chain complex $Q(2L)$ of the linearized contact homology of $2L$ with the differential $\widehat{\partial}_1^{\widehat{\varepsilon}}$. We decompose $Q(2L)$ into four summands

$$Q(2L) = Q^1 \oplus C^1 \oplus P^1 \oplus Q^0;$$

where Q^1 is generated by mixed q generators, C^1 by mixed Morse generators, P^1 by mixed p-generators, and Q^0 by pure generators. Some components of the matrix of $\widehat{\partial}_1^{\widehat{\varepsilon}}$ corresponding to this decomposition will turn the summands into chain complexes in their own right, while other components will become maps between these complexes. In the next section, we will reinterpret the complexes corresponding to the summands as the original linearized complex, its cochain complex, and the Morse-Witten complex of L with respect to the Morse function f. The maps between these complexes will give the exact sequence of Theorem 1.1.

Remark 3.2. For simpler notation below, we will suppress the augmentations from the notation for the differential writing simply $\widehat{\partial}_1$ and ∂_1 instead of $\widehat{\partial}_1^{\widehat{\varepsilon}}$ and ∂_1^{ε}, respectively.

We next consider the decomposition of the differential corresponding to the direct sum decomposition above. Lemma 3.1 implies that $Q^0 \subset Q(2L)$ is a subcomplex and that the component of $\widehat{\partial}_1$ which maps $Q^1 \oplus C^1 \oplus P^1$ to Q^0 vanishes. Thus $Q^1 \oplus C^1 \oplus P^1$ is a subcomplex as well. It is the subcomplex $Q^1 \oplus C^1 \oplus P^1$ which carries the important information for the the exact sequence. We therefore restrict attention to this subcomplex and leave Q^0 aside.

Lemma 2.6 and Equation (3.1) imply that the linearized differential takes on the following lower triangular form with respect to the splitting $Q^1 \oplus C^1 \oplus P^1$:

$$\widehat{\partial}_1 = \begin{bmatrix} \widehat{\partial}_q & 0 & 0 \\ \rho & -\widehat{\partial}_c & 0 \\ \eta & \sigma & \widehat{\partial}_p \end{bmatrix}. \tag{3.6}$$

Since $\widehat{\partial}_1^2 = 0$, the diagonal entries in the matrix of $\widehat{\partial}_1^2$ give

$$\widehat{\partial}_q^2 = 0, \qquad \widehat{\partial}_c^2 = 0, \quad \text{and} \quad \widehat{\partial}_p^2 = 0. \tag{3.7}$$

The subdiagonal entries give

$$\rho\widehat{\partial}_q - \widehat{\partial}_c\rho = 0 \quad \text{and} \quad \widehat{\partial}_p\sigma - \sigma\widehat{\partial}_c = 0. \tag{3.8}$$

The last interesting entry in the matrix for $\widehat{\partial}_1^2$ is

$$\eta\widehat{\partial}_q + \widehat{\partial}_p\eta + \sigma\rho = 0. \tag{3.9}$$

Proposition 3.3. *With notation as above, we have:*

1. *$(Q^1, \widehat{\partial}_q)$, $(C^1, \widehat{\partial}_c)$, and $(P^1, \widehat{\partial}_p)$ are all chain complexes.*
2. *The maps ρ and σ are chain maps of degree -1.*
3. *$\sigma_*\rho_* = 0$, where η acts as a chain homotopy between $\sigma\rho$ and the zero map.*

Proof. This follows from (3.7), (3.8), and (3.9). □

Combining Q^1 and C^1 into $QC^1 = Q^1 \oplus C^1$ gives another useful view of $\widehat{\partial}_1$. Let $\widehat{\partial}_{qc} = -\widehat{\partial}_q - \rho + \widehat{\partial}_c$ and let $H = \eta + \sigma$. With respect to the splitting $QC^1 \oplus P^1$, then, $\widehat{\partial}_1$ takes the following form:

$$\widehat{\partial}_1 = \begin{bmatrix} -\widehat{\partial}_{qc} & 0 \\ H & \widehat{\partial}_p \end{bmatrix}. \tag{3.10}$$

In parallel to Proposition 3.3, we obtain the following proposition.

Proposition 3.4. *With notation as above we have:*

1. *$(QC^1, \widehat{\partial}_{qc})$ is a chain complex.*
2. *$H : (QC^1, \widehat{\partial}_{qc}) \to (P^1, \widehat{\partial}_p)$ is a chain map of degree -1.*

Another way to look at the Proposition 3.4 is as follows:

Corollary 3.5. *The complex* $(QC^1 \oplus P^1, -\widehat{\partial}_{qc} + H + \widehat{\partial}_p)$ *is the mapping cone of* H*. Further,* QC^1 *itself is the mapping cone of* ρ*.*

As we shall see, the mapping cone of H is acyclic and hence H is an isomorphism. In fact, we will use this isomorphism to get the linearized contact cohomology into the exact sequence.

3.3. **Identifying the Complexes in** $Q(2L)$**.** In this section, we identify the subcomplexes of $Q(2L)$ discussed in the previous section with the original linearized chain complex of L, its cochain complex, and the Morse-Witten complex of L. The proofs of these identifications rest on an analytic theorem which describes all rigid holomorphic disks for a sufficiently small perturbation function f. We will describe that theorem here, but defer the detailed analytic treatment and its proof to Section 6.

3.3.1. *Generalized Disks.* In order to state the analytic theorem for disks of the two-copy, we first introduce terminology for objects built from holomorphic disks with boundary on L and flow lines of the Morse function f on L.

We define a *lifted disk* to be a holomorphic disk u in $\mathcal{M}_A(a; b_1, \dots, b_k)$ or in $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k)$ with boundary on L together with an assignment of a component L or $\tilde{L}$ of $2L$ to the portion of ∂D_{m+1} counter-clockwise of the puncture x_0 and clockwise of the puncture x_j for some $j > 0$ and an assignment of the other component $\tilde{L}$ or L, respectively, to the complementary portion of ∂D_{m+1}. If $u \in \mathcal{M}_A(a_1, a_2; b_1, \dots, b_k)$, then we require both x_0 and x_j to be positive punctures of u.

We define a *generalized disk* to be a pair (u, γ) consisting of a holomorphic disk $u \in \mathcal{M} = \mathcal{M}_A(a; b_1, \dots, b_k)$ with boundary on L and a negative gradient flow line γ of the Morse function f beginning or ending at the boundary of u. We call the point on the boundary of u where the flow line γ begins or ends the *junction point* of (u, γ). If the flow line starts at the junction point then (u, γ) has a *negative Morse puncture*; otherwise it has a *positive Morse puncture.*

We define the *formal dimension* $\dim((u, \gamma))$ of a generalized disk (u, γ) as follows.

$$\dim((u,\gamma)) = \begin{cases} \dim \mathcal{M} + 1 + (I(p) - n), & p \text{ a positive Morse puncture,} \\ \dim \mathcal{M} + 1 - I(p), & p \text{ a negative Morse puncture,} \end{cases} \tag{3.11}$$

where $I(p)$ is the index of the critical point p at the end of γ which is not the junction point.

If f is a Morse function and g is a Riemannian metric for which a suitable genericity condition holds (see Lemma 6.2, below), then we say that a generalized disk of formal dimension 0 is *rigid*. Under this genericity condition, the set of rigid generalized disks is finite. Further, rigid generalized disks correspond to transverse intersections of a stable/unstable manifold of f with the evaluation map $\mathrm{ev}\colon \overline{\mathcal{M}}^* \to L$. Here, $\overline{\mathcal{M}}^*$ denotes the compactified moduli space of holomorphic disks with one positive puncture and one marked point on the boundary, and ev denotes the evaluation map at the marked point.

We define a *lifted generalized disk* as follows. Let (u, γ) be a generalized disk. The Morse flow line γ in L and its orientation reversed z-translate $\tilde{\gamma}$ in $\tilde{L}$ are two oriented curves, one which is oriented away from the junction point and one which is oriented toward it. The positive puncture and the junction point subdivide the boundary of the domain of u into two parts. One of these parts is oriented toward the junction point and the other one away from it. We assign the component L to the part oriented toward (away from) the junction point if the curve

γ is oriented away from (toward) the junction point, and we assign the component $\tilde{L}$ to the other part.

We note that lifted disks, lifted generalized disks, and negative flow lines of f give rise to continuous maps from the boundary of a punctured disk to $2L$ in a natural way. We call such maps *lifted boundary maps.*

Theorem 3.6. *For a sufficiently small perturbing Morse function f, there are the following bijective correspondences between sets of holomorphic disks with boundary on $2L$ and sets of disks, lifted disks, and lifted generalized disks. The correspondences are such that the continuous lift of the boundary of a holomorphic disk is contained in a small neighborhood of the lifted boundary map of the object to which it corresponds, and vice versa.*

(1) *The set of rigid disks with boundary on $2L$ without mixed punctures corresponds to two copies of the set rigid disks with boundary on L, one with boundary on L and the other with boundary on $\tilde{L}$.*
(2) *The set of rigid disks with boundary on $2L$ with mixed punctures but without mixed Morse punctures corresponds to the set of rigid lifted disks.*
(3) *The set of rigid disks with boundary on $2L$ with exactly one Morse puncture corresponds to the set of rigid lifted generalized disks.*
(4) *The set of rigid disks with boundary on $2L$ with two mixed Morse punctures corresponds to the the set of rigid negative gradient flow lines of f.*

3.3.2. *Chain Complexes.* Theorem 3.6 allows us to prove the following correspondences between chain complexes.

Proposition 3.7. *With notation as established in Section 3.2 we have:*

(1) $(Q^1, \widehat{\partial}_q)$ *is isomorphic to the original linearized chain complex* $(Q(L), \partial_1)$ *and*
(2) $(C^1, \widehat{\partial}_c)$ *is isomorphic to the Morse-Witten complex* $CM_*(L; f)$ *with respect to the perturbing Morse function f, with degrees shifted so that* $C^1_k \cong CM_{k+1}(L; f)$.

Proof. For both parts of the proposition, there is an obivous bijective correspondence between the generators of the underlying graded groups (up to the indicated shift in the second part). We will now show that the disks contributing to the differentials also correspond.

For $(Q^1, \widehat{\partial}_q)$, a disk u that contributes q^1_k to $\widehat{\partial}_q q^1_j$ has one positive mixed q puncture q^1_j, one negative mixed q puncture q^1_k, and possibly other augmented negative pure q punctures. By Theorem 3.6, this corresponds to a lifted disk with a positive puncture at q_i, a negative puncture at q_j, and possibly other augmented negative punctures. The rigid disk underlying the lifted disk contributes q_k to $\partial_1 q_j$ in $Q(L)$, as required. Conversely, exactly one of the two lifts of such a rigid disk yields a disk with mixed q punctures at q_j and q_k and appropriate signs at the punctures.

