

PLANAR OPEN BOOK DECOMPOSITIONS AND CONTACT STRUCTURES

JOHN B. ETNYRE

ABSTRACT. In this note we observe that while all overtwisted contact structures on compact 3-manifolds are supported by planar open book decompositions, not all contact structures are. This has relevance to the Weinstein conjecture [1] and invariants of contact structures.

1. INTRODUCTION

In [14], Emmanuel Giroux showed that all contact structures on compact 3-manifolds come from open book decompositions via the Thurston-Winkelnkemper [27] construction. That is, given any contact structure ξ on M there is an open book decomposition of M such that ξ is transverse to the binding of the open book and can be isotoped arbitrarily close to the pages. This fundamental breakthrough has provided the basis for a much greater understanding of contact structures, see [4, 8, 9, 25], and 3-manifold topology, see [20, 24].

A first obvious question concerning the connection between open book decompositions and contact structures is what is the minimal genus of a page of an open book decomposition that supports a given contact structure. It is interesting to observe that *any* 3-manifold has an open book decomposition with planar pages (that is, the pages are S^2 with a finite number of disjoint open disks removed) [26]. Even given this it seems unlikely that every contact structure can be supported by a planar open book. However, for overtwisted contact structures we have:

Theorem 3.5. *Any overtwisted contact structure on a closed 3-manifold is supported by a planar open book decomposition.* \square

Thus there is no homotopy theoretic obstruction to a contact structure admitting a planar open book, but there are nonetheless obstructions. To state them we first recall that given a 4-manifold with boundary X we can consider its intersection form q_X on $H_2(X, \mathbb{Z})$. We denote by b_2^\pm the maximal dimension of a subspace of $H_2(X, \mathbb{Z})$ on which q_X is \pm definite and we denote $b_2^0(X)$ the dimension of the subspace on which q_X is degenerate.

Theorem 4.1. *If X is a symplectic filling of a contact 3-manifold (M, ξ) supported by a planar open book decomposition then $b_2^+(X) = b_2^0(X) = 0$ and the boundary of X is connected. Moreover, if M is an integral homology sphere then the intersection form on X is diagonalizable.* \square

We now give several simple applications of this theorem.

Key words and phrases. open book decompositions, contact structure, planar.

Examples 1.1. Let (M, ξ) be a contact manifold obtained by performing Legendrian surgery on a Legendrian link in S^3 with one component having Thurston–Bennequin invariant larger than 0, then (M, ξ) is not supported by an planar open book decomposition.

Examples 1.2. In [21], McDuff gave the first examples of a symplectic manifold (X, ω) with two convex boundary components. For example, take the unit disk bundle in the cotangent bundle of a surface of high genus, perturb the symplectic form so that the 0-section is symplectic, and let (X, ω) be the complement of a symplectic neighborhood of this perturbed section. It is not hard to check that both boundary components of X are convex. Let (M, ξ) be either boundary component (X, ω) then (M, ξ) cannot be supported by a planar open book.

Examples 1.3. Let (X, ω) be the symplectic manifold obtained by plumbing together symplectic disk bundles over S^2 with Euler number -2 according to the $-E_8$ graph. The boundary of X is convex and topologically the Poincaré homology sphere P . Let ξ be the tight contact structure induced on P by (X, ω) . According to Theorem 4.1, (P, ξ) cannot be supported by a planar open book decomposition since the intersection form of X is non-diagonalizable.

It would be very interesting to find obstructions to a contact structure being supported by a planar open book that were more intrinsically 3-dimensional. In particular, an obstruction that could be used to show some non-fillable contact structures are not supported by planar open books. It would also be interesting to find bounds on the minimal genus of a supporting open book for a contact structure. It is still conceivable that all contact structures are supported by open book decompositions with genus one pages.

