CONTACT MANIFOLDS

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1. Introduction

Contact geometry has been seen to underly many physical phenomena and be related to many other mathematical structures. Contact structures first appeared in the work of Sophus Lie on partial differential equations. They reappeared in Gibbs' work on thermodynamics, Huygens' work on geometric optics and in Hamiltonian dynamics. More recently contact structures have been seen to have relations with fluid mechanics, Riemannian geometry, low dimensional topology and provide an interesting class of subelliptic operators.

After summarizes the basic definitions, examples and facts concerning contact geometry this article will proceed to discuss the connections between contact geometry and symplectic geometry, Riemannian geometry, complex geometry, analysis and dynamics. The article ends discussing two of the most studied connections with physics: Hamiltonian dynamics and geometric optics. References for other important topics in contact geometry (thermodynamics, fluid dynamics, holomorphic curves and open book decompositions) are provided at the end of the article.

2. Basic definitions and examples

A *hyperplane field* ξ on a manifold M is a codimension one sub-bundle of the tangent bundle TM. Locally, hyperplane fields can always be described as the kernel of a 1-form. In other words, for every point in M there is a neighborhood U and a 1-form α defined on U such that the kernel of the linear map $\alpha_x\colon T_xM \to \mathbb{R}$ is ξ_x for all x in U. The form α is called a local defining form for ξ. A *contact structure* on a $(2n+1)$ dimensional manifold M is a "maximally non-integrable hyperplane field" ξ. The hyperplane field ξ is *maximally non-integrable* if for any (and hence every) locally defining 1-form α for ξ the following equation holds

$$\alpha \wedge (d\alpha)^n \neq 0 \tag{1}$$

(this means the form is pointwise never equal to 0). Geometrically the non-integrability of ξ means that no hypersurface in M can be tangent to ξ along an open subset of the hypersurface. Intuitively this says the hyperplanes "twist too much" to be tangent to hypersurfaces. See Figure 1. The pair

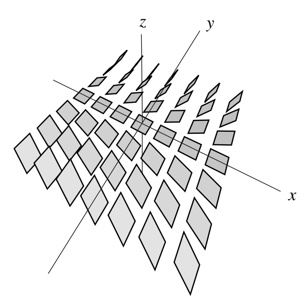

Figure 1. The standard contact structure on $\mathbb{R}^3$ given as the kernel of $dz - y\,dx$. Figure courtesy of Stephan Schönenberger.

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(M, ξ) is called a *contact manifold* and any locally defining form α for ξ is called a *contact form* for ξ.

Example 2.1. The most basic example of a contact structure can be seen on $\mathbb{R}^{2n+1}$ as the kernel of the 1-form $\alpha = dz - \sum_{i=1}^n y_i dx_i$ where the coordinates on $\mathbb{R}^{2n+1}$ are $(x_1, y_1, \ldots, x_n, y_n, z)$. This example is shown in Figure 1 when $n = 1$.

Example 2.2. Recall that on the cotangent space of any n-manifold M there is a canonical 1-form λ, called the *Liouville form.* If $(q_1, \ldots, q_n)$ are local coordinates on M then any 1-from can be expressed as $\sum_{i=1}^n p_i dq_i$ so $(q_1, p_1, \ldots, q_n, p_n)$ are local coordinates on T^*M. In these coordinates

$$\lambda = \sum_{i=1}^n p_i \pi^* dq_i, \tag{2}$$

where $\pi\colon T^*M \to M$ is the natural projection map. The *1-jet space* of M is the manifold $J^1(M) = T^*M \times \mathbb{R}$ and can be thought of as a bundle over M. The 1-jet space has a natural contact structure given as the kernel of $\alpha = dz - \lambda$, where z is the coordinate on $\mathbb{R}$. Note that if $M = \mathbb{R}^n$ then we recover the previous example.

Example 2.3. The *(oriented) projectivized cotangent space* of a manifold M is the set P^*M of non-zero covectors in T^*M where two covectors are identified if they differ by a positive real number, that is

$$P^*M = (T^*M \setminus \{0\})/\mathbb{R}_+ \tag{3}$$

where $\{0\}$ is the zero section of T^*M and $\mathbb{R}_+$ denotes the positive real numbers. If M has a metric then P^*M can easily be identified with the space of unit covectors. Thinking of P^*M as unit covectors we can restrict the canonical 1-from λ to P^*M to get a 1-form α whose kernel defines a contact structure ξ on P^*M. (Though there is no canonical contact form on P^*M the contact structure ξ is still well defined.) Note that if M is compact then so is P^*M so this gives examples of contact structures on compact manifolds.