For $(C^1, \widehat{\partial}_c)$, a disk u that contributes c^1_k to $\widehat{\partial}_c c^1_j$ has a positive mixed Morse puncture at c^1_k and a negative mixed Morse puncture at c^1_k. Lemma 3.1 and Theorem 3.6 imply that this disk has only these two mixed Morse punctures and corresponds to a flowline from the critical point c_j to c_k, as in the Morse-Witten differential. The converse is clear. $\square$

We next relate $(P^1, \widehat{\partial}_p)$ to the cochain complex of $(Q^1, \widehat{\partial}_q)$. Define a pairing $\langle , \rangle_q$ on the generators of $P^1_{n-k-2} \otimes Q^1_k$ by

$$\langle p^1_i, q^1_j \rangle_q = \delta_{ij}. \tag{3.12}$$

Proposition 3.8. *The complex $(P^1, \widehat{\partial}_p)$ is isomorphic to the cochain complex of $(Q^1, \widehat{\partial}_q)$ with respect to the pairing $\langle , \rangle_q$.*

Proof. We need to show that

$$\langle \widehat{\partial}_p p_j, q_k \rangle_q = \langle p_j, \widehat{\partial}_q q_k \rangle_q. \tag{3.13}$$

A disk contributing to the left-hand side of (3.13) has a mixed positive puncture at p_j, a mixed negative puncture at p_k, and possibly other pure augmented punctures. By Theorem 3.6, such a disk corresponds to a unique rigid lifted disk. Reversing assignment of components of $2L$ in the lifted rigid disk, we obtain a lifted disk which contributes to the right-hand side. Similarly, to each disk contributing to the right hand side, we find a unique disk contributing to the left hand side. □

3.3.3. *The Maps ρ_* and σ_*.* Poincaré duality for $H_*(L)$ can be realized on the chain level in the Morse-Witten complex as follows (see [15] or [25], for instance): there is an obvious correspondence

$$\begin{aligned} \Delta : CM_k(L; -f) &\to CM^{n-k}(L; f) \\ x &\mapsto \langle x, \cdot \rangle. \end{aligned}$$

It induces an isomorphism $\Delta_* : HM_k(L; -f) \to HM^{n-k}(L; f)$ on homology which, when combined with the continuation map $h : CM_k(L; f) \to CM_k(L; -f)$, yields the Poincaré duality isomorphism $\Delta_* h_*$. We can thus define a Poincaré pairing on $HM_k(L; f) \otimes HM_{n-k}(L; f)$ by the following pairing at the chain level:

$$\langle x, y \rangle_c = (\Delta \circ h(x))(y). \tag{3.14}$$

Using this interpretation of Poincaré duality, we get the following relationship between the maps ρ and σ.

Proposition 3.9. *The maps ρ_* and σ_* are adjoints in the following sense:*

$$\langle x, \rho_* q \rangle_c = \langle \sigma_* x, q \rangle_q.$$

Proposition 3.9 is proved at the end of this section. The proof uses the following lemma.

Lemma 3.10. *The maps ρ_* and σ_* are independent of the perturbation function f. That is, if f_0 and f_1 are two generic perturbations and $h_* : HM_*(L; f_0) \to HM(L; f_1)$ is the continuation map, then:*

$$\rho^1_* = h_* \rho^0_* \quad \text{and} \quad \sigma^1_* h_* = \sigma^0_*,$$

where ρ^i and σ^i are the maps corresponding to the perturbation function f_i.

Lemma 3.10 uses some results from Morse theory which we present before giving its proof. Similar Morse theoretic questions were considered in [13, 26]. Our approach here is a modification of that in [26].

Denote the singular chain complex of L by $C_*(L)$ and the corresponding homology by $H_*(L)$. Define a map $\Phi^f : CM_*(L; f) \to C_*(L)$ as follows. If x is a critical point of f, then $\Phi^f(x)$ is the singular chain carried by the closure of the unstable manifold $\overline{W^{\mathrm{u}}(x)}$.

Lemma 3.11. *The map Φ^f is a chain map that is compatible with the continuation map $h_* : HM_*(L; f_0) \to HM_*(L; f_1)$ in the sense that $\Phi^1_* h_* = \Phi^0_*$.*

Proof. By Lemma 4.2 of [26], the boundary of $\overline{W^{\mathrm{u}}(x)}$ consists of products of rigid flow lines from x to y with $\overline{W^{\mathrm{u}}(y)}$. The first part of the lemma follows. The second part follows by analogy with the proof of Lemma 4.8 of [26], using smooth chains instead of pseudo-cycles. □

Remark 3.12. For technical reasons, we assume from here on that $C_*(L)$ is generated by a set of smooth chains that is the union of both the chains in the image of Φ^f and the chains in the image of the evaluation maps of all moduli spaces of holomorphic disks with boundary on L, one marked point, and at most two positive punctures.[3] Further, we assume that the smooth chains are transverse to the closures of the stable manifolds $\overline{W^{\mathrm{s}}(x)}$. That this chain complex exists follows from arguments similar to those in [13, §19].

The inverse map $\Psi^f : C_*(L) \to CM_*(L; f)$ is defined using intersections between chains and the stable manifolds of the critical points as follows. Let:

$$\Psi^f(\chi) = \sum_x (\chi \bullet W^{\mathrm{s}}(x))x, \tag{3.15}$$

where χ is a chain, $\bullet$ denotes the intersection product on chains, and where the sum ranges over all critical points x of f. Note that the intersection product on chains is well-defined by our assumptions on the chains generating $C_*(L)$. Using the same proof as for Lemma 3.11, though this time based on ideas from Section 4.2 of [26], we obtain the following.

Lemma 3.13. *The map Ψ^f is a chain map compatible with the continuation map.*

Since it is clear that $\Psi^f \circ \Phi^f$ is the identity on $CM_*(L; f)$, and since the Morse and singular homologies are isomorphic, it follows that Φ^f_* and Ψ^f_* are mutual inverses.

We can now prove that ρ_* and σ_* are independent of the perturbation.

Proof of Lemma 3.10. Let $\mathcal{M}(q)$ be the moduli space of holomorphic disks with positive puncture at q and possibly negative punctures at augmented crossings. Let K_q be the image in $C_*(L)$ of $\overline{\mathcal{M}}(q)$ under the evaluation map. Theorem 3.6 implies that ρ may be defined using lifted generalized disks, so we obtain:

$$\rho^i(q) = \Psi^i(K_q).$$

If $\alpha \in H_*(Q^1, \widehat{\partial}_q)$, then:

$$\begin{aligned}\rho^0_*\alpha &= \Psi^0_*(K_\alpha)\\ &= h_*\Psi^1_*(K_\alpha)\\ &= h_*\rho^1_*\alpha.\end{aligned}$$

The proof for σ_* is similar. □

[3] See Section 2.2.3, above, and [4, §7.7].

We are now ready to prove the original proposition.

Proof of Proposition 3.9. Denote by ρ^f and σ^{-f} the maps induced by the perturbations $\pm f$. By Lemma 3.10, it suffices to prove that, on the chain level,

$$\langle \tilde{h}(y), \rho^f q\rangle_c = \langle \sigma^{-f} y, q\rangle_q \tag{3.16}$$

for any given $y \in CM_k(L; -f)$, where $\tilde{h}$ is the continuation map from $CM_k(L;-f) \to CM_k(L; f)$.

Theorem 3.6 says that a contribution to the left-hand side is given by a lifted generalized disk with a mixed positive puncture at q and a negative Morse puncture at y (and possibly other negative augmented pure punctures), where the flowline γ in the generalized disk is a negative gradient flowline for f. By reversing the lifting, we obtain a lifted generalized disk with a mixed negative puncture at the p corresponding to q and a positive Morse puncture at y. This time, however, the flowline γ points in the opposite direction, and hence is a negative gradient flowline for $-f$. Thus, this disk contributes to $\sigma^{-f}y$, and hence to the right hand side. Therefore, there is a bijective correspondence between the disks on the left- and right-hand sides of Equation (3.16). □

4. The duality sequence

In this section we prove Theorem 1.1 in two steps. First, we establish an isomorphism between QC^1 and P^1 (using notation as in Section 3.2) for $L \subset P \times \mathbb{R}$. Second, we collect the information from all identifications in Section 3.3.

4.1. **The Duality Isomorphism.**

Assume that $L \subset J^1(M \times \mathbb{R})$ is a Legendrian submanifold. The first step will be to show that H_* is an isomorphism between QC^1 and P^1.

Proposition 4.1. *The map $H_* : H_*(QC^1) \to H_*(P^1)$ is a degree -1 isomorphism.*

Proof. By Corollary 3.5, it suffices to prove that the complex

$$(QC^1 \oplus P^1, -\widehat{\partial}_{qc} + H + \widehat{\partial}_p)$$

is acyclic. Since $(Q(2L), \widehat{\partial}_1)$ splits as the direct sum of two subcomplexes Q^0 and $QC^1 \oplus P^1$, it is clear that

$$H_*(Q(2L)) \cong H_*(Q^0) \oplus H_*(QC^1 \oplus P^1). \tag{4.1}$$

In order to prove that $QC^1 \oplus P^1$ is acyclic, we show that the linearized contact homology of $2L$ comes from Q^0 only, as follows.

Modify the two-copy by a Legendrian isotopy that leaves the bottom component fixed and horizontally displace the top component by the lift of a Hamiltonian isotopy in P of the projection of L off of itself. For any augmentation, the linearized contact homology of this shifted two-copy is clearly isomorphic to the homology of Q^0 since all of the other components of $Q(2L)$ are trivial. Since the set of linearized contact homologies is invariant under Legendrian isotopy, this implies that, for any augmentation, the linearized contact homology of $2L$ comes from Q^0 only, as desired. □

Remark 4.2. It should not be surprising that H turns out to be an isomorphism essential to the proof of duality. In finite-dimensional Morse theory, rigid gradient flow trees with two positive ends and one vertex define the Poincaré duality isomorphism (see [1], for example). Using Theorem 3.6, it is straightforward to see that the disks that define H — or at least η —

correspond to disks with two positive punctures and no non-augmented negative punctures in L. Thus, the disks defining H are combinatorially similar to the gradient flow trees that give Poincaré duality.

4.2. **The Proof of Theorem 1.1.** The complex QC^1 is the mapping cone of ρ. The corresponding long exact sequence is:

$$\cdots \to H_k(C^1) \xrightarrow{i_*} H_k(QC^1) \to H_k(Q^1) \xrightarrow{\rho_*} \cdots, \tag{4.2}$$

where $i : C^1 \to QC^1$ is the natural inclusion. By Proposition 3.7(2), we have $H_k(C^1) \cong H_{k+1}(L)$. By Propositions 3.8 and 4.1, the middle term $H_k(QC^1)$ is isomorphic to $H^{n-k-1}(Q^1)$. Further, we have $H_* i_* = \sigma_*$. Inserting these facts into the exact sequence above yields:

$$\cdots \to H_{k+1}(L) \xrightarrow{\sigma_*} H^{n-k-1}(Q^1) \to H_k(Q^1) \xrightarrow{\rho_*} \cdots. \tag{4.3}$$

Proposition 3.7(1) now finishes the proof of the first part of the duality theorem. The second part of the duality theorem follows directly from Proposition 3.9.

5. Applications and Examples

5.1. **Proof of Theorem 1.2.** Let $L \subset P \times \mathbb{R}$ be a Legendrian submanifold with linearizable contact homology over a field Λ. To set notation, let:

$$\begin{aligned} b_k &= \dim H_k(L), \\ r_k &= \dim \operatorname{Im} \rho_* \subset H_k(L), \\ s_k &= \dim \operatorname{Im} \sigma_* \subset H^{n-k}(Q(L)). \end{aligned}$$

Note that r_k is at most the dimension of $H_k(Q(L))$, which in turn bounds below c_k, the number of Reeb chords of grading k. By the second part of Theorem 1.1, we obtain $s_k = r_{n-k}$, so:

$$b_k = r_k + s_k = r_k + r_{n-k} \leq c_k + c_{n-k}.$$

5.2. **Basic Examples.** We study the relation between the linearized contact homology and the Morse homology implied by Theorem 1.1 in several simple examples.

Example 5.1 (The Flying Saucer, revisited). Recall the flying saucer of Example 2.9, whose linearized homology (and cohomology) is $\mathbb{Z}$ in degree n and trivial otherwise. The relevant part of the the duality exact sequence in Theorem 1.1 is:

$$\cdots \to H^{-1}(Q(L)) \to H_n(Q(L)) \xrightarrow{\rho_*} H_n(L) \to H^0(Q(L)) \to \cdots.$$

This reduces to $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$. Thus, we see that ρ_* gives an isomorphism between the degree n linearized homology and the group generated by the fundamental class $[L]$ of $L \cong S^n$.