Given a contact form α for the contact structure ξ on M the *Reeb vector field* v_α is the unique vector field satisfying $\alpha(v_\alpha) = 1$ and $\iota_{v_\alpha} d\alpha = 0$. Recall the well known (extension of the) Weinstein conjecture [28] asserts any Reeb vector field v_α for ξ has a closed periodic orbit. This conjecture was proven of any contact structure on S^3 or on any reducible 3-manifold by Hofer in [18]. That paper also establishes the conjecture for all contact structure that are virtually overtwisted (that is have a finite cover that is overtwisted). Abbas, Cieliebak and Hofer have a program for proving the Weinstein conjecture based on open book decompositions. Currently [1] they can prove the Weinstein conjecture is true for any contact structure supported by a planar open book decomposition. While this greatly enlarges the class of contact structures for which the Weinstein conjecture is known the above examples and Theorem 4.1 show there are contact structures for which the conjecture is still not yet known.

In Section 2 we recall the necessary background about open book decompositions, contact surgeries and symplectic surface bundles. In the following section we establish Theorem 3.5 concerning open books for overtwisted contact structures. In Section 4 we derive obstructions for a contact structure to be supported by a planar open book.

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2. OPEN BOOK DECOMPOSITIONS AND CONTACT STRUCTURES

In this section we recall various facts about open book decompositions, contact surgeries and symplectic surface bundles.

2.1. Open book decompositions and contact structures. Recall an *open book decomposition* of a 3-manifold M is a triple (B, Σ, ϕ) where B is a link in M such that $M \setminus B$ fibers over the circle with fiber Σ and monodromy ϕ so that ϕ is the identity near the boundary and each fiber of the fibration is a Seifert surface for B . By saying ϕ is the monodromy of the fibration we mean that $M \setminus B = \Sigma \times [0, 1] / \sim$ where $(1, x) \sim (0, \phi(x))$. The fibers in the fibration are called *pages* of the open book and B is called the *binding*. Note, given a diffeomorphism of a surface ϕ that is fixed near the boundary we may form its mapping torus and glue in solid tori to get a closed three manifold having an open book decomposition ϕ . So sometimes we will designate an open book decomposition simply by ϕ without reference to the binding B . (The main difference here is whether we are thinking of the open book as inside a pre-existing 3-manifold or whether we are defining the 3-manifold by the open book.)

Two open books for a manifold M are said to be *equivalent* if there is an ambient isotopy of M taking the binding and pages of one to the binding and pages of the other. Given an open book (B, Σ, ϕ) for M , let Σ' be Σ with a 1-handle attached. Let α be a simple closed curve in Σ' that intersects the co-core of the attached 1-handle exactly once. Set $\phi' = \phi \circ D_\alpha^\pm$, where D_α^\pm is a right/left handed Dehn twist along α . The mapping torus of ϕ' has torus boundary components each with a canonical product structure $S^1 \times S^1$ where the first S^1 bounds the fibers in the fibration of the mapping torus. Let M' be the mapping torus with $S^1 \times D^2$ glued to each boundary component so that the product structure (and ordering of the S^1 factors) is preserved. Let B' be the cores of the added tori. One may check that M' is diffeomorphic to M and thus (B', Σ', ϕ') is another open book for M . If D_α^+ was used (B', Σ', ϕ') is called the *positive stabilization* of (B, ϕ) otherwise it is called the *negative stabilization*.

A contact structure ξ on M is *compatible with*, or *supported by* an open book (B, Σ, ϕ) of M if B is transverse to ξ and on the complement of B the contact planes ξ can be isotoped to be arbitrarily close to the pages of the open book (while keeping B transverse). Thurston and Winkelnkemper [27] showed that any open book supports a contact structure. It is not too hard to see that if two contact structures are supported by a fixed open book then they are isotopic as contact structures. One may also check that if (B, Σ, ϕ) is compatible with a contact structure then so is any open book obtained from (B, Σ, ϕ) by *positive stabilization*. Giroux made the following fundamental observation (for a discussion of the proof see [7, 16]).

