OPEN BOOKS AND PLUMBINGS

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ABSTRACT. We construct, somewhat non-standard, Legendrian surgery diagrams for some Stein fillable contact structures on some plumbing trees of circle bundles over spheres. We then show how to put such a surgery diagram on the pages of an open book for S^3, with relatively low genus. Thus we produce open books with low genus pages supporting these Stein fillable contact structures, and in many cases it can be shown that these open books have minimal genus pages.

1. INTRODUCTION

A closed and oriented 3-manifold Y which is described by a plumbing tree Γ of oriented circle bundles over S^2, all having Euler numbers less than or equal to -2, admits many Stein fillable contact structures. A Kirby diagram of Y is given by a collection of unknotted circles in S^3 corresponding to the vertices of Γ, linked with respect to the edges of Γ so that the smooth framing of a circle in the diagram is exactly the Euler number $n_i \leq -2$ corresponding to the circle bundle it represents. To find Stein fillable contact structures on Y one can simply put all the circles into Legendrian position (with respect to the standard contact structure in S^3) in such a way that the contact framing, i.e., the Thurston-Bennequin number $tb(K_i)$ of a circle K_i is given by $n_i + 1$. Then by applying Legendrian surgery on these Legendrian circles in S^3 we get a Stein fillable contact structure on Y. Note that, the freedom to Legendrian realize each K_i with different rotation numbers (but fixing $tb(K_i) = n_i + 1$) will enable us to find different Stein fillable contact structures on Y.

We will call a plumbing tree "non-positive" if $d_i + n_i \leq 0$ for every vertex i, where d_i denotes the degree of the i-th vertex. We will refer to a vertex in a tree with $d_i + n_i > 0$ as a bad vertex. A planar open book supporting the contact structure obtained by a Legendrian realization of a non-positive plumbing tree was presented in [9]. In this article we will generalize the methods in [9] to find an open book supporting the contact structure obtained by a Legendrian realization of a plumbing tree which is not necessarily non-positive. The genus of the open book we will construct for a tree Γ is given by a number $g(\Gamma)$, which we

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define in Section 3. As a preliminary step in the construction of the open books we first derive special Legendrian surgery diagrams for Y in Section 3. In the following section we show how to realize these Legendrian surgery diagrams onto the pages of an open book for S^3. Thus after Legendrian surgery we have an open book supporting the desired contact structure. We also discuss how to apply these ideas to more general contact surgery diagrams. These constructions lead to open books decompositions supporting all tight contact structures on small Seifert fibered spaces with $e_0 \neq -2, -1$ having page genus zero or one. In the last section we exhibit various examples of our construction.

2. Open books and contact structures

Suppose that for an oriented link L in a closed and oriented 3–manifold Y the complement $Y \setminus L$ fibers over the circle as $\pi\colon Y \setminus L \to S^1$ such that $\pi^{-1}(\theta) = \Sigma_\theta$ is the interior of a compact surface bounding L, for all $\theta \in S^1$. Then (L, π) is called an *open book decomposition* (or just an *open book*) of Y. For each $\theta \in S^1$, the surface Σ_θ is called a *page*, while L the *binding* of the open book. The monodromy of the fibration π is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class of this diffeomorphism is independent of the chosen flow and we will refer to that as the *monodromy* of the open book decomposition.

An open book (L, π) on a 3–manifold Y is said to be *isomorphic* to an open book (L', π') on a 3–manifold Y', if there is a diffeomorphism $f : (Y, L) \to (Y', L')$ such that $\pi' \circ f = \pi$ on $Y \setminus L$. In other words, an isomorphism of open books takes binding to binding and pages to pages.

An open book can also be described as follows. First consider the mapping torus

$$\Sigma_\phi = [0, 1] \times \Sigma/(1, x) \sim (0, \phi(x))$$

where Σ is a compact oriented surface with r boundary components and ϕ is an element of the mapping class group Γ_Σ of Σ. Since ϕ is the identity map on $\partial\Sigma$, the boundary $\partial\Sigma_\phi$ of the mapping torus Σ_ϕ can be canonically identified with r copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0, 1]/(0 \sim 1)$ and the second one comes from a component of $\partial\Sigma$. Now we glue in r copies of $D^2 \times S^1$ to cap off Σ_ϕ so that ∂D^2 is identified with $S^1 = [0, 1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of $\partial\Sigma$. Thus we get a closed 3-manifold $Y = \Sigma_\phi \cup_r D^2 \times S^1$ equipped with an open book decomposition whose binding is the union of the core circles $D^2 \times S^1$'s that we glue to Σ_ϕ to obtain Y. In conclusion, an element $\phi \in \Gamma_\Sigma$ determines a 3-manifold together with an "abstract" open book decomposition on it. Notice that by conjugating the monodromy ϕ of an open book on a 3-manifold Y by an element in Γ_Σ we get an isomorphic open book on a 3-manifold Y' which is diffeomorphic to Y.