Since $H_0(Q(L)) = 0$, and hence the image of ρ_* in $H_0(L)$ is trivial, it is clear that the Poincaré dual of $[L]$ evaluates to 0 on the image of ρ_* in $H_0(L)$, as guaranteed by the Theorem 1.1. In fact, as we can see from the following part of the exact sequence, $H_0(L)$ is isomorphic to the dual of $H_n(Q(L))$:

$$\cdots \to H_0(Q(L)) \to H_0(L) \xrightarrow{\sigma_*} H^n(Q(L)) \to H_{-1}(Q(L)) \to \cdots.$$

In other words, we see that the Morse homology of L gets split between the linearized homology and the linearized cohomology.

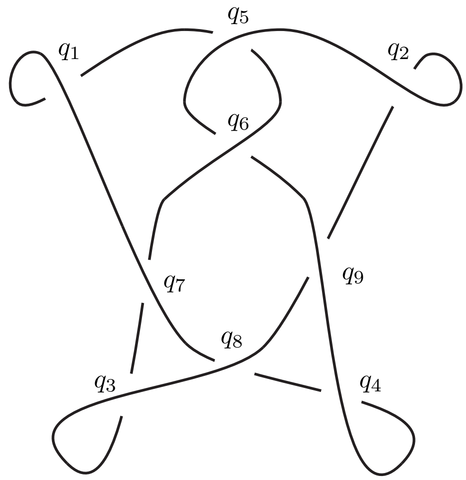

FIGURE 4. The knot L.

Example 5.2 (Chekanov's Example in $\mathbb{R}^3$, reinterpreted). Consider the Legendrian knot L in $\mathbb{R}^3$ whose Lagrangian projection is shown if Figure 4. Working over $\mathbb{Z}$, the algebra for L is $\mathcal{A}(L) = \mathbb{Z}\langle q_1, \dots q_9\rangle$ with $|q_i| = 1$ for $i = 1, \dots, 4$, $|q_5| = 2 = -|q_6|$, and the other q_i having grading 0. We have

$$\partial q_i = \begin{cases} 1 + q_7 - q_7 q_6 q_5 & i = 1, \\ 1 - q_9 + q_5 q_6 q_9 & i = 2, \\ 1 + q_8 q_7 & i = 3, \\ 1 + q_9 q_8 & i = 4, \\ 0 & i \geq 5. \end{cases}$$

This differential is not augmented, but there is a unique augmentation ε that sends q_7, q_8, and q_9 to 1. The linearized differential is:

$$\partial_1^\varepsilon q_i = \begin{cases} q_7 & i = 1, \\ -q_9 & i = 2, \\ q_8 + q_7 & i = 3, \\ q_9 + q_8 & i = 4, \\ 0 & i \geq 5. \end{cases} \tag{5.1}$$

Thus, we have the following ranks for the linearized homology:

k	-2	-1	$\mathbf{0}$	1	2
$\dim H_k(Q(L))$	1	0	0	1	1

This computation agrees with the predictions of the duality theorem in [24]: off of a class in degree 1, the linearized homology is symmetric about degree 0.

The first interesting part of the duality exact sequence is:

$$\cdots \to H_{-1}(L) \to H^2(Q(L)) \to H_{-2}(Q(L)) \to H_{-2}(L) \to \cdots .$$

This sequence reduces to $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ and shows that $H^2(Q(L))$ and $H_{-2}(Q(L))$ are isomorphic; this is the "duality" in Theorem 1.1.

The next interesting parts of the exact sequence are:

$$\cdots \to H^{-1}(Q(L)) \to H_1(Q(L)) \to H_1(L) \to H^0(Q(L)) \to \cdots .$$

and

$$\cdots \to H_0(Q(L)) \to H_0(L) \to H^1(Q(L)) \to H_{-1}(Q(L)) \to \cdots .$$

As above, these sequences both reduce to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$. As in the flying saucer example, we see that the homology of $L \cong S^1$ is split between the linearized contact homology (in this case, $H_1(L) \cong H_1(Q(L))$) and cohomology (in this case, $H_0(L) \cong H^1(Q(L))$) in Poincaré dual pairs.

Example 5.3 (Front Spinning). Another source of examples in which the manifold classes follow an interesting pattern is the front spinning construction from [5]. Given a Legendrian submanifold L in $\mathbb{R}^{2n+1}$, we construct the *suspension* $\Sigma(L)$ of L by "spinning the front of L". More specifically, suppose $\phi\colon L \to \mathbb{R}^{2n+1}$ is a parameterization of L, and for $p \in L$, we write $\phi(p) = (x_1(p), y_1(p), \ldots, x_n(p), y_n(p), z(p))$. The front projection $\Pi_F(L)$ of L is parameterized by $\Pi_F \circ \phi(p) = (x_1(p), \ldots, x_n(p), z(p))$. We may assume that L has been translated so that the x_1 coordinates of all points in $\Pi_F(L)$ are positive. If we embed $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ via $(x_1, \ldots, x_n, z) \mapsto (x_0 = 0, x_1, \ldots x_n, z)$, then $\Pi_F(\Sigma L)$ is obtained by revolving $\Pi_F(L) \subset \mathbb{R}^{n+1}$ around the subspace $\{x_0 = x_1 = 0\}$ as in Figure 5. That is, we can parameterize $\Pi_F(\Sigma L)$ by

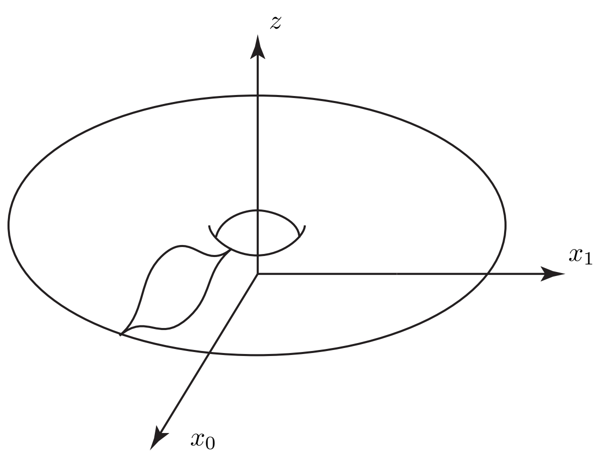

FIGURE 5. The front of ΣL.

$(x_1(p)\sin\theta, x_1(p)\cos\theta, x_2(p), \ldots, x_n(p))$, for $\theta \in S^1$. Thus, $\Pi_F(\Sigma L)$ is the front for a Legendrian embedding $L \times S^1 \to \mathbb{R}^{2n+3}$. We denote the corresponding Legendrian submanifold by ΣL.

We can derive the following facts about the contact homology DGA over $\mathbb{Z}_2$ of ΣL from Proposition 4.17 of [5]: if $\{q_1, \ldots, q_n\}$ are the generators of $\mathcal{A}(L)$, then $\mathcal{A}(\Sigma L)$ is stable tame isomorphic to an algebra generated by the set

$$\left\{q_1[\alpha], \ldots, q_n[\alpha], \hat{q}_1[\beta], \ldots \hat{q}_n[\beta]\right\}_{\alpha=0,2,\beta=1,3}.$$

Let $\Delta_\alpha : \mathcal{A}(L) \to \mathcal{A}(\Sigma L)$ take q_i to $q_i[\alpha]$, and similarly for Δ_β. The gradings of the new generators are given by $|q_i[\alpha]| = |q_i|$ and $|\hat{q}_i[\beta]| = |q_i| + 1$. Using the form of the differential in [5], we can see that if ε is an augmentation of $(\mathcal{A}(L), \partial^L)$, then the following formula defines an augmentation of $(\mathcal{A}(\Sigma L), \partial^\Sigma)$:

$$\varepsilon_\Sigma(q_i[\alpha]) = \varepsilon(q_i),$$
$$\varepsilon_\Sigma(\hat{q}_i[\beta]) = 0.$$

We will use this form of augmentation from here on, though there might be others. Finally, the linearized differential takes the following form, where the augmentations have been supressed from the notation:

$$\partial_1^\Sigma q_i[\alpha] = \Delta_\alpha(\partial_1^L q_i),$$
$$\partial_1^\Sigma \hat{q}_i[\beta] = q_i[0] + q_i[2] + \Delta_\beta(\partial_1^L q_i).$$

Claim 5.4. *The linearized homology of ΣL with respect to the augmentation ε_Σ may be computed using a Künneth-like formula:*

$$H_*(Q(\Sigma L)) \simeq H_*(Q(L)) \otimes H_*(S^1).$$

Proof. To prove the claim, we change basis by replacing $\hat{q}_i[3]$ by $\hat{q}_i[1] + \hat{q}_i[3]$. The subspace $Q[1+3]$ spanned by these new basis elements forms a subcomplex that is isomorphic to the original $(Q(L), \partial_1^L)$, but with gradings shifted up by 1. Similarly, the subspaces $Q[0]$ spanned by the $q_i[0]$ and $Q[2]$ spanned by the $q_i[2]$ also form subcomplexes isomorphic to $(Q(L), \partial_1^L)$. Finally, the restriction of the differential to the subspace $Q[1]$ yields no new cycles. Further, if z is a cycle in the original chain complex, then the differential of $\Delta_1(z)$ identifies the cycles $z[0]$ and $z[2]$ in homology. Thus, the linearized homology of ΣL comes from $Q[1+3]$ and $Q[0]$, each of which are copies of the original chain complex $Q(L)$, with degrees in $Q[1+3]$ shifted up by one. □

If we apply the spinning construction k times to the n-dimensional "flying saucer" of Example 5.1, the claim above shows that the linearized homology corresponds to the homology of T^k with degrees shifted up by n. This is half of the homology of $S^n \times T^k$, and hence the linearized homology consists entirely of manifold classes.

For a more interesting example, apply the spinning construction k times to the Legendrian knot L in Example 5.2 to obtain a Legendrian $T^{k+1} \subset \mathbb{R}^{2k+3}$. Setting $k = 2$ and using the calculation of the linearized homology in Example 5.2 and Claim 5.4, we easily obtain the following dimensions:

k	-2	-1	0	$\mathbf{1}$	2	3	4
$\dim H_k(Q(\Sigma^2 L))$	1	2	1	1	3	3	1

The duality theorem implies that the non-manifold classes are symmetric about $\frac{3-1}{2} = 1$, and we can see that every non-manifold class in $H_k(Q(L))$ has been replaced by $H_*(T^3)$, shifted by k. The remaining classes are the manifold classes, corresponding to $H_*(T^3)$ shifted up by one degree.

5.3. **Finding Manifold Classes.** As indicated by the examples above, it is sometimes possible to use symmetry arguments to predict the degree in which manifold classes appear. In particular, working over $\mathbb{Z}_2$, we can generalize the fact from [24] that for a Legendrian *knot*, there is always a manifold class in degree 1 as follows.

Theorem 5.5. *Let $L \subset P \times \mathbb{R}$ be a closed, connected, horizontally displacable Legendrian submanifold. If the contact homology DGA of L is good, then ρ_* is trivial in degree 0 and onto in degree n. That is, the linearized homology of L (with respect to any augmentation) always has a $\mathbb{Z}_2$ factor in degree n corresponding to the fundamental class of L.*

Proof. By the second part of Theorem 1.1, it suffices to prove that ρ_* is trivial in degree 0. To compute ρ in degree zero, we need to find lifted generalized disks (u, γ) in L with one negative Morse puncture and a positive mixed puncture at q with $|q| = 0$. The fact that this generalized disk is rigid implies, via Equation (3.11), that u belongs to a moduli space of dimension -1. That is, u must be a constant map to the double point q with one positive and one negative puncture. Further, q must be augmented, or else the generalized disk will not contribute to ρ.