Theorem 2.1 (Giroux [14]). *Every contact structure is supported by an open book. Moreover, if two open books support the same contact structure then they each may be positively stabilized some number of times so that the open books are equivalent.*

Suppose (B_1, Σ_1, ϕ_1) and (B_2, Σ_2, ϕ_2) are open book decompositions supporting the contact manifolds (M_1, ξ_1) and (M_2, ξ_2) , respectively. Let α_i a properly embedded arc in Σ_i . The Murasugi sum of (B_1, Σ_1, ϕ_1) and (B_2, Σ_2, ϕ_2) is obtained as follows: let $\Sigma_1 * \Sigma_2$ be the surface $\Sigma_1 \cup_{R_1=R_2} \Sigma_2$ where R_i is a rectangular neighborhood of α_i in Σ_i and R_1 is identified to R_2 in such a way that $\partial R_1 \cap \partial \Sigma_1 = \overline{\partial R_2} \setminus \partial \Sigma_2$. Each of the ϕ_i 's can be extended to $\Sigma_1 * \Sigma_2$. The Murasugi sum is now defined to be the open book obtained from $\phi_1 \circ \phi_2$.

Lemma 2.2. *If (B_1, Σ_1, ϕ_1) and (B_2, Σ_2, ϕ_2) are open book decompositions supporting the contact manifolds (M_1, ξ_1) and (M_2, ξ_2) , respectively, then the Murasugi sum of (B_1, Σ_1, ϕ_1) and (B_2, Σ_2, ϕ_2) is an open book decomposition supporting the contact manifold $(M_1, \xi_1) \# (M_2, \xi_2)$.*

2.2. Contact Surgeries. Let (M, ξ) be a contact manifold and $L \subset M$ a closed Legendrian curve. Let $N(L)$ be a *standard tubular neighborhood* of the Legendrian curve L . This means the neighborhood has convex boundary and two parallel dividing curves (see [10]). Choose a framing for L so that the meridian has slope 0 and the dividing curves have slope ∞ . With respect to this choice of framing, a ± 1 *contact surgery* is a ± 1 Dehn surgery, where a copy of $N(L)$ is glued to $M \setminus N(L)$ so that the new meridian has slope ± 1 . Even though the boundary characteristic foliations may not exactly match up a priori, we may use Giroux's Flexibility Theorem [12, 19] and the fact that they have the same dividing set to make the characteristic foliations agree. This gives us a new manifold (M', ξ') . It is common to call -1 contact surgery *Legendrian surgery*. For a detailed discussion of contact surgery see [2]. The following is a well known theorem, see for example [9].

Theorem 2.3. *Suppose the L is a Legendrian knot in the contact manifold (M, ξ) , ξ is supported by the open book (B, ϕ) and L is contained in a page of the open book. The contact manifold obtained from (M, ξ) by ± 1 contact surgery on L is equivalent to the one compatible with the open book with monodromy $\phi \circ D_\alpha^\mp$.*

2.3. Symplectic surface bundles. In this paper a *symplectic fibration over S^1* will mean a 3-manifold M that fibers over the circle together with a closed 2-form ω which is positive on each fiber. The kernel of ω defines a line field l that is transverse to the fibers of the fibration. An orientation on M and on the fibers induces an orientation on l and thus we can fix a fiber Σ_0 of the fibration and use l to define a return map $H_{(M, \omega)}: \Sigma_0 \rightarrow \Sigma_0$ called the *holonomy* of the symplectic fibration. If we normalize ω so that it integrates to 1 on each fiber of the fibration then the holonomy determines (M, ω) up to fiber preserving diffeomorphism. It is also important to notice that the holonomy determines a symplectic neighborhood of (M, ω) or more precisely we have the following lemma.

Lemma 2.4. *Suppose (X_i, ω_i) is a symplectic 4-manifold and $f_i: M \rightarrow X_i$ is an embedding of a fibered 3-manifold into X_i for $i = 0, 1$. Suppose $f_i^* \omega_i$ define symplectic fibrations on M with the same holonomy. Then $f_0(M)$ and $f_1(M)$ have symplectomorphic neighborhoods.*

We will also find the following lemma useful.