If α and α' are two locally defining 1-forms for ξ then there is a non-zero function f such that $\alpha' = f\alpha$. Thus $\alpha' \wedge (d\alpha')^n = f^{n+1}\alpha \wedge (d\alpha)^n$ is a non-zero top dimensional form on M and if n is odd then the orientation defined by the local defining form is independent of the actual form. Hence when n is odd a contact structure defines an orientation on M (this is independent of whether or not ξ is orientable!). If M had a preassigned orientation (and n is odd) then the contact structure is called *positive* if it induces the given orientation and *negative* otherwise. One should be careful when reading the literature as some authors build *positive* into their definition of contact structure, especially when $n = 1$. If there is a globally defined 1-form α whose kernel defines ξ then ξ is called *transversally orientable* or *co-orientable.* This is equivalent to the bundle ξ being orientable when n is odd or when n is even and M is orientable. We restrict ourselves to discussing transversely orientable contact structures in the article.

Suppose that α is a contact form for ξ, then Equation (1) implies that $d\alpha|_\xi$ is a symplectic form on ξ. This is one sense in which a contact structure is like an odd dimensional analog of a symplectic structure.

A submanifold L of a contact manifold (M, ξ) is called *Legendrian* if $\dim M = 2\dim L + 1$ and $T_pL \subset \xi_p$.

Example 2.4. A fiber in the unit cotangent bundle with the contact structure form Example 2.3 is a Legendrian sphere.

Example 2.5. Let $f\colon M \to \mathbb{R}$ be a function. Then $j_1(f)(q) = (q, df_q, f(q))$ is a section of the 1-jet space $J^1(M)$ of M, it is called the 1-jet of f. If s is any section of the 1-jet space then it is Legendrian if and only if it is the 1-jet of a function.

This observation is the basis for Lie's study of partial differential equations. More specifically, a first order partial differential equation on M can be thought of as giving an algebraic equation on $J^1(M)$. Then a section of $J^1(M)$ satisfying this algebraic equation corresponds to the 1-jet of a solution to the original partial differential equation if and only if it is Legendrian.

Recently, Legendrian submanifolds have been much studied. There are various classification results in dimension three and several striking existence results in higher dimensions.

3. Local theory

The natural equivalence between contact structures is contactomorphism. Two contact structures ξ_0 and ξ_1 on manifolds M_0 and M_1, respectively, are *contactomorphic* if there is a diffeomorphism $f\colon M_0 \to M_1$ such that $f_*(\xi_0) = \xi_1$. All contact structures are locally contactomorphic. In particular we have the following theorem.

Theorem 3.1 (Darboux's Theorem). *Suppose ξ_i is a contact structure on the manifold $M_i, i = 0, 1$, and M_0 and M_1 have the same dimension. Given any points p_0 and p_1 in M_0 and M_1, respectively, there are neighborhoods N_i of p_i in M_i and a contactomorphism from $(N_0, \xi_0|_{N_0})$ to $(N_1, \xi_1|_{N_1})$. Moreover, if α_i is a contact form for ξ_i near p_i then the contactomorphism can be chosen to pull α_1 back to α_0.*

Thus locally all contact structures (and contact forms!) look like the one given in Example 2.1 above.

Furthermore contact structures are "local in time". That is compact deformations of contact structures do not produce new contact structures.

Theorem 3.2 (Gray's Theorem). *Let M be an oriented $(2n+1)$-dimensional manifold and $\xi_t, t \in [0, 1]$ a family of contact structures on M that agree off of some compact subset of M. Then there is a family of diffeomorphisms $\phi_t\colon M \to M$ such that $(\phi_t)_*\xi_t = \xi_0$.*

In particular, on a compact manifold all deformations of contact structures come from diffeomorphisms of the underlying manifold. The theorem is not true if the contact structures do not agree off of a compact set. For example there is a 1-parameter family of non-contactomorphic contact structures on $S^1 \times D^2$.