There are two possibilities for the flowline γ. There are two sheets of $\Pi_P(L)$ incident to q, and the flowline γ can start on either of these two sheets. In a suitably generic setup, the self-intersection points of $\Pi_P(L)$ are disjoint from the stable manifolds of the critical points of f with index greater than 0, so the two flow lines starting at q both descend to a minimum of f. It follows that $\rho(q)$ is either zero or the sum of two minima of f. Thus, given a homology class $\alpha \in H_0(Q^1)$, its image under ρ_* is the sum of an even number of minima. As L is connected, these minima are all homologous, so $\rho_*\alpha = 0$. □

Remark 5.6. Provided certain orientation conventions are employed, Theorem 5.5 hold for more general coefficients in the case that L is spin. More precisely, the two disks corresponding to the flow lines in the last argument of the proof should cancel with signs. The ordered punctures of these two disks are of the forms (q^1, c^1, q^0) and $(q^1, \tilde{q}^0, c^1)$, respectively. Choosing capping operators as in [6], the capping operator of $\tilde{q}^0$ is identical to that of q^0 and both have index 1. The capping operator of c^1 has index 0 if $n = \dim(L)$ is odd and index 1 if n is even. Thus, the disks cancel also with signs if n is even. If n is odd, they cancel with signs provided we redefine the augmentation $\hat{\epsilon}$ on the $\tilde{q}^0$-chords by declaring that $\widehat{\varepsilon}(\tilde{q}^0) = -\varepsilon(q)$ instead. Here q is the Reeb chord of L which corresponds to the Reeb chord $\tilde{q}^0$ of $\tilde{L}$, and ε is the augmentation on $\mathcal{A}(L)$.

Corollary 5.7. *Let L be a Legendrian n-sphere embedded in $P \times \mathbb{R}$. If the contact homology DGA of L is good, then the linearized homology of L (with respect to any augmentation) always has a $\mathbb{Z}_2$ factor in degree n and, off of this factor, the remaining homology is symmetric about $\frac{n-1}{2}$. Said another way, if $P(t)$ is the Poincaré-Chekanov polynomial of the linearlized homology, then:*

$$P(t) - t^{n-1}P(t^{-1}) = (t^n - t^{-1}).$$

Theorem 5.5 and Corollary 5.7 can greatly ease computations, as can be seen in the next example; see also [14].

Example 5.8 (Example 2.10, revisited). Recall Example 2.10, in which two flying saucers are attached by a tube. For $n > 2$, the algebra is good for degree reasons, with three generators in degree n, three in degree $n-1$, and one is degree 0. The fact that the generator of degree 0 is isolated shows that $\dim H_0(Q(L)) = 1$, and Corollary 5.7 immediately implies that

$$P(t) = 1 + t^{n-1} + t^n.$$

In particular, the duality theorem forces nonzero differentials between degrees n and $n-1$.

Example 5.9 (A Non-Spun Torus). The previous example may be combined with Claim 5.4 to produce an example of a Legendrian $S^1 \times S^n$ that is not spun from a Legendrian S^n for $n > 1$. Let L be the Legendrian $n+1$-sphere constructed in the previous example, and let L' be standard n-dimensional flying saucer. Taking the connect sum $L\#\Sigma L'$ as in [5] yields a Legendrian submanifold with four generators in degree $n+1$, five generators in degree n, and one in degree 0. As before, the fact that the generator of degree 0 is isolated shows that $\dim H_0(Q(L)) = 1$, and Corollary 5.7 immediately implies that

$$P(t) = 1 + t^n + t^{n+1}.$$

This homology, however, is not of the form $H_*(Q(L'')) \otimes H_*(S^1)$ near degree 0, so $L\#\Sigma L'$ cannot be a spun submanifold.

The final example shows that Theorem 5.5 is the best that we can hope for in terms of pinning down the degrees of the manifold classes.

Example 5.10 (Superspun Products of Spheres). Construct a generic front diagram in $\mathbb{R}^{p+k+1}$ as follows: let $S^p \times S^k \hookrightarrow \mathbb{R}^{p+k+1}$ be the standard embedding, with $S^p \times S^k = \partial(S^p \times B^{k+1})$. Deform the embedding so that the S^k cross-sections are fronts for the flying saucer (see Example 2.9). Let $F : S^p \to \mathbb{R}$ be a Morse function with one maximum and one minimum, and scale each cross section $\{x\} \times S^k$ by $1 + \epsilon F(x)$ for some small $\epsilon > 0$.

The Legendrian embedding L coming from this front has exactly two Reeb chords, one of degree $k+p$ and the other of degree k. Suppose, for convenience, that $p > k > 1$, so that the algebra $\mathcal{A}(L)$ is good for degree reasons. Direct calculation then shows that the linearized homology is $\mathbb{Z}$ in degrees $k+p$ and k, and zero otherwise. Theorem 1.1 shows that the homology class of degree k is a manifold class coming from $H_k(S^p \times S^k)$.

Reversing the roles of k and p yields another Legendrian embedding of $S^p \times S^k$, but this time with a linearized homology class in degree p is a manifold class that comes from $H_p(S^p \times S^k)$. Thus, it is impossible to determine a priori which classes in $H_*(L)$ in degrees other than 0 or n will be manifold classes.

6. Proof of the Main Analytic Theorem

This section is devoted to the proof of Theorem 3.6, which describes the rigid holomorphic disks with boundary on $2L$ in terms of holomorphic disks with boundary on L and Morse flow lines of f.

Remark 6.1. We will occasionally make reference to a "flow tree" in this section. Flow trees are generalizations of gradient flow lines that arise when one takes more that two copies of L and perturbs them by several different functions. We use the flow tree terminology when it seems appropriate in reference to some of the work in [3], where more general versions of some of the results below are proven, but we will not give a rigorous definition as we are only interested in flow lines here.

6.1. **Evaluation maps and moduli spaces as manifolds with boundary with corners.**

The generalized disks that appear in Theorem 3.6 require the use of the evaluation map on a moduli space of J-holomorphic disks, so we begin with a description of those maps. Assume that the almost complex structure J is sufficiently generic so that Lemma 2.2 holds. As in previous sections, a "J-holomorphic disk with boundary on L" is short hand for a J-holomorphic disk in P with boundary on $\Pi(L)$ whose restriction to the boundary has a continuous lift into L. As in Section 2.2, let $\mathcal{M}_A(a; b_1, \dots, b_k)$ denote the moduli space of J-holomorphic disks with boundary on L, positive puncture at the Reeb chord a, negative punctures at Reeb chords $b_1, \dots, b_k$, and boundary data corresponding to the homology class $A \in H_1(L)$.

By Gromov compactness,[4] a moduli space $\mathcal{M}_A(a; b_1, \dots, b_k)$ has a natural compactification consisting of broken J-holomorphic disks with one positive puncture. Furthermore, the smooth moduli spaces corresponding to the pieces of the broken disks fit together into a compact manifold with boundary and corners $\overline{\mathcal{M}}$, the interior of which is $\mathcal{M} = \mathcal{M}_A(a; b_1, \dots, b_k)$. We only give a brief sketch of this fact, as the full arguments are quite standard and technically simpler than those for actual disks corresponding to generalized disks (which appear later in this section).

The codimension k part of the boundary of $\mathcal{M}_A(a; b_1, \dots, b_k)$ corresponds to $(k+1)$-component broken disks $v^0 \cup v^1 \cup \dots \cup v^k$. The positive puncture of v^0 is at a and v^j is adjoined at its positive puncture to some v^l, $l < j$, at one of its negative punctures. To construct $\overline{\mathcal{M}}$, we must describe a neighborhood of this broken configuration.

As a first step, consider the following description of a neighborhood in the moduli space of an unbroken disk v. Such a neighborhood consists of a bundle of Sobolev spaces of vector fields along v over a neighborhood of the conformal structure of the source of v in the space of conformal structures. The $\bar{\partial}_J$-operator gives a Fredholm section of the bundle of complex anti-linear maps from the tangent space of the source to the pull-back of the tangent space of the target. The zero-set of this section gives a neighborhood of v in the moduli space.

Moving back to the general picture, we employ Floer's Picard lemma (see [4, Lemma 6.17]) in order to provide a neighborhood for the broken disk $v^0 \cup \dots \cup v^k$. This lemma is an infinite-dimensional version of Newton iteration. It provides zero-sets of Fredholm mappings of Banach spaces which are manifolds in a neighborhood of the start point of the iteration provided the following three conditions hold:

(1) the start point is sufficiently close to being a solution,
(2) the differential of the map admits a bounded right inverse, and
(3) the non-linear remainder term of the first order Taylor expansion can be estimated by a quadratic polynomial.

To satisfy these three conditions, we think of the domains of the disks $v^0, \dots, v^k$ as standard domains $\Delta_m \subset \mathbb{C}$ which are horizontal strips in the complex plane of integer width with small slits around horizontal half lines with integer vertical coordinates removed; see [3, Section 2.1.1] for a more precise definition. This allows us to represent the space of conformal structures $\mathcal{C}_m$ on the m-punctured disk as $\mathbb{R}^{m-3} = \mathbb{R}^{m-2}/\mathbb{R}$ where $\mathbb{R}^{m-2}$ encodes the horizontal coordinates of the slit minima and where $\mathbb{R}$ acts by over all horizontal translation. Furthermore, we require that the standard domains are stable.[5] We join the standard domains of the pieces v^k into one new standard domain Δ^ρ depending on k parameters $\rho = (\rho_1, \dots, \rho_k)$, where ρ_j is the length of the strip region joining cut off versions of the disks v^l and v^j. Interpolating between the solutions using their exponential decay near double points, we produce approximate solutions w^ρ such that $\bar{\partial}_J w^\rho \to 0$ exponentially fast as $\rho \to \infty$, thus satisfying condition (1) of Floer's

[4] See [4, 7] for proofs of Gromov compactness in the present setting.

[5] Disks with less than three punctures can be stabilized by adding marked point near the positive puncture as in [6, Section 4.2.3] and [4, Section 7].

Picard Lemma. Arguments similar to those that will be used to prove Lemma 6.15 below show that the linearization of the $\bar{\partial}_J$-operator at w_ρ is uniformly invertible off of a finite dimensional subspace spanned by cut off versions of the kernels of the pieces and certain "newborn" conformal variations arising in the gluing process and thus condition (2) is satisfied. The argument referred to in Lemma 6.16 below shows that the non-linear estimate satisfies a quadratic bound and thus condition (3) is satisfied . Floer's Picard Lemma now provides a solution space of the expected dimension near each w_ρ for all sufficiently large ρ.

For moduli spaces of dimension 0 and 1, gluing theorems of the sort just discussed were proved in [4]. Furthermore, the argument establishing a uniformly bounded right inverse shows that the kernel of the differential projects non-trivially to the k-dimensional subspace of the space of conformal structures of components going to ∞ corresponding the breaking of the domain. Thus, patching the local coordinate neighborhoods in the solution spaces near w_ρ as $\rho \to \infty$, we get an induced manifold with corner structure from the corresponding structure on the space of conformal structures on the disk with boundary punctures, which is compatible with the corner structure corresponding to broken subdisks of the broken disk.

Since the areas of J-holomorphic disks with boundary on L are a priori bounded above (there are only finitely many Reeb chords), we conclude that the collection of all J-holomorphic disks fit together to a compact manifold with boundary with corners $\overline{\mathcal{M}}$. Note that restricting to disks with one positive puncture rules out multiple covers, so $\overline{\mathcal{M}}$ is a manifold rather than an orbifold.