Lemma 2.5. *Suppose the holonomy of the symplectic fibration (M, ω) is a Hamiltonian diffeomorphism. Then there is a symplectic form Ω on $X = \Sigma \times D^2$, such that $\partial(X, \Omega) = (M, \omega)$, where Σ is the fiber of the fibration.*

We now have the fundamental result of Eliashberg [4].

Theorem 2.6. *Suppose (M, ξ) is a contact 3-manifold and ω is a closed 2-form on M that is positive on ξ . Furthermore suppose (B, ϕ) is an open book supporting ξ . Use the pages of the open book to define a framing on the components of the binding B . Let X be the 4-manifold obtained from $M \times [0, 1]$ by attaching 2-handles to $M \times \{1\}$ along B with framing 0. The manifold X is oriented so that $\partial X = (-M) \cup M'$, where M' is the fibered three manifold obtained by 0-surgery on B in M . Then X admits a symplectic form Ω such that $\Omega|_M = \omega$ and $\Omega|_{M'}$ defines a symplectic fibration on M' .*

3. OVERTWISTED CONTACT STRUCTURES

In order to prove all overtwisted contact structures are supported by open books we need a few preliminary observations.

Lemma 3.1. *Suppose (B, ϕ) is an open book for the 3-manifold M . Let $\alpha_1, \dots, \alpha_k$ be curves on a page of the open book that generate the homology of the page. Then the homology of M is generated by $\alpha_1, \dots, \alpha_k$. \square*

The proof of this lemma is an easy exercise that is left to the reader. Recall that given an oriented Legendrian knot L then there is a positive L_+ and negative L_- transverse push off, [5, 10]. Moreover, there is a positive $S_+(L)$ and negative $S_-(L)$ stabilization of L , defined by “adding down or up zig zags”. (This is completely accurate for Legendrian knots in the standard contact structure on \mathbb{R}^3 represented by their front projection. For Legendrian knots in a more general contact structure one must take a little more care in the definition [10].)

Proposition 3.2 (Ding, Geiges and Stipsicz [3]). *Let L be a Legendrian knot in a contact 3-manifold (M, ξ) . Let L' be a parallel copy of L that has been stabilized positively (respectively negatively) twice. Let T be L_+ (respectively L_-). The contact structure obtained from ξ by performing a Lutz twist along T is isotopic to the one obtained from ξ by performing +1-contact surgery along both L and L' .*

The proposition was first hinted at in Proposition 4.3 of [9]. It was explicitly stated and given a nice proof by Ding, Geiges and Stipsicz [3]. Here we sketch a somewhat different proof.

Sketch of Proof. Note that both surgeries take place in a solid torus. So we will restrict our attention to a neighborhood of L . Let N be a standard neighborhood of a Legendrian knot L . That is ∂N is convex with two dividing curves of slope ∞ . Given that ξ restricted to N is tight, this uniquely determines a contact structure on N (see [13, 19]). We can write N as $N_1 \cup N'$ where N_1 is a solid torus with convex boundary having two dividing curves of slope $-\frac{1}{2}$ and $N' = \overline{N \setminus N_1}$. There are actually three ways of splitting N like this. They come from stabilizing the core Legendrian knot in L twice positively, twice negatively or once positively and once negatively. See [10]. Now split N' into two pieces $N_2 \cup N_3$ where N_3 is a vertically invariant neighborhood of ∂N that is contained in N' and N_2 is the complement of this neighborhood. One may easily check that a Legendrian divide on $N_3 \cap N_2$ is Legendrian isotopic to L . Also N_1 is a standard neighborhood of a Legendrian knot obtained from L by stabilizing twice. That is, with the appropriate choice of N_1 , N_1 is a standard neighborhood of L' . For this and the following facts about contact structures on toric annuli and Legendrian knots see [10].

We can identify slopes $s \in \mathbb{R} \cup \{\infty\}$ of linear foliations on T^2 with their respective angles, $[\theta_s] \in \mathbb{R}/\pi\mathbb{Z}$. In order to distinguish the different amounts of twisting, we will choose a lift $\theta_s \in \mathbb{R}$ instead. There exists an exhaustion of N by concentric T^2 with linear foliations where the angles of the foliation on the tori monotonically increase over the interval $(\frac{\pi}{2}, \pi)$ as the T^2 move towards the core of N .