4. Existence and classification

The existence of contact structures on closed odd dimensional manifolds is quite difficult. However, Gromov has shown that contact structures on open manifolds obey an h-principle. To explain this we not that if (M^{2n+1}, ξ) is a co-oriented contact manifold then the tangent bundle of M can be written $\xi \oplus \mathbb{R}$ and thus the structure group of TM can be reduces to $U(n)$ (since ξ has a conformal symplectic structure on it). Such a reduction of the structure group is called an *almost contact structure* on M. Clearly a contact structure on M induces an almost contact structure. If M is an open manifold Gromov proved that the inclusion of the space of co-oriented contact structure on M into the space of almost contact structures on M is a weak homotopy equivalence. In particular, if an open manifold meets the necessary algebraic condition for the existence of an almost contact structure then the manifold has a co-oriented contact structure.

Lutz and Martinet proved a similar, but weaker, result for oriented closed 3-manifolds. More specifically, every closed oriented 3-manifold admits a co-oriented contact structure and in fact has at least one for every homotopy class of plane field. There has been much progress on classifying contact structures on 3-manifolds and here an interesting dichotomy has appeared. Contact structures break into one of two types: *tight* or *overtwisted*. Overtwisted contact structures obey an h-principle and are in general easy to understand. Tight contact structures have a more subtle, geometric nature. In higher dimensions there is much less known about the existence (or classification of) contact structures.

5. Relations with symplectic geometry

Let (X, ω) be a symplectic manifold. A vector field v satisfying

$$L_v \, \omega = \omega, \tag{4}$$

where $L_v\omega$ is the Lie derivative of ω in the direction of v, is called a *symplectic dilation.* A compact hypersurface M in (X, ω) is said to have *contact type* if there exists a symplectic dilation v in a neighborhood of M that is transverse to M. Given a hypersurface M in (X, ω) the *characteristic line field* LM in the tangent bundle of M is the symplectic complement of TM in TX. (Since M is codimension one it is coisotropic and thus the symplectic complement lies in TM and is one dimensional.)

Theorem 5.1. *Let M be a compact hypersurface in a symplectic manifold (X, ω) and denote the inclusion map $i : M \to X$. Then M has contact type if and only if there exists a 1-form α on M such that $d\alpha = i^*\omega$ and the form α is never zero on the characteristic line field.*

If M is a hypersurface of contact type, then the 1-form α is obtained by contracting the symplectic dilation v into the symplectic form: $\alpha = \iota_v\omega$. It is easy to verify the 1-form α is a contact from on M. Thus a hypersurface of contact type in a symplectic manifold inherits a co-oriented contact structure.

Given a co-orientable contact manifold (M, ξ) its *symplectization* $\mathrm{Symp}(M, \xi) = (X, \omega)$ is constructed as follows. The manifold $X = M \times (0, \infty)$ and given a global contact form α for ξ the symplectic form is $\omega = d(t\alpha)$, where t is the coordinate on $\mathbb{R}$. (The symplectization is also equivalently defined as $(M \times \mathbb{R}, d(e^t\alpha))$.)

Example 5.2. The symplectization of the standard contact structure on the unit cotangent bundle (see Example 2.3) is the standard symplectic structure on the complement of the zero section in the cotangent bundle.

The symplectization is independent of the choice of contact from α. To see this fix a co-orientation for ξ and note the manifold X can be identified (in may ways) with the subbundle of T^*M whose fiber over $x \in M$ is

$$\{\beta \in T^*_x M : \beta(\xi_x) = 0 \text{ and } \beta > 0 \text{ on vectors positively transverse to } \xi_x\} \tag{5}$$

and restricting $d\lambda$ the this subspace yields a symplectic form ω, where λ is the Liouville form on T^*M defined in Example 2.2. A choice of contact form α fixes an identification of X with the subbundle of T^*M under which $d(t\alpha)$ is taken to $d\lambda$.

The vector field $v = \frac{\partial}{\partial t}$ on (X, ω) is a symplectic dilation that is transverse to $M \times \{1\} \subset X$. Clearly $\iota_v\omega|_{M\times\{1\}} = \alpha$. Thus we see that any co-orientable contact manifold can be realized as a hypersurface of contact type in a symplectic manifold. In summary we have the following theorem.