In order to define evaluation maps, we generalize the above discussion slightly. Let $\mathcal{M}^* = \mathcal{M}^*_A(a; b_1, \dots, b_k)$ denote the moduli space of holomorphic disks with punctures as before, but with one marked point on the boundary; see [4, §7.7]. Evaluation at the marked point gives a map $\mathrm{ev}\colon \mathcal{M}^*_A(a; b_1, \dots, b_k) \to L$. In order to define the compactification of $\mathcal{M}^*$, we consider all broken disks $v^0 \cup \dots \cup v^k$ with one marked point, where the marked point may belong to any one of the components of the broken curve. The same argument as in the previous case gives a compactification $\overline{\mathcal{M}^*}$ which is a manifold with boundary and corners. The evaluation map extends nicely to $\overline{\mathcal{M}^*}$. To see why, consider configurations in which the marked point lies close to the positive puncture of v^j which is attached to a negative puncture of v^l and configurations where the marked point lies close to the negative puncture of v^l. Neighborhoods of approximate solutions corresponding to these configurations are overlapping and, by construction, compatible. In conclusion, the evaluation maps on the smooth strata induce an evaluation map $\mathrm{ev}\colon \overline{\mathcal{M}^*} \to L$ which is a C^1-map from a manifold with boundary with corners, though the C^1-structure on $\overline{\mathcal{M}}$ is not canonical. See [13] for a discussion of evaluation maps in a similar situation.

6.2. **Morse flows and generalized disks.** Consider $L \subset P \times \mathbb{R}$ as above. If g is a Riemannian metric and $f\colon L \to \mathbb{R}$ is a smooth function, we write ∇f to denote the g-gradient of f. If the pair (f, g) is Morse-Smale and if $p \in L$ is a critical point of f, then the stable and unstable manifolds $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$ are submanifolds of L with natural compactifications consisting of broken (and constant) flow lines.

Lemma 6.2. *There exists a Riemannian metric g and a Morse function f such that, for each critical point p of f, $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$ are stratumwise transverse to* $\mathrm{ev}\colon \overline{\mathcal{M}^*} \to L$.

Proof. This follows from a standard transversality argument along the lines of [27]. □

6.3. **Adjusting the metric and the Morse function.** Let $L \subset P \times \mathbb{R}$ be a Legendrian submanifold, let J be an almost complex structure on P, and let $f: L \to \mathbb{R}$ be a Morse function and g a Riemannian metric on L so that Lemmas 2.2 and 6.2 hold. We will produce a different metric and a family of functions f_λ, $0 \leq \lambda \leq 1$, for which Lemma 6.2 continues to hold, but for which the function and the metric have special properties near the flow lines of rigid lifted generalized disks; see Subsection 3.3 for the notion of lifted generalized disks and for terminology related to such objects. The properties we want are as follows:

(1) In a neighborhood of any flow line in a rigid lifted generalized disk, the metric is flat and the function f_λ is affine in some neighborhood of the junction point, and quadratic near any critical point.
(2) Outside an $\mathbf{O}(\lambda)$-neighborhood of finitely many points along any flow line of a rigid lifted generalized disk, the function f_λ is linear or quadratic along the flow line and constant in directions perpendicular to it. Inside the $\mathbf{O}(\lambda)$-neighborhoods, the function interpolates between the pieces, and its derivative is bounded by $\mathbf{O}(\lambda)$.

Remark 6.3. Although not explicitly mentioned in [3], the $\mathbf{O}(\lambda)$-neighborhoods mentioned in (2) should be taken to have diameter $K\lambda$ where K is some large constant so that the Lagrangian is sufficiently close to the 0-section on the λ-scale. See Remark 6.6 for details.

Lemma 6.4. *There exist a metric and a function which satisfy Lemma 6.2 and which have properties* (1) *and* (2).

Proof. We first adjust the data near junction points. Note that there exists a coordinate map from $(\mathbb{C}^n, \mathbb{R}^n) \to (P, L)$ such that the push forward of the standard complex structure on $\mathbb{C}^n$ is arbitrarily C^1-close to the given complex structure on P at the junction point. Furthermore, the Morse flows in the given metric and in the flat metric can be taken to be C^1-close. Thus, changing from one almost complex structure to the other and changing the Morse flow in a sufficiently small neighborhood, property (1) near junction points follows.

(If $\dim(L) = 2$, we adjust the data in the same way at all other points where flow lines in rigid generalized disks intersect holomorphic disk parts of rigid generalized disks. Our transversality assumptions guarantee that there are only finitely many such points and that the intersections are transverse.)

We next adjust the metric along the remaining parts of the flow lines. Here we use a similar diffeomorphism generated by flow coordinates as in [3]. This gives a diffeomorphism which induces a complex structure which is only C^0-close to the given one. Viewing the $\bar{\partial}_J$-operator as an operator from a Banach manifold of maps with one derivative in L^p, $p > 2$ to L^p-sections of the bundle of complex anti-linear maps, the distance between the operators are controlled by the C^0-distance between the corresponding almost complex structures. Since there are no holomorphic parts of generalized disks in a neighborhood of the flow line and since that is a C^0-open condition, Lemma 6.2 continues to hold for the new data. □

6.4. **From holomorphic to generalized disks.** Fix a metric and a Morse function as in Lemma 6.4. Let $\tilde{L}_\lambda$ denote the Legendrian submanifold obtained by shifting L a distance s upwards in the z-direction and then a small distance of magnitude $\mathbf{O}(\lambda)$ along f_λ. We will consider holomorphic disks with boundary on $L \cup \tilde{L}_\lambda$. The Reeb chords of $L \cup \tilde{L}_\lambda$ can be separated into pure chords, mixed chords, and Morse chords as in Section 3.1.

By Lemma 3.1, a holomorphic disk with one positive puncture must have either zero or two mixed Reeb chords. Holomorphic disks with all their punctures at pure Reeb chords correspond naturally to holomorphic disks of L. We therefore concentrate on disks with mixed punctures. We first analyze the cases where both of the mixed Reeb chords are Morse chords and when one of them is. Once this analysis is complete, the cases when no Reeb chords are Morse chords can be easily handled; see the end of Section 6.5.

Lemma 6.5. *If both the positive and some negative puncture of a holomorphic disk are Morse chords, then the disk has no other punctures and converges to a (possibly broken) Morse flow line of f as $\lambda \to 0$. Furthermore, if the disk is rigid, so is the flow line.*

Proof. The area of the disk is of size $\mathbf{O}(\lambda)$. By monotonicity, the disk cannot leave an $\mathbf{O}(\lambda^{\frac{1}{2}})$-neighborhood of $\Pi_P(L)$. Since the complex structure is standard in a such neighborhood of $\Pi_P(L)$, it follows from [3, Lemmas 5.13] that the holomorphic disk converges to a flow line. This flow line must furthermore be rigid by comparison of dimension formulas for holomorphic disks and flow lines, see [3, Proposition 3.18]. □

Remark 6.6. In the proof of [3, Lemma 5.13], the edge point regions (see [3, Section 4.3.8]) were not explicitly mentioned. For completeness, we give the explicit argument here. An edge point region is a region of size $\mathbf{O}(\lambda)$ around a point on a flow line in a rigid generalized disk where the Lagrangian interpolates between its nearby affine pieces. In fact, these edge point regions should be chosen to have size $K\lambda$, where K is a sufficiently large constant. With such a choice, the derivative of the interpolation function can be taken as small as $\mathbf{O}(\frac{1}{K})$ after rescaling of base and fiber by λ^{-1} as compared to all other gradient differences nearby. In the proof of the convergence result [3, Lemma 5.12], one uses a split coordinate system near edge points. In the directions perpendicular to the flow line the argument is the one given in the lemma. In the directions along the flow line the re-scaled correction function f in [3, Equation (5-8)] is now $\mathbf{O}(\frac{1}{K})$ rather than $\mathbf{O}(\lambda)$. However, changes in this direction just correspond to reparameterization of the holomorphic disk (i.e. variations along the flow direction). The fact that the error term is small after rescaling shows that the time spent by a holomorphic map u_λ in the $K\lambda$-neighborhood (the length of the part of the domain mapping to the $K\lambda$-neighborhood) is $\mathbf{O}(1)$. Consequently, the disk lies at most $\mathbf{O}(\lambda)$ from a flow line after having passed through the edge point region.

We next work with disks that have one mixed Morse puncture and another mixed puncture. The first step is to show that for any sequence of J-holomorphic disks u_λ with boundary on $L \cup \tilde{L}_\lambda$, there are neighborhoods of each Morse puncture where the disk converges to a (possibly constant) flow line. We think of the domains of our holomorphic disks as standard domains $\Delta_m^\lambda \subset \mathbb{C}$ and the space of conformal structures in the corresponding way, as explained in Subsection 6.1.

The key to establishing this convergence is a certain derivative bound. More precisely, we have the following. Fix a constant $M > 0$ and let $l_\lambda \approx [0,1]$ be a vertical segment in Δ_m^λ with the following properties:

(l1) $u_\lambda(l_\lambda)$ is contained in an $\mathbf{O}(\lambda)$-neighborhood of L.

(l2) $\left|\int_{l_\lambda} u_\lambda^*(p\,dq)\right| \le M\lambda$.

(l3) $\big||z(1) - z(0)| - s\big| \le M\lambda$, where we think of 0 and 1 as the endpoints of l_λ and z as the $\mathbb{R}$ coordinate in $P \times \mathbb{R}$.

Note that l_λ subdivides Δ^λ_m into two components. Let $\Delta^\lambda(l_\lambda)$ denote the component which contains the Morse puncture. Also, for $d > 0$, let $\Delta^\lambda(l_\lambda, d)$ denote the subset of points in $\Delta^\lambda(l_\lambda)$ which are at distance at least d from l_λ.

Lemma 6.7. *For all sufficiently small $\lambda > 0$,*

$$|Du_\lambda(z)| = \mathbf{O}(\lambda), \quad z \in \Delta^\lambda(l_\lambda, 1).$$

Proof. The idea of the proof is to localize the situation and then use technology from the analysis of flow lines (trees) in [3].

Let b denote the Morse chord. We first show that the area of $u_\lambda(\Delta^\lambda(l_\lambda))$ is of order $\mathbf{O}(\lambda)$. If the puncture of u_λ at b is negative, then the area of $u_\lambda(\Delta^\lambda(l_\lambda))$ is smaller than

$$\int_{l_\lambda} u_\lambda^*(p\,dq) + |z(1) - z(0)| - \ell(b) - \sum \ell(c),$$

where the sum ranges over all negative punctures in $\Delta^\lambda(l_\lambda)$. Since the second and third terms are both of size $s + \mathbf{O}(\lambda)$, since the first term is of size $\mathbf{O}(\lambda)$, and since $\ell(c) > \ell_{\min} > 0$ for all Reeb chords c, we find that the sum must be empty, and consequently that the area of $u_\lambda(\Delta^\lambda(l_\lambda))$ is of size $\mathbf{O}(\lambda)$.

If the puncture of u_λ at b is positive, the area of $u_\lambda(\Delta^\lambda(l_\lambda))$ is smaller than

$$\ell(b) - \int_{l_\lambda} u_\lambda^*(p\,dq) - |z(1) - z(0)| - \sum \ell(c),$$

and an analogous argument shows that there are no negative punctures and that the area of $u_\lambda(\Delta^\lambda(l_\lambda))$ is also of size $\mathbf{O}(\lambda)$.