It has been known for a long time that every closed and oriented 3–manifold admits an open book decomposition. Our interest in finding open books on 3-manifolds arises from their connection to contact structures, which we will describe very briefly. We will assume throughout this paper that a contact structure $\xi = \ker \alpha$ is coorientable (i.e., α is a global 1–form) and positive (i.e., $\alpha \wedge d\alpha > 0$).

Definition 2.1. *An open book decomposition (L, π) of a 3–manifold Y* supports *a contact structure ξ on Y if ξ can be represented by a contact form α such that $\alpha(L) > 0$ and $d\alpha > 0$ is on every page.*

In [10], Thurston and Winkelnkemper show that every open book supports a contact structure.

Suppose that an open book decomposition with page Σ is specified by $\phi \in \Gamma_\Sigma$. Attach a 1-handle to the surface Σ connecting two points on $\partial\Sigma$ to obtain a new surface Σ'. Let γ be a closed curve in Σ' going over the new 1-handle exactly once. Define a new open book decomposition with $\phi' = \phi \circ t_\gamma \in \Gamma_{\Sigma'}$, where t_γ denotes the right-handed Dehn twist along γ. The resulting open book decomposition is called a *positive stabilization* of the one defined by ϕ. If we use a left-handed Dehn twist instead then we call the result a *negative stabilization*. The inverse of the above process is called *positive* (*negative*) *destabilization*. Notice that although the resulting monodromy depends on the chosen curve γ, the 3–manifold specified by (Σ', ϕ') is diffeomorphic to the 3–manifold specified by (Σ, ϕ).

A converse to the Thurston-Winkelnkemper result is given by

Theorem 2.2 (Giroux [6])**.** *Every contact 3–manifold is supported by an open book. Two open books supporting the same contact structure admit a common positive stabilization. Moreover two contact structures supported by the same open book are isotopic.*

We refer the reader to [2] and [8] for more on the correspondence between open books and contact structures.

3. Legendrian surgeries and plumbings

We assume that all the circle bundles we consider are oriented with Euler numbers less than or equal to -2. We will call a plumbing tree of circle bundles over S^2 *non-positive* if the sum of the degree of the vertex and the Euler number of the bundle corresponding to that vertex is non-positive for every vertex of the tree. In this section we describe Legendrian surgery diagrams of some contact structures on plumbings of circle bundles over S^2 according to trees which are not necessarily non-positive. These surgery diagrams will be transformed into open books in the following section.

Let us denote a circle bundle over S^2 with Euler number n by Y_n. Given a plumbing tree Γ of circle bundles Y_{n_i}, denote the boundary of the plumbed sphere bundles by Y_Γ. A vertex with $n_i + d_i > 0$ will be called a bad vertex, where d_i denote the degree (or the valence) of that vertex. We will call a connected linear subtree $\widehat{\Gamma} \subset \Gamma$ maximal if there is no connected linear subtree $\widetilde{\Gamma} \subset \Gamma$ such that $\widehat{\Gamma}$ is a proper subset of $\widetilde{\Gamma}$. The set $\Gamma \setminus \widehat{\Gamma}$ will denote the subtree where we remove from Γ all the edges emanating from any vertex in $\widehat{\Gamma}$ as well as all the vertices of $\widehat{\Gamma}$. Take a maximal linear subtree $\Gamma_1 \subset \Gamma$ which includes at least one bad vertex. Then take a maximal linear subtree $\Gamma_2 \subset \Gamma \setminus \Gamma_1$ which includes at last one bad vertex of Γ. It is clear that by iterating this process we will end up with a subtree $\Gamma \setminus \bigcup_{j=1}^{s} \Gamma_j \subset \Gamma$ without any bad vertices, for some disjoint subtrees $\Gamma_1, \ldots, \Gamma_s$ such that $\Gamma_{j+1} \subset \Gamma \setminus \bigcup_{t=1}^{j} \Gamma_t$, for $j = 1, \ldots, s-1$. Note, however, that $\Gamma_1, \ldots, \Gamma_s$ may not be uniquely determined by Γ. In particular, given any tree Γ, the number s above is not uniquely determined. Nevertheless there is certainly a minimum s, associated to Γ, over all possible choices of subtrees in the above process. We will refer to this number as the *genus* of Γ and denote it by $g(\Gamma)$. If there is no bad vertex in Γ then we define $g(\Gamma)$ to be zero.