We can think of L as sitting in N_3 as a Legendrian divide on a convex torus in N_3 . One may check that performing $+1$ -surgery on L gives a tight minimally twisting contact structure on N_3 . Moreover, the dividing curves on both boundary components still have infinite slope. (The product structure on the $T^2 \times I$ has changed though.) The contact structure, after surgery on L , on $N = N_1 \cup N_2 \cup N_3$ is tight and we can find concentric T^2 where the characteristic foliation are linear and run over the interval $(\frac{\pi}{8}, \pi)$. Performing $+1$ -surgery on L' yields a contact structure on N with twisting from $(-\frac{\pi}{2}, \pi)$. Thus the result of performing the $+1$ -contact surgeries on L and L' is equivalent to replacing the contact structure on N that twists over $(\frac{\pi}{2}, \pi)$ by one that twists over $(-\frac{\pi}{2}, \pi)$. But this is exactly the change that happens when one does a Lutz twist on a transverse push off of the core of N . \square

In our discussion below it will be useful to see how to stabilize a Legendrian knot on a page of an open book so that the stabilized knot is also on a page of an open book.

Lemma 3.3. *Let (B, Σ, ϕ) be an open book decomposition supporting the contact structure ξ on M . Suppose L is a Legendrian knot in M that lies on a page of the open book. If we positively stabilize (B, Σ, ϕ) twice as shown in Figure 1 then we may isotop the page of the open book so that $S_+(L)$ and $S_-(L)$ appear on the page as seen in Figure 1.*

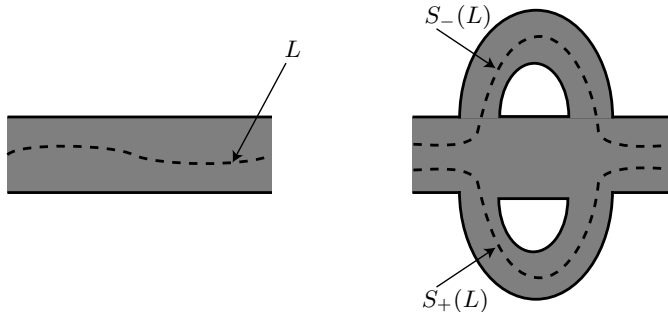


FIGURE 1. A neighborhood of a piece of L in Σ , left. (L is oriented so it points towards the left.) The twice stabilized open book, right.

This Lemma is relatively easy to prove, see [7].

We first prove our main theorem for the special case of $M = S^3$.

Lemma 3.4. *Any overtwisted contact structure on S^3 is supported by a planar open book decomposition.*

Proof. We simply need to exhibit a planar open book for S^3 realizing each homotopy class of plane field. Using the notation of [2] (cf [15]) the homotopy classes of plane

fields correspond to half-integers. That is given a plane field ξ on S^3 let (X, J) be an almost complex 4-manifold with $\partial X = S^3$ and ξ the set of complex tangencies to the boundary. Then we associate the invariant

$$d_3 = \frac{1}{4}(c_1(X, J)^2 - 3\sigma(X) - 2\chi(X)).$$

One may compute (see [2]) that the indicated contact surgeries in Figure 2 represent two overtwisted contact structures on S^3 with d_3 equal to $\frac{1}{2}$ and $-\frac{3}{2}$, respectively. From Lemma 3.3 one may easily see that the two contact structures in Figure 2

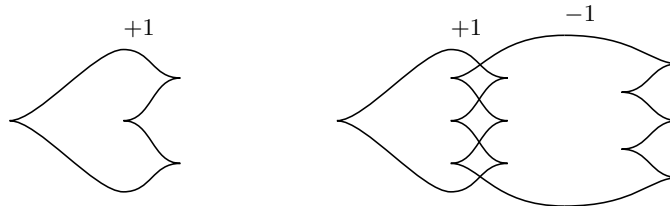


FIGURE 2. Surgery diagrams for overtwisted contact structures with $d_3 = \frac{1}{2}$, left, and $d_3 = -\frac{3}{2}$, right.