We conclude by monotonicity that $u_\lambda(\Delta^\lambda(l_\lambda))$ must lie in an $\mathbf{O}(\lambda^{\frac{1}{2}})$-neighborhood of L. Since J agrees with the almost complex structure induced by the metric on L in such a neighborhood, [3, Lemma 5.4] shows that the function $|p|^2$, where p is the fiber coordinate in T^*L composed with u_λ is subharmonic on $\Delta^\lambda(l_\lambda)$ and therefore attains its maximum on the boundary. The lemma then follows from [3, Lemma 5.6]. $\square$

Corollary 6.8. *The restriction $u_\lambda|\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))$ converges to a flow line of f_λ. (Here we also allow constant flow lines.)*

Proof. Note that any region of diameter $\log(\lambda^{-1})$ maps inside a disk of radius $\mathbf{O}(\lambda\log(\lambda^{-1}))$. Moreover, along any strip region in $\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))$, the map converges to a flow line at rate $\mathbf{O}(\lambda)$ by the proof of [3, Theorem 1.2]. $\square$

Remark 6.9. If the limiting flow line in Corollary 6.8 is constant, then it lies at $\Pi_P(c)$ for some Reeb chord c of L. To see this, note that $\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))$ always contains a half infinite strip and that if the starting point of this strip does not converge to the projection of a Reeb chord, then the flow line is non-constant.

6.4.1. *Blow up analysis.* We next show that for any sequence of J-holomorphic disks u_λ with boundary on $L \cup \tilde{L}_\lambda$, we can choose conformal representatives Δ^λ_m of their domains such that the derivatives $|Du_\lambda|$ are uniformly bounded. When a bubble forms in a sequence of maps on such domains, some coordinate of the domains Δ^λ_m in $\mathcal{C}_m \approx \mathbb{R}^{m-3}$ goes to ∞ (rather than that the derivative of u_λ blowing up).

Lemma 6.10. *Let $u_\lambda \colon \Delta_m^\lambda \to P$ be a sequence of J-holomorphic disks with boundary on $L \cup \tilde{L}_\lambda$. After addition of a finite number of punctures in Δ_m^λ, creating new domains Δ_{m+k}^λ, the induced maps $u_\lambda \colon \Delta_{m+k}^\lambda \to P$ satisfy a uniform derivative bound.*

Proof. The proof is a standard blow up argument, so we only sketch the details. Assume that $M_\lambda = \sup_{\Delta_m^\lambda} |Du_\lambda|$ is not bounded. The asymptotics near the punctures of u_λ show that there exist points $p_\lambda \in \Delta_m^\lambda$ at which $|Du_\lambda| = M_\lambda$. Consider the sequence of maps $g_\lambda = u_\lambda\left(p_\lambda + \frac{z}{M_\lambda}\right)$ defined on $\left\{z \in \mathbb{C} : (p_\lambda + \frac{z}{M_\lambda}) \in \Delta_m^\lambda\right\}$. Note that the derivatives of these maps are uniformly bounded. Therefore we can extract a convergent subsequence. This gives a non-constant holomorphic disk with boundary on L which has one positive puncture and no other puncture. Denote this limit disk $v^{[1]} \colon D \to P$ and fix a local hypersurface H transversely intersecting $v^{[1]}(\partial D)$ at a point far from all Reeb chords. It follows from the convergence $g_\lambda \to v^{[1]}$ that there exist a point in a neighborhood of p_λ which u_λ maps to H. Puncturing Δ_m^λ at this point induces a new sequence of maps $u_\lambda^{[1]} \colon \Delta_{m+1}^\lambda \to P$. If $|Du_\lambda^{[1]}|$ is uniformly bounded then the lemma follows.

Assume that $\sup_{\Delta_{m+1}^\lambda} |Du_\lambda^{[1]}|$ is unbounded. Arguing as above, we find another bubble $v^{[2]}$ in the limit with one positive puncture and no other punctures. Adding another puncture in the domain which corresponds to some point in $v^{[2]}$ in the limit, we get new maps and domains $u_\lambda^{[2]} \colon \Delta_{m+2}^\lambda \to P$. Repeating this, we either have no derivative blow up in which case the lemma follows or we add punctures as above. To see that this is a finite process note that each bubble has area bounded from below by the length of the shortest Reeb chord and that the sum of the areas of all bubbles must be smaller than the length of the longest Reeb chord of L. □

Remark 6.11. Let p be the image point corresponding to an additional puncture added in the procedure described above. Note that there exists a disk of finite radius around p which does not contain any Reeb chords. It follows from monotonicity that for $\lambda > 0$ small enough, the area contribution of any subdisk obtained by cutting off a vertical segment which connects L to $\tilde{L}$ and which contains the puncture corresponding to the marked point is uniformly bounded from below. It follows that the disk $\Delta^\lambda(l_\lambda)$ in Lemma 6.7 cannot contain any additional punctures.

6.4.2. *Generalized disk convergence.* Consider a sequence u_λ of J-holomorphic disks with boundary on $L \cup \tilde{L}_\lambda$ with one puncture at a Morse chord.

Using Lemma 6.10, we assume that these are maps $u_\lambda \colon \Delta_m^\lambda \to P$ with uniformly bounded derivatives. It is a consequence of Gromov compactness that this sequence converges to a broken disk v on L, uniformly on compact subsets. In the present situation, this kind of convergence does not give the full picture since the areas of holomorphic disks with boundary on $L \cup \tilde{L}_\lambda$ are not uniformly bounded from below as $\lambda \to 0$. For example, on a compact neighborhood of a point which lies in a flow line segment to which the disk converges, the disk converges to a constant map.

Let ∂v denote the image in L of the boundary of the possibly broken limit disk v. Lemma 6.8 and Remark 6.11 imply that if a vertical line segment $l_\lambda \subset \Delta_m^\lambda$ satisfies (l1)–(l3) then the l_λ-half of the disk which contains a Morse puncture converges to a flow line.

Lemma 6.12. *There exist vertical segments l_λ which satisfies* (l1)–(l3) *and such that $u_\lambda(l_\lambda)$ converges to a point in ∂v.*

Proof. We prove this lemma by contradiction: if the statement of the lemma does not hold, then the area difference between a limit disk and the disks before the limit violates an $\mathbf{O}(\lambda)$ bound derived from Stokes' theorem.

Thus, we assume that the lemma does not hold. Then there exists $\epsilon > 0$ such that for any sequence of l_λ which satisfies (l1)–(l3), some point on l_λ maps a distance at least $\epsilon > 0$ from ∂v. Consider a strip region $[-d,d] \times [0,1] \subset \Delta_m^\lambda$ for which some point converges to a point a distance δ from ∂v, where $\frac{\epsilon}{4} < \delta < \frac{\epsilon}{2}$. Let $\sup_{[-d,d]\times[0,1]} |Du_\lambda| = K$. Then K is not bounded by $M\lambda$ for any $M > 0$. Since the difference between the area of the limit disk v and that of u_λ is of order of magnitude $\mathbf{O}(\lambda)$ it follows that $|Du_\lambda| = \mathbf{O}(\lambda^{\frac{1}{2}})$ from the usual bootstrap estimate. Thus $K = \mathbf{O}(\lambda^{\frac{1}{2}})$. Consider next the scaling of the target by K^{-1}. We get a sequence of maps $\hat{u}_\lambda$ from $[-d,d] \times [0,1]$ with bounded derivative. Note moreover that the boundary condition is $\mathbf{O}(\lambda^{\frac{1}{2}})$ from the 0-section. Changing coordinates to the standard $(\mathbb{C}^n, \mathbb{R}^n)$ respecting the complex structure at the limit point, we find that there are maps $f_\lambda \colon [-d,d] \times [0,1] \to \mathbb{C}^n$ with the following properties

- $\sup_{[-d,d]\times[0,1]} |D^k f_\lambda| = \mathbf{O}(\lambda^{\frac{1}{2}})$, $k = 0, 1$.
- $u_\lambda + f_\lambda$ satisfies $\mathbb{R}^n$ boundary conditions
- $\bar{\partial}(u_\lambda + f_\lambda) = \mathbf{O}(\lambda^{\frac{1}{2}})$.

It follows that $u_\lambda + f_\lambda$ converges to a holomorphic map with boundary on $\mathbb{R}^n$, which takes 0 to 0 and which has derivative of magnitude 1 at 0. Using solvability of the $\bar{\partial}$-equation in combination with L^2-estimates in terms of area we find that the area of $\hat{u}_\lambda$ must be uniformly bounded from below by a constant C. The area contribution to the original disks near the limit is thus at least K^2C. Since $[-d,d] \times [0,1]$ covers a length along L of at most $2Kd$, we may repeat the argument with many disjoint finite strips which together cover a finite length and with maximal derivatives K_j. We find that the area contribution is bounded from below by $C\sum K_j^2$ and that, since the length contribution is finite, we get:

$$2d\sum K_j \geq \frac{\epsilon}{100}.$$

Now,

$$C\sum K_j^2 \geq C\inf_j\{K_j\}\sum K_j \geq C'\inf_j\{K_j\}.$$

For any $M > 0$, $\inf_j\{K_j\} \geq M\lambda$. To see this assume that it does not hold true. Then there is a sequence of vertical segments l_λ such that $|Du_\lambda| \leq 2M\lambda$ with the property that the distance between $u(l_\lambda)$ and ∂v is at most $\frac{3}{4}\epsilon$. This however contradicts our hypothesis. Consequently, the area contribution from the remaining part of the disk is not $\mathbf{O}(\lambda)$, which contradicts Stokes' theorem. □

We get the following:

Corollary 6.13. *Any sequence of rigid holomorphic disk with boundary on $L \cup \tilde{L}_\lambda$ and with one Morse puncture converges to a rigid generalized disk.*

Proof. It follows from Lemmas 6.8 and 6.12 and from Gromov compactness that the limit gives a generalized disk. This generalized disk must furthermore have formal dimension 0. By our choice of Morse function it follows that the holomorphic part is not broken and that the generalized disk is transversely cut out. □

6.5. **From generalized to holomorphic disks.** In order to produce a unique holomorphic disk near each generalized holomorphic disk, we will first associate a family of domains to a generalized holomorphic disk equipped with approximately holomorphic maps with boundary on $L \cup \tilde{L}_\lambda$. We then define a functional analytic space of variations of the approximately holomorphic map, show that the linearized $\bar{\partial}_J$-operator is uniformly invertible on this space, and derive a second derivative estimate. The invertibility and second derivative estimate gives a unique holomorphic disk in the functional analytic neighborhood of the approximate solution, and we show that any solution must lie in this neighborhood.

6.5.1. *Approximate solutions.* Consider a rigid generalized disk (u, γ), where $u\colon \Delta_{m-1} \to P$ is the holomorphic disk part with $m-1$ punctures mapping to Reeb chords and γ the Morse flow line part. Consider the Morse flow line γ. As in [3, Section 6.1], we associate a half strip $[0,\infty) \times [0,1]$ with almost holomorphic maps $w_\lambda^\gamma\colon [0,\infty)\times[0,1] \to P$ which agree with the natural holomorphic strip over the gradient flow lines in the flat metric outside the edge point regions where it interpolates between these solutions and satisfies $|\bar{\partial}_J w_\lambda^\gamma| = \mathbf{O}(\lambda)$.

Consider the holomorphic disk u. Add a puncture to u at the junction point and view it as a map $u\colon \Delta_m \to P$. (Here, Δ_m is a standard domain with one puncture mapping to the marked junction point.) Fix a reference point 0 in Δ_m which u maps to a point far from any puncture. Note that the derivative of u is bounded.