Proposition 3.1. *Suppose that we are given a plumbing tree Γ of l circle bundles Y_{n_i} such that $n_i \leq -2$ for all i. There are $(|n_1|-1)(|n_2|-1)\cdots(|n_l|-1)$ special Legendrian surgery diagrams giving Stein manifolds with boundary Y_Γ. These all have different c_1's so the associated Stein fillable contact structures are distinct.*

Remark 3.2. We do not claim these are all possible Stein fillable contact structures on Y_Γ, but in some cases (like when Y_Γ is a small Seifert fibered space with $e_0 < -2$) we do construct all Stein fillable (and all tight) contact structures. This follows from the classification of tight contact structures in [11].

Remark 3.3. There are other, possibly more obvious, Legendrian surgery diagrams for these contact structures on Y_Γ, but the diagrams we derive here are the key to our constructions of open books in the next section.

Proof. From [9] we recall how to "role up" a linear plumbing tree Γ. Let Γ be the linear plumbing tree for $Y_{n_1}, \ldots, Y_{n_k}$ where each Y_{n_i} is plumbed to $Y_{n_{i-1}}$ and $Y_{n_{i+1}}, i = 2, \ldots, k-1$. See the left hand side of Figure 1. The standard surgery diagram for Γ is a chain of unknots $U_1, \ldots, U_k$ with each U_i simply linking U_{i-1} and $U_{i+1}, i = 2, \ldots, k$ and with U_i having framing n_i. We think of this chain as horizontal with components labeled from left to right. Let $U_1' = U_1$. Start with U_2 and slide it over U_1 to get a new link with U_2 replaced by an unknot U_2' that now links U_1, $n_1 + 1$ times. Now slide U_3 over U_2'. Continue in this way until U_k is slid over U_{k-1}'. The new link L is called the "rolled up" surgery diagram. See the right hand side of Figure 1. We observe a few salient features of this construction. First, each U_i' links U_j' for $j > i$ the same number of times. Denote this linking number by l_i. Secondly, $l_i \geq l_{i+1}$ for all i. (Recall l_i is negative.) In fact, $l_i = n_1 + \ldots + n_{i-1} + 2i - 1$. Thirdly, the framings m_i on the U_i''s are non-increasing and decrease only when $n_i < -2$. In fact, $m_{i+1} - m_i = n_{i+1} + 2$. Fourthly, the meridians μ_i for U_i simple link $U_i' \cup \ldots \cup U_k'$. And lastly, L sits in an unknotted solid torus neighborhood of U_1. There is an obvious

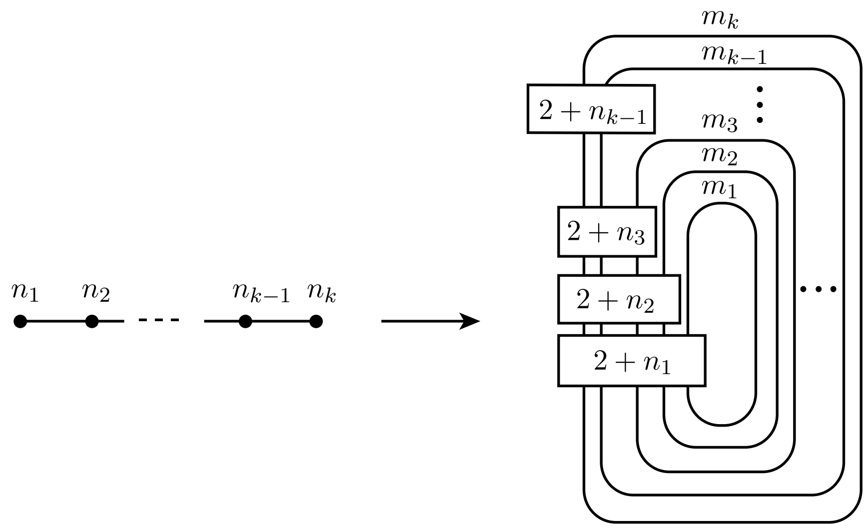

FIGURE 1. A linear plumbing of circle bundles and its rolled-up version. (The number inside a box denotes the number of full–twists we should apply to the knots entering into that box.)