are supported by the open books shown in Figure 3. Note these are both planar

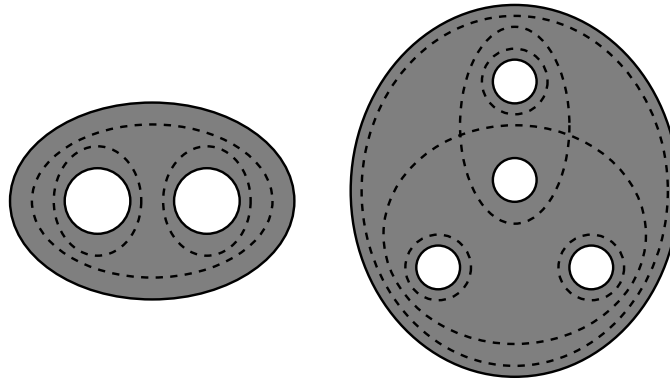


FIGURE 3. The fibers of open book decompositions for the contact structures in Figure 2. The monodromies come from Dehn twists about the dotted curves in the picture. All Dehn twists are right handed except for the Dehn twist about the outer most curve in both pictures which is left handed.

open books. The invariant d_3 is additive under contact connected sum, so by taking the connected sum of these examples we can realize any overtwisted contact structure on S^3 . These connected sums can be obtained by Murasugi summing the corresponding open books (see Lemma 2.2). Since Murasugi sums can be performed in a way so that the genus of the fibers adds, we see that we have planar open books supporting all overtwisted contact structures on S^3 . \square

We are now ready for the main theorem of this section.

Theorem 3.5. *Let (M, ξ) be an overtwisted contact 3-manifold. Then there is an open book (B, ϕ) for M that supports ξ and has planar pages.*

Proof. Two plane field are homotopic if and only if their $(\frac{1}{2})$ Euler classes d_2 and their three dimensional invariants d_3 are the same (see [11] for a description of this using the notation we use here, also see [15]). We show that all such invariants can be realized on a manifold by overtwisted contact structures supported by planar open books. Thus Eliashberg [6] tells us that all overtwisted contact structures will then be realized by planar open books.

It is well known, see [26], that any three manifold M has an open book with planar leaves (B, Σ, ϕ) . The associate contact structure ξ may or may not be overtwisted, but all overtwisted contact structures can be obtained from it. We begin by showing that any possible d_2 can be realized by an overtwisted contact structure supported by a planar open book. To this end let $\alpha_1, \dots, \alpha_k$ be the simple closed curves on Σ that generate the homology of the page. We know that the α_i generate the homology of M by Lemma 3.1. If we orient α_i , make it transverse (respecting this orientation) and perform a Lutz twist on α_i then one may compute that the difference between d_2 for the two contact structures is Poincaré dual to the homology class given by α_i . Thus by performing Lutz twists along the α_i 's with various orientation (and transverse realizations) we can realize any possible d_2 by an overtwisted contact structure. Using Proposition 3.2 we may obtain all invariants d_2 by overtwisted contact structures by performing various contact surgeries on Legendrian realizations of the α_i 's and their stabilizations. We can Legendrian realize any α_i on the page of our open book. To realize all the possible stabilization of α_i that we will need we must positively stabilize the open book four times for each α_i . (Two stabilizations of the open book to realize $S_+^2(\alpha_i)$ and two to realize $S_-^2(\alpha_i)$.) We can do this keeping the open book planar. Now all the Legendrian knots on which we need to do contact surgeries can be Legendrian realized on the pages of the open book. Performing a ± 1 contact surgery on one of these Legendrian knots will yield a new open book whose monodromy is the old monodromy composed with a \mp Dehn twist along the knot in the page. Moreover, this open book supports the contact structure obtained from ξ by the contact surgeries which were performed. Thus we have realized all possible invariants d_2 by overtwisted contact structures supported by planar open books.