Since L and J have a standard form near its Reeb chords, there are $\mathbb{C}^n$-coordinates near each Reeb chord such that the map $u = (u_1, \dots, u_n)$ looks like

$$u_j(z) = \sum_{n \le 0} c_{j,n} e^{(-\theta_j + n\pi)z}, \quad z \in [0,\infty)\times[0,1],$$

where $0 < \theta_j < \pi$ and where $c_{j,n} \in \mathbb{R}$. Consequently, there is a vertical segment $l_\lambda^{\mathrm{Reeb}}$ of distance $\mathbf{O}(\log(\lambda^{-1}))$ from $0 \in \Delta_m$ such that a finite neighborhood of $l_\lambda^{\mathrm{Reeb}}$ maps into an $\mathbf{O}(\lambda)$ neighborhood of the Reeb chord. Also, by our choice of complex structure near the marked junction point, there exists $\mathbb{C}^n$-coordinates such that the map looks like

$$u(z) = \sum_{n<0} c_n e^{n\pi z}, \quad z \in [0,\infty)\times[0,1],$$

where $c_n \in \mathbb{R}^n$. We find a vertical segment $l_\lambda^{\mathrm{mark}}$ at finite distance from $0 \in \Delta_m$ which maps into an ϵ-neighborhood of the junction point.

We next construct a family of approximately holomorphic maps $w_\lambda\colon \Delta_m \to P$ with boundary on $L \cup \tilde{L}_\lambda$. Consider the region $\Delta_m(d) \subset \Delta_m$ which contains 0 and which is obtained by cutting at $l_\lambda^{\mathrm{mark}}$. Adjoin a segment $[0, \frac{d}{\lambda}] \times [0,1]$ and then adjoin the domain $[0,\infty)\times[0,1]$ of w_λ^γ. We think of this domain as of our new Δ_m. We equip Δ_m with a weight function h of the following form. The function h equals 1 inside the vertical segment l^{mark}, between l^{mark} and the vertical segment at $\frac{d}{2\lambda}$ in the gluing strip it grows with exponential weight δ then it decreases back to 1 at the start of $[0,\infty)\times[0,1]$ and continues as in [3], between edge points of γ. Let $\|\cdot\|_{k,\delta}$ denote the Sobolev norm with weight h.

We define an almost holomorphic map w_λ^u on $\Delta_m(\lambda)$ which meets the boundary conditions of $L \cup \tilde{L}_\lambda$ and which agrees with constant maps close to Reeb chord punctures in a neighborhood of each corresponding l_λ. To this end we must move the boundary of $u|_{\Delta_m(d)}$ a distance $\mathbf{O}(\lambda)$.

Supporting such a deformation in an $\mathbf{O}(\lambda^{\frac{1}{2}})$-neighborhood of the boundary, one can achieve this while changing the derivative of u by at most $\mathbf{O}(\lambda^{\frac{1}{2}})$.

We then define a holomorphic map $w_\lambda^{\mathrm{jun}}\colon [0, \frac{d}{\lambda}] \times [0,1] \to P$ which satisfies the boundary conditions. To define this map we assume that the $\mathbb{C}^n$-coordinates have been chosen so that L corresponds to $\mathbb{R}^n = \{y_1 = \cdots = y_n = 0\}$ and $\tilde{L}_\lambda$ corresponds to $\{y_1 = \lambda, y_2 = \cdots = y_n = 0\}$. The holomorphic map is then

$$w_\lambda^{\mathrm{jun}}(z) = (\lambda z, 0, \dots, 0) + \sum_{n<0} c_n e^{n\pi z}, \tag{6.1}$$

where the sum is as in the expansion of $u(z)$.

Near the junction point, the maps w_λ^u and w_λ^{jun} (as well as w_λ^γ and w_λ^{jun}) are then of distance $\mathbf{O}(\lambda)$ apart, and we interpolate between them using a function on a finite rectangle of size $\mathbf{O}(\lambda)$. The function resulting from this interpolation is $w_\lambda\colon \Delta_m \to P$.

Lemma 6.14. *The approximately holomorphic function w_λ satisfies*

$$\|\bar\partial_J w_\lambda\|_{1,\delta} = \mathbf{O}(\lambda^{\frac{3}{4}}(\log\lambda^{-1})^{\frac{1}{2}})$$

Proof. The map is non-holomorphic only in regions where the weight is finite. These are of two kinds: finite rectangles where the size of the derivatives are $\mathbf{O}(\lambda)$ and an $\mathbf{O}(\lambda^{\frac{1}{2}})$-neighborhood of the boundary in $\Delta_m(d)$ cut off at all $l_\lambda^{\mathrm{Reeb}}$. The former regions give a contribution of size $\mathbf{O}(\lambda)$ the latter gives the contribution

$$\sqrt{\mathbf{O}(\lambda)\cdot\mathbf{O}(\lambda^{\frac{1}{2}})\cdot\mathbf{O}(\log\lambda^{-1})},$$

where the first factor is the square of the size of the deformation and the product of the last two is the area over which it is integrated. □

We associate a variation space $\hat{\mathcal{H}}_{2,\delta}$ to w_λ. This is a direct sum

$$\hat{\mathcal{H}}_{2,\delta} = \mathcal{H}_{2,\delta} \oplus V_{\mathrm{con}} \oplus V_{\mathrm{sol}},$$

where the summands are the following:

- $\mathcal{H}_{2,\delta}$ is a Sobolev space of vector fields v along w_λ with two derivatives in L^2 weighted by h which satisfy the following additional conditions: v is tangent to L along the boundary, $\bar\nabla_J v = 0$ along the boundary (see [7]), and v vanishes at one boundary point midway between neighboring edge points as well as in on a boundary point in the middle of the gluing strip at the junction point.
- V_{con} is the space of conformal variations of Δ_m (see [4] for a concrete description).
- V_{sol} is a finite dimensional space consisting of certain cut off constant solutions of the $\bar\partial_J$-equation supported in the regions where the weight h is large, as follows. Along the flow line part, there are n-dimensional cut-off constant solutions between any two edge points exactly as in [3]. There is also an n-dimensional space corresponding to the junction point. More precisely, in the coordinates of (6.1), this n-dimensional space is spanned by

$$\beta(1,0,\dots,0),\ \beta(0,1,0,\dots,0),\ \dots\ ,\beta(0,\dots,0,1),$$

where β is a cut off function equal to 1 in the gluing region $[0, \frac{d}{\lambda}] \times [0,1]$ corresponding to the junction point and equal to 0 outside a uniformly finite neighborhood of it. We equip V_{sol} with the supremum norm.

6.5.2. *Uniform invertibility.* Let $L\bar{\partial}_J$ denote the linearization of the $\bar{\partial}_J$-operator acting on elements $v \in \hat{\mathcal{H}}_{2,\delta}$.

Lemma 6.15. *The differential*

$$L\bar{\partial}_J \colon \hat{\mathcal{H}}_{2,\delta} \to \mathcal{H}_{1,\delta}$$

is uniformly invertible.

Proof. The proof is similar to the proof of [3, Proposition 6.20] so we just give an outline. Consider a variation v_λ of w_λ. Write $v_\lambda = v_\lambda^{\mathrm{di}} + v_\lambda^{\mathrm{jun}} + v_\lambda^{\mathrm{mo}}$ where we use cut off functions to subdivide v into a disk-piece, a junction-piece, and a Morse-piece. If the operator is not uniformly invertible, then there is a sequence of variation maps v_λ such that

$$\|v_\lambda\|_{2,\delta} = 1, \quad \|L\bar{\partial}_J v_\lambda\|_{1,\delta} \to 0. \tag{6.2}$$

Parameterize a neighborhood of the holomorphic disk part using a Sobolev space with small positive exponential weight near the added marked point and cut off constant solutions transverse to the disk. We infer from the properties on the disk part that v_λ^{di} has non-zero component along the cut-off and conformal solutions at the junction point. Similarly, v_λ^{mo} has non-zero component in the tangent directions of the flow manifold. The components of the cut-off solutions correspond to cut-off solutions at the junction point in the space of variations of w_λ. Since the evaluation map from the moduli space of holomorphic disks is transverse to the (un)stable manifold at the junction point, it follows that the component of v_λ^{jun} along the cut off solutions must go to 0 with λ. We conclude from this that $v_\lambda^{\mathrm{di}} \to 0$ and $v_\lambda^{\mathrm{mo}} \to 0$. Since the $\bar{\partial}$-operator on the middle piece is invertible on the complement of cut off solutions, we conclude that $v_\lambda^{\mathrm{jun}} \to 0$ as well. This contradicts (6.2). □

6.5.3. *Existence and uniqueness of solutions near generalized disks.* Consider the $\bar{\partial}_J$-operator on maps in a neighborhood of w_λ as a map $F \colon \hat{\mathcal{H}}_{2,\delta} \to \mathcal{H}_{1,\delta}$, with w_δ corresponding to $0 \in \hat{\mathcal{H}}_{2,\delta}$.

Lemma 6.16. *There exists a constant $C > 0$ such that*

$$F(v) = F(0) + dF(v) + N(v),$$

where

$$\|N(v_1) - N(v_2)\|_{1,\delta} \leq C(\|v_1\|_{2,\delta} + \|v_2\|_{2,\delta})\|v_1 - v_2\|_{2,\delta}.$$

Proof. This is a standard argument in the case there are no weights and no cut-off solutions. However, as the weight function is ≥ 1, the left hand side is linear in the weight and the right hand side is quadratic, so the weight does not interfere with the estimate. The cut-off solutions are true solutions except in finite regions where the weight is bounded and we conclude that the estimate holds. □

Corollary 6.17. *There exists a unique disk in a finite $\|\cdot\|_{2,\delta}$-neighborhood of w_λ for all sufficiently small $\lambda > 0$.*

Proof. This is a consequence of Lemma 6.16 in combination with Floer's Picard Lemma discussed in Sectoin 6.1, for the precise statement see e.g. [3, Lemma 6.17]. □

Lemma 6.18. *For sufficiently small $\lambda > 0$, if a holomorphic disk lies in a sufficiently small finite neighborhood of a generalized disk, then it lies inside a small $\|\cdot\|_{2,\delta}$-neighborhood of w_λ.*

Proof. The proof of this lemma is analogous to the last argument in the proof of [3, Theorem 1.3], so we only sketch it. Fix a generalized disk. It is a consequence of the convergence result Corollary 6.13 that for sufficiently small λ, any holomorphic disk in a finite neighborhood of the generalized disk converges to it as $\lambda \to 0$. Further, the domains of such a sequence of disks can be subdivided into three pieces: a subset of a domain $\Delta_m(\lambda)$ which converges to the domain Δ_m of the limit disk on which the map converges to the limit map, a strip part of length $\mathbf{O}(\lambda^{-1})$ mapping to a small neighborhood of the junction point, and a half infinite strip converging to a Morse flow line. We estimate the $\|\cdot\|_{2,\delta}$-distance by considering these three pieces separately. Over the big-disk part closeness to w_λ in the $\|\cdot\|_{2,\delta}$-norm follows from the convergence result just mentioned using the decomposition of holomorphic functions into cut-off constant solutions and exponentially decaying functions near the marked point puncture. Over the gradient line part, the $\|\cdot\|_{2,\delta}$-norm can be controlled exactly as the corresponding part of a flow tree in [3, Theorem 1.3]. Finally, in the region over the junction point we note that if y_λ is any solution and if u_λ^{jun} is the local solution which was used to build w_λ then $y_\lambda - u_\lambda^{\mathrm{jun}}$ maps into $\mathbb{C}^n$, is holomorphic, and satisfies $\mathbb{R}^n$ boundary conditions. As in the proof of [3, Theorem 1.3], we find that the C^0-distance near the endpoints of the strip region controls the $\|\cdot\|_{2,\delta}$-norm. The C^0-distance goes to 0 by Corollary 6.13. □

Corollary 6.19. *For all sufficiently small $\lambda > 0$, there is a unique rigid holomorphic disk with at most two Morse chords corresponding to each rigid generalized disk.*

Proof. This follows from Corollary 6.17 and Lemma 6.18. □

6.6. **Proof of Theorem 3.6.** After Corollary 6.19, it remains only to deal with disks with two mixed punctures, neither of which are Morse punctures. Consider a mixed puncture of such a disk. At such a puncture, the disk has an incoming and an outgoing sheet. Identify these sheets of $L \cup \tilde{L}$ with the corresponding sheets of L. This gives asymptotic data for a holomorphic disk with boundary on L at the Reeb chord corresponding to the mixed Reeb chord we started with. That asymptotic data may correspond to a positive or a negative puncture. We call it the *induced asymptotic data* at the mixed puncture. There are the following two cases to consider.