Legendrian representation of L such that U_i' is the Legendrian push off of U_{i-1}' with $|n_i+2|$ stabilizations. Thus Legendrian surgery produces exactly the number of Stein manifolds claimed in the statement of the theorem.

Returning to the topological situation consider a tree Γ with one valence three vertex, then we can decompose Γ as above into linear trees Γ_1 and Γ_2, where the first sphere bundle of Γ_2 is plumbed into the i-th sphere bundle of Γ_1. Let $L_1 = U_1' \cup \ldots \cup U_k'$ and $L_2 = V_1' \cup \ldots, V_{k'}'$ be the rolled up surgery links for Γ_1 and Γ_2, respectively. It is clear that if the neighborhood of V_1 in which L_2 sits is identified with a neighborhood of the meridian μ_i for U_i then the resulting surgery link will describe Y_Γ. As above we can Legendrian realize L_1 and L_2. Moreover, if $n_i < -2$ then there will be a zig-zag from the stabilization of U_i' and we may link L_2 into U_i' using this zig-zag as shown in Figure 2. If $n_i = -2$ then there is no zig-zag and no apparent way to hook L_2 to U_i'. However we can preform a type 1 Legendrian Reidemeister move to create a cusp edge that can be used to hook L_2 to U_i, as shown in Figure 2. Thus we have Legendrian realized $L_1 \cup L_2$ and have the desired number of Stein fillings of Y_Γ.

We can continue in this way to obtain rolled up surgery diagrams and Legendrian surgery diagrams for any plumbing tree. Note that we will need to add $(n_i + d_i)$ type 1 Legendrian Reidemeister moves to each Legendrian knot corresponding to a bad vertex. □

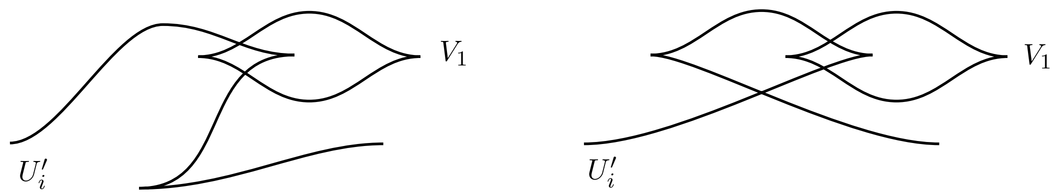

FIGURE 2. Linking L_2 (which is in a neighborhood fo V_1) to U_i'.

4. OPEN BOOKS FOR PLUMBINGS WHICH ARE NOT NECESSARILY NON-POSITIVE.

Using the notation established at the beginning of Section 3 we are ready to state our main result.

Theorem 4.1. *Suppose that we are given a plumbing tree* Γ *of circle bundles* Y_{n_i} *such that* $n_i \leq -2$ *for all* i*. Then the Legendrian realizations of* Γ *from Proposition 3.1 give rise to Stein fillable contact structures that are supported by open books of genus* $g(\Gamma)$.

This theorem was proven in [9] for the case with no bad vertices. We generalize the ideas there for our current proof.

Remark 4.2. In [3] it was shown that if a contact structure is filled by a symplectic 4-manifold whose intersection pairing does not embed in a negative definite form then the contact structure cannot be supported by a planar open book. We observe that the intersection forms of some plumbings can embed in negative definite forms but the above theorem still gives an open book with genus larger than zero. For example if a plumbing graph has one bad vertex with Euler number $-n < 0$ and valence $v < 2n - l$, where l is the number of branches from the bad vertex with length greater than 1, then the intersection form of this plumbing can embed into a negative definite form. It would be very interesting to see if the genus of these open books can indeed be reduced.

Remark 4.3. The ideas in Theorem 4.1 are much more general. Given any contact surgery diagram for a contact structure if one can embed the individual knots in the surgery link into an open book for the tight contact structure on S^3 then the ideas of “rolling up”, “hooking into zig-zag’s” and “hooking into a type 1 Legendrian Reidemeister move” can be used to construct open books for the resulting manifolds. While algorithms for constructing open books have been known for some time, see for example [1], this algorithm seems to produce much smaller genus open books. We demonstrate this by constructing an open book for each tight contact structure on small Seifert fibered spaces with $e_0 \neq -1, -2$. For notation see [5].

Proposition 4.4. *Consider the small Seifert fibered space* $M = M(e_0; r_1, r_2, r_3)$*. Any tight contact structure on* M *is supported by an open book with planar pages if* $e_0 \leq -3, e_0 \geq 0$ *or if* $e_0 = -1, \frac{1}{2} \leq r_1, r_2 < 1$ *and* $0 < r_3 < 1$.