Theorem 5.3. *If (M, ξ) is a co-oriented contact manifold, then there is a symplectic manifold* $\mathrm{Symp}(M, \xi)$ *in which M sits as a hypersurface of contact type. Moreover, any contact form α for ξ gives an embedding of M into* $\mathrm{Symp}(M, \xi)$ *that realizes M as a hypersurface of contact type.*

We also note that all the hypersurfaces of contact type in (X, ω) look locally, in X, like a contact manifold sitting inside its symplectification.

Theorem 5.4. *Given a compact hypersurface M of contact type in a symplectic manifold (X, ω) with the symplectic dilation given by v there is a neighborhood of M in X symplectomorphic to a neighborhood of $M \times \{1\}$ in* $\mathrm{Symp}(M, \xi)$ *where the symplectization is identified with $M \times (0, \infty)$ using the contact form $\alpha = \iota_v \, \omega|_M$ and $\xi = \ker \alpha$.*

6. The Reeb vector field and Riemannian geometry

Let (M, ξ) be a contact manifold. Associated to a contact form α for ξ is the *Reeb vector field* v_α. This is the unique vector field satisfying

$$\iota_{v_\alpha}\alpha = 1 \quad \text{and} \quad \iota_{v_\alpha}d\alpha = 0. \tag{6}$$

One may readily check that v_α is transverse to the contact hyperplanes and the flow of v_α preserves ξ (in fact, it preserves α). These two conditions characterize Reeb vector fields; that is a vector field v is the Reeb vector field for some contact form for ξ if and only if it is transverse to ξ and its flow preserves ξ.

The fundamental question concerning Reeb vector fields asks if its flow has a (contractible) periodic orbit. A paraphrazing of the *Weinstein conjecture* asserts a positive answer to this question. Most progress on this conjecture has been made in dimension 3 where H. Hofer has proven the existence of periodic orbits for all Reeb fields on S^3 and on 3 manifolds with essential spheres (that is embedded S^2's that do not bound a 3-ball in the manifold). Relations with Hamiltonian dynamics are discussed below.

Recall, from Example 2.3, that a Riemannian metric g on a manifold M provides an identification of the (oriented) projectivized cotangent bundle P^*M with the unit cotangent bundle. Thought of as a subset of T^*M, P^*M inherits not only a contact structure but also a contact form α (by restricting the Liouville form). Let v_α be the associated Reeb vector field. The metric g also provides an identification of the tangent and cotangent bundles of M. Thus P^*M may be thought of as the unit tangent bundle of M. Let w_g be the vector field on the unit tangent bundle generating the geodesic flow on M.

Theorem 6.1. *The Reeb vector field v_α is identified with geodesic flow field w_g when P^*M is identified with the unit tangent space using the metric g.*

7. Relations with complex geometry and analysis

Let X be a complex manifold with boundary and denote the induced complex structure on TX by J. The complex tangencies ξ to $M = \partial X$ are described by the equation $d\phi \circ J = 0$, where ϕ is a function defined in a neighborhood of the boundary such that 0 is a regular value and $\phi^{-1}(0) = M$. The form $L(v, w) = -d(d\phi \circ J)(v, Jw)$, for $v, w \in \xi$, is called the *Levi form*, and when $L(v, w)$ is positive (negative) definite then X is said to have *strictly pseudoconvex (pseudococave) boundary.* The hyperplane field ξ will be a contact structure if and only if $d(d\phi \circ J)$ is a non-degenerate 2-form on ξ (if and only if $L(v, w)$ is definite). A well studied source of examples comes from Stein manifolds.

Example 7.1. Let X be a complex manifold and again let J denote the induced complex structure on TX. From a function $\phi\colon X \to \mathbb{R}$ we can define a 2-form $\omega = -d(d\phi \circ J)$ and a symmetric form $g(v, w) = \omega(v, Jw)$. If this symmetric form is positive definite the function ϕ is called *strictly plurisubharmonic*. The manifold X is a *Stein manifold* if X admits a proper strictly plurisubharmonic function $\phi\colon X \to \mathbb{R}$. An important result says that X is Stein if and only if it can be realized as a closed complex submanifold of $\mathbb{C}^n$. Clearly any non-critical level set of ϕ gives a contact manifold.