Now recall that the invariant d_3 is additive under contact connected sum. Thus to realized any pair (d_2, d_3) by an overtwisted contact structure supported by a planar open book we first take (M, ξ') realizing the appropriate d_2 invariant. Then there is a contact structure ξ'' on S^3 such that $\xi' \# \xi''$ on M realizes (d_2, d_3) . Since ξ'' and ξ' are both supported by planar open books, we can perform the Murasugi sum in such a way that the resulting open book is also planar. Thus we have a planar open book supporting the contact structure $\xi' \# \xi''$. \square

4. THE GENUS OF OPEN BOOKS

Theorem 4.1. *If X is a symplectic filling of a contact 3-manifold (M, ξ) supported by a planar open book decomposition then $b_2^+(X) = b_2^0(X) = 0$ and the boundary of X is connected. Moreover, if M is an integral homology sphere then the intersection form on X is diagonalizable.*

Proof. Let (X, ω) be a weak filling of (M, ξ) and assume ξ is supported by the planar open book (B, Σ, ϕ) . For now assume the boundary of X is connected. Theorem 2.6

and Lemma 2.4 say we may attach 2–handles to X along $B \subset M$, with 0 framing measured with respect to the fibers of the open book, so that the new 4–manifold X' has a symplectic structure ω' such that $\omega'|_X = \omega$ and the new boundary component of X' is $M' = S^2 \times S^1$ and $S^2 \times \{p\}$ is symplectic for each $p \in S^1$. That is ω' gives $S^2 \times S^1$ the structure of a symplectic fibration.

Let C be $S^2 \times D^2$ and ω_c the symplectic form on C coming from Lemma 2.5 such that $\partial(C, \omega_c) = -(M', \omega'|_{M'})$. (Note we can apply the lemma since any symplectomorphism of S^2 is Hamiltonian.) Thus using Lemma 2.4 we can construct the closed symplectic 4–manifold $(W, \Omega) = (M', \omega') \cup (C, \omega_c)$. Clearly $S = S^2 \times \{p\}$ is a symplectic sphere in W with square 0. Thus a theorem of McDuff [22] implies that (W, Ω) is the blow up of a ruled surface. The cocore of a 2–handle H attached to X in forming X' can be glued to the $\{q\} \times D^2$ in C to form a topological sphere S' with square k and intersecting S geometrically one time. If B_1 is the component of the binding B that H is attached to then k depends on the Dehn twists in the monodromy that are parallel to B_1 . Thus by adding positive Dehn twists to the monodromy if necessary we can assume that k is even. Now there is a sphere S'' in the homology class $S' - \frac{k}{2}S$ that has self intersection 0 and intersects S exactly one time. Thus a neighborhood N of $S \cup S''$ is a punctured $S^2 \times S^2$ in W . McDuff’s result now implies that $W = S^2 \times S^2 \#_n \overline{\mathbb{C}P}^2$. Moreover $X \subset W$ is disjoint from N so we may embed X in $\#_n \overline{\mathbb{C}P}^2$ and since this manifold has a negative definite, diagonalizable intersection form we see that $b_2^+(X) = b_2^0(X) = 0$ and if M is a homology sphere then the intersection form on X is diagonalizable.

Now if the boundary of X was not connected then we can find symplectic caps for the other boundary components [4, 8]. That is, for each extra boundary component M_i of X we can construct a symplectic manifold (C_i, ω_i) such that $\partial C_i = -M_i$ and (C_i, ω_i) may be glued to (X, ω) along M_i to form a new closed symplectic manifold with one less boundary components. Moreover, examining the construction in [8] one can easily arrange that $b_2^+(C_i) > 0$. Thus after capping off the extra boundary components of (X, ω) we have a filling of (M, ξ) with one boundary component and $b_2^+ > 0$ contradicting the first part of our proof. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 SOUTH 33RD ST., PHILADELPHIA, PA 19104-6395

E-mail address: etnyre@math.upenn.edu

URL: <http://math.upenn.edu/~etnyre>