Proof of Theorem 4.1. We recall the idea used in [9] to find open books supported by a contact structure obtained by a Legendrian realization of the linear plumbing tree described in Proposition 3.1. (See the proof of this proposition for notation used here.) Consider the core circle γ of the open book $\mathfrak{ob}_H$ in S^3 given by the positive Hopf link H. The page of $\mathfrak{ob}_H$ is an annulus and its monodromy is a right-handed Dehn twist along γ. First Legendrian realize γ on a page of $\mathfrak{ob}_H$. In [3], it was shown that to stabilize a Legendrian knot on a page of an open book, in general, one can first stabilize the open book and then push the knot over the 1–handle which is attached to stabilize the open book. Apply this trick to stabilize $\mathfrak{ob}_H$, $(|m_1|-1)$–times, by successively attaching 1–handles while keeping the genus of the page to be zero. As a result, by pushing γ over all the attached 1–handles, we can embed the inner-most knot in the rolled-up diagram on a page of the stabilized open book in S^3 as a Legendrian knot. Then by iterating this process from innermost to outermost knot, we can find an open book in S^3 which contains all the knots in the rolled-up diagram as Legendrian knots in distinct pages. Applying Legendrian surgery on these knots yields a Stein fillable contact structure together with a planar open book supporting it.

In a general tree Γ of circle bundles Y_{n_i} $(n_i \leq -2)$ without bad vertices we can take a maximal linear subtree $\Gamma_1 \subset \Gamma$ to start with and apply the algorithm above to roll it up and construct a corresponding open book. Then take a maximal linear subtree $\Gamma_2 \subset \Gamma \setminus \Gamma_1$ splitting off at a vertex of Γ_1. Note that there is a stabilization used in the open book for Γ_1, at the splitting vertex, with its core circle so that we can apply our algorithm to find an open book for Γ_2 starting from this annulus and extend the previous open book. This is the translation of the left hand side of Figure 2 from a front projection to Legendrian knots on pages of open books. It is clear that we can continue this process to cover all the vertices in Γ. As observed in [9], this will work as long as the tree does not have any bad vertices, since the condition $n_i + d_i \leq 0$ guarantees that there are as many "free" annulus in that vertex as we need to hook in a subtree splitting off at that vertex. It should be clear that we will always get a planar open book as a result.

To understand the situation with bad vertices we need to translate the right hand side of Figure 2 into Legendrian knots on pages of open books. Specifically, we need a lemma that tells us how one can embed a type 1 Legendrian Reidemeister move into the page of an open book.

Lemma 4.5. *Let (Σ, ϕ) be an open book supporting a contact structure ξ on M and K an oriented Legendrian knot on a page of the open book. Suppose $R = [0,1] \times [-1,1]$ is a rectangle in the page of an open book such that $(\partial\Sigma) \cap R = [0,1] \times \{-1,1\}$ and $[0,1] \times \{0\} = K \cap R$ with the orientation on K agreeing with the standard orientation on $[0,1]$. Stabilize the open book by adding a 1–handle to R such that the 1–handle connects $[0,1] \times \{1\}$ to $[0,1] \times \{-1\}$ and the new monodromy has an extra Dehn twist along $\{\frac{1}{2}\} \times [-1,1]$ union the core of the 1–handle. Call this curve γ and orient γ so that the* *orientation on γ and $[-1,1]$ agree. The homology class $K \pm \gamma$ can be represented by an embedded Legendrian curve $K_\pm$ on the page. The curve K_+ is Legendrian isotopic to K and "corresponds" to a type 1 Legendrian Reidemeister move. The curve K_- is isotopic to the positive and negative stabilization of K.*

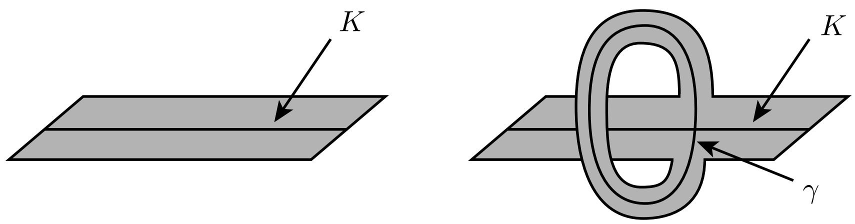

FIGURE 3. The rectangle R on the left and the stabilized open book on the right (the embedding shown on the right is not correct there will