Contact manifolds also give rise to an interesting class of differential operators. Specifically, a contact structure ξ on M defines a symbol-filtered algebra of pseudodifferential operators $\Psi^*_\xi(M)$, called the "Heisenberg Calculus". Operators in this algebra are modeled on smooth families of convolution operators on the Heisenberg group. An important class of operators of this type are the "sum of squares" operators. Locally, the highest order part of such an operator takes the form:

$$L = \sum_{j=1}^{2n} v_j^2 + iav_\alpha, \tag{7}$$

where $\{v_1, \ldots, v_{2n}\}$ is a local framing for the contact field and v_α is a Reeb vector field. This operator belongs to $\Psi^2_\xi(M)$ and is subelliptic for a outside a discrete set.

8. Hamiltonian dynamics

Given a symplectic manifold (X, ω) a function $H\colon X \to \mathbb{R}$ will be called a *Hamiltonian*. (Only autonomous Hamiltonians are discussed here.) The unique vector field satisfying

$$\iota_{v_H}\omega = -dH$$

is called the *Hamiltonian vector field* associated to H. Many problems in classical mechanics can be formulated in terms of studying the flow of v_H for various H.

Example 8.1. If $(X, \omega) = (\mathbb{R}^{2n}, d\lambda)$, where λ is form Example 2.2, then the flow of the Hamiltonian vector field is given by

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

A standard fact says that the flow of v_H preserves the level sets of H.

Theorem 8.2. *If M is a level set of H corresponding to a regular value and M is a hypersurface of contact type then the trajectories of v_H and of the Reeb vector field (associated to M in Theorem 5.1) agree.*

Thus under suitable hypothesis Hamiltonian dynamics is a reparametrization of Reeb dynamics. In particular, searching for periodic orbits in such a Hamiltonian systems is equivalent to searching for periodic orbits in a Reeb flow. Thus in this context Weinstein's conjecture asserts a positive answer to the questions: Does the Hamiltonian flow along a regular level set of contact type have a periodic orbit? Viterbo proved the answer was yes if the hypersurface is compact and in $(\mathbb{R}^{2n}, \omega = d\alpha)$. Other progress has been made by studying Reeb dynamics, see the comments on the Weinstein conjecture above.

9. Geometric Optics

In this section we study the propagation of light (or various other disturbances) in a medium (for the moment we do not specify the properties of this medium). The medium will be given by a 3-dimensional manifold M. Given a point p in M and $t > 0$ let $I_p(t)$ be the set of all points to which light can travel in time less than or equal to t. The *wave front* of p at time t is the boundary of this set and is denoted $\Phi_p(t) = \partial I_p(t)$.

Theorem 9.1 (Huygens' Principle)**.** *$\Phi_p(t + t')$ is the envelope of the wave fronts $\Phi_q(t')$ for all $q \in \Phi_p(t)$.*

This is best understood in terms of contact geometry. Let $\pi\colon (T^*M \setminus \{0\}) \to P^*M$ be the natural projection (see Example 2.3) and let S be any smooth subbundle of $T^*M \setminus \{0\}$ that is transverse to the radial vector field in each fiber and for which $\pi|_S\colon S \to P^*M$ is a diffeomorphism. The restriction of the Liouville form to S gives a contact form α and a corresponding Reeb vector field v. Given a subset F of M with a well defined tangent space at every point set

$$L_F = \{p \in S : \pi(p) \in F \text{ and } p(w) = 0 \text{ for all } w \in T_{\pi(p)}F\}. \tag{8}$$

The set L_F is a Legendrian submanifold of S and is called the *Legendrian lift* of F. If L is a generic Legendrian submanifold in S then $\pi(L)$ is called the *front projection* of L and $L_{\pi(L)} = L$. Given a Legendrian submanifold L let $\Psi_t(L)$ be the Legendrian submanifold obtained from L by flowing along v for time t.

Example 9.2. Given a metric g on M, Fermat principle says light travels along geodesics. Thus if S is the unit cotangent bundle then using g to identify the geodesic flow with the Reeb flow one sees that light will travel along trajectories of the Reeb vector field. Given a point p in M the Legendrian submanifold L_p is a sphere sitting in T_p^*M. Huygens Principle follows from the observation that $\Phi_p(t) = \pi(\Psi_t(L_p))$.

Using the more general S discussed above one can generalize this example to light flowing in a nonhomogeneous (*i.e.* the speed differs from point to point in M) and anisotropic (*i.e.* the speed differs depending on the direction of travel) media.

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