- (i) The asymptotic data of one of the mixed punctures corresponds to a positive puncture and that of the other corresponds to a negative puncture.
- (ii) The asymptotic data of both mixed punctures correspond to positive punctures.

To get the correspondence between rigid disks of type (i) and liftings of rigid disks in $\mathcal{M}(a; b_1, \dots, b_k)$, we argue as follows. View the boundary condition for an (i)-disk as a perturbation of the boundary condition for a disk in $\mathcal{M}(a; b_1, \dots, b_k)$. As the latter moduli space is transversely cut out, it follows by Gromov compactness that for $\lambda > 0$ small enough there is a bijective correspondence between rigid (i)-disks and liftings of rigid disks in $\mathcal{M}(a; b_1, \dots, b_k)$.

Fix a small $\lambda > 0$ so that this bijective correspondence exists and so that Corollary 6.19 holds. Assume that the Morse function is sufficiently generic so that Lemma 2.3 holds for $L_0 = L$ and $L_1 = \tilde{L}_\lambda$. Then, by definition, rigid holomorphic disks of type (ii) corresponds to liftings of rigid disks in $\mathcal{M}(a_1, a_2; b_1, \dots, b_k)$. This finishes the proof.

Appendix A. Torsion Framings and the Grading

The goal of this section is to show how a choice of sections of TP over the 3-skeleton of some fixed triangulation of P determines a loop $\mathbb{Z}_g$-framing used for grading in Section 2.2.2. We will prove these results in the slightly more general setting of a rank k complex vector bundle $\eta \to M$, where $k \geq 2$. See Section 2.2.2 for the definitions of the greatest divisor $g(\eta)$ and a $\mathbb{Z}_g$-framing of η along a closed curve $\gamma \subset M$.

Fix a triangulation T of M and let $T^{(j)}$ denote the j-skeleton of T. Fix a Hermitian metric on η.

Lemma A.1. *The complex vector bundle η admits $k-1$ everywhere orthonormal sections over $T^{(3)}$. Any two restrictions of such $(k-1)$-frames over $T^{(2)}$ are homotopic.*

Proof. The first statement follows from the fact that the higher Chern classes of $\eta|_{T^{(3)}}$ vanish as $H^i(T^{(3)}) = 0$ for $i \geq 4$.

We next consider the uniqueness of such frames. Let $V(r, r-1)$ denote the Stiefel manifold of orthonormal $(r-1)$-frames in $\mathbb{C}^r$ and consider the natural fibration

$$V(r-1, r-2) \xrightarrow{\ \iota\ } V(r, r-1) \xrightarrow{\ \pi\ } S^{2r-1},$$

where π is the projection which maps a frame to its first vector and where ι is the inclusion of the fiber. The long exact homotopy sequences of these fibrations show that for $j = 1, 2$ we have

$$\pi_j(V(r, r-1)) \cong \pi_j(V(r-1, r-2)) \cong \ldots \cong \pi_j(V(2,1)) = 0, \quad j = 1, 2.$$

A $(k-1)$-frame in η over $T^{(2)}$ gives a section in the bundle $V(\eta, k-1)|_{T^{(2)}}$, where $V(\eta, k-1)$ is the bundle with fiber $V(k, k-1)$ naturally associated to η. The obstructions to finding a homotopy between two such sections over $T^{(j)}$ lie in

$$H^j\Big(T^{(2)}; \pi_j\big(V(k, k-1)\big)\Big) = 0, \quad j = 1, 2$$

and the lemma follows. $\square$

Pick $k-1$ orthonormal vector fields $(v_1, \ldots, v_{k-1})$ of η over $T^{(3)}$. This induces a decomposition

$$\eta|_{T^{(3)}} = \epsilon_1 \oplus \cdots \oplus \epsilon_{k-1} \oplus L,$$

where ϵ_j, $j = 1, \ldots, k-1$ are trivial and trivialized line bundles and where L is a line bundle. The line bundle L has Chern class $c_1(L) = c_1(\eta) = g(\eta)a$ for some $g(\eta) \geq 0$ and some $a \in H^2(M; \mathbb{Z}) = H^2(T^{(3)}; \mathbb{Z})$. Let K be a line bundle over $T^{(3)}$ with $c_1(K) = a$ and let w be a generic section of K. (Here we say that w is generic if it does not vanish along $T^{(1)}$ and if its 0-set is transverse to $T^{(2)}$.) Since two line bundles are isomorphic if and only if they have the same Chern class, it follows that $L \cong K^{\otimes g(\eta)}$. In particular, $\big(v_1, \ldots, v_{k-1}, w^{g(\eta)}\big)$ gives a framing $\zeta^{(1)}$ of η over $T^{(1)}$.

Lemma A.2. *If $\delta_j \colon S^1 \to T^{(1)}$, $j = 1, \ldots, m$ are curves in $T^{(1)}$ and if $C \colon \Sigma \to T^{(2)}$ is any map of an orientable surface with m boundary components $\partial\Sigma = \partial\Sigma_1 \cup \cdots \cup \partial\Sigma_m$ such that $C|\partial\Sigma_j = \delta_j$, then the obstruction to extending the trivialization $C^*(\zeta^{(1)})$ of $C^*(\eta)$ from $\partial\Sigma$ to Σ is a class $b \in H^2\big(\Sigma, \partial\Sigma; \mathbb{Z}\big)$ which is divisible by $g(\eta)$.*

Proof. The obstruction to finding such a trivialization is equal to the obstruction to extending the section $w^{g(\eta)}$ of $C^*(L)$ over Σ, which equals $g(\eta)$ times the obstruction of extending w. $\square$

The first cohomology group $H^1(M;\mathbb{Z}_g)$ of M acts naturally on $\mathbb{Z}_g$-framings as follows. Let $g \geq 0$ and let γ be a closed curve in M. Assume that γ is equipped with a $\mathbb{Z}_g$-framing Ξ_γ of the complex vector bundle η. If $b \in H^1(M;\mathbb{Z}_g)$, then we define an action of b on the $\mathbb{Z}_g$-framing of γ, $\Xi_\gamma \mapsto \Xi'_\gamma$ as follows. Pick a framing Z_γ of $\eta|_\gamma$ representing Ξ_γ. Then the $\mathbb{Z}_g$-framing Ξ'_γ is represented by any framing Z'_γ with $d(Z_\gamma, Z'_\gamma) = \langle b, [\gamma]\rangle \mod g$, where $\langle a, \beta\rangle$ denotes the homomorphism $H_1(M;\mathbb{Z}) \to \mathbb{Z}_g$ corresponding to the cohomology class a evaluated on the homology class β. Note that in the special case $g = 0$, a $\mathbb{Z}_g$-framing is simply a framing and $H^1(M;\mathbb{Z})$ acts on framings.

Lemma A.3. *Let M be a manifold with a complex vector bundle η. Let $g = g(\eta)$ be the greatest divisor of $c_1(\eta)$, $c_1(\eta) = g \cdot a$. Let γ be any closed curve in M. Then there is an induced $\mathbb{Z}_g$-framing Ξ_γ of γ which is unique up to the action of $H^1(M;\mathbb{Z}_g)$. In particular, the $\mathbb{Z}_g$-framing of any γ which represents a homology class in $H_1(M;\mathbb{Z})$ which generates a subgroup isomorphic to $\mathbb{Z}_m$ where m and g are relatively prime is unique. In the special case $g = 0$, the framing of any curve representing a torsion class is unique.*

Proof. Fix a triangulation T of M. Construct a framing $\zeta^{(1)}$ over $T^{(1)}$ as described above. Pick a homotopy C which connects γ to a curve in $T^{(1)}$. Transporting the trivialization $\zeta^{(1)}$ over C gives a trivialization Z_γ over γ. Let C' be another homotopy inducing another trivialization Z'_γ. Let C'' be a homotopy in $T^{(2)}$ connecting the end-curve of C to the end-curve of C'. Then the cylinder D constructed by joining C to C'' and C'' to C' connects γ to itself. The obstruction to finding a framing over the torus corresponding to D is on the one hand equal to $\langle c_1(\eta), [D]\rangle = g\langle a, [D]\rangle$. On the other hand, this obstruction equals the sum of $d(Z_\gamma, Z'_\gamma)$ and the obstruction $o \in \mathbb{Z}$ to extending the framing $\zeta^{(1)}$ over C''. The obstruction o is divisible by g by Lemma A.2, $o = go'$. Thus

$$d(Z_\gamma, Z'_\gamma) = g\big(\langle a, [D]\rangle - o'\big),$$

and existence of Ξ_γ follows.

We next consider uniqueness. Consider applying the above construction in two ways for a fixed triangulation T. This gives two generic frames $(v_1, \dots, v_{k-1}, w)$ and $(v'_1, \dots, v'_{k-1}, w')$ over $T^{(2)}$. Lemma A.1 implies that there is a homotopy of the $(k-1)$-frames. Note that such a homotopy induces 1-parameter family of bundle isomorphisms on orthonormal complements of the $(k-1)$-frame. Deforming $(v'_1, \dots, v'_{k-1})$ to $(v_1, \dots, v_{k-1})$ thus gives a new generic frame $(v_1, \dots, v_{k-1}, w'')$. The generic frames give the same $\mathbb{Z}_g$-framing of all loops in $T^{(1)}$ provided the corresponding difference classes of the sections w and w'' of the line bundle L are divisible by g. It follows that $H^1(T^{(2)};\mathbb{Z}_g)$ acts transitively on homotopy classes of $\mathbb{Z}_g$-framings of loops in $T^{(1)}$. Thus all loop $\mathbb{Z}_g$-framings obtained from a fixed triangulation form a principal homogeneous space over $H^1(T^{(2)};\mathbb{Z}_g) = H^1(M;\mathbb{Z}_g)$.

Let S be some other triangulation of M. After small perturbation of S we may assume that the triangulations S and T have a common refinement U. Using U we find that there exist trivializations $\zeta_T^{(1)}$ and $\zeta_S^{(1)}$ over the 1-skeleta of T and S, respectively such that the corresponding loop $\mathbb{Z}_g$-framings agree. Since the action of $H^1(M;\mathbb{Z}_g)$ on $\mathbb{Z}_g$-framings is independent of triangulation it follows that the set of loop $\mathbb{Z}_g$-framings is independent of the chosen triangulation. □

Remark A.4. Let M be an orientable manifold. Consider the tangent bundle of the cotangent bundle $T(T^*M)$. After fixing an almost complex structure J on $T(T^*M)$ which is compatible with the standard symplectic form, we consider $T(T^*M)$ as complex vector bundle. Since $T^*M \simeq M$ and since the restriction of $T(T^*M)$ to M is the complexification of a real vector bundle, we see that $c_1(T(T^*M)) = 0$. Furthermore, an orientation of M gives a non-zero section of $\Lambda^{\max}TM$, which gives a non-zero section of $\Lambda^{\max}T(T^*M)$. This, in turn, induces a trivialization over the 3-skeleton of M following the construction above by requiring that the section $v_1 \wedge \cdots \wedge v_{n-1} \wedge w$, where v_j and w give the trivialization $\zeta^{(1)}$ in Lemma A.2, is homotopic to the one induced by the orientation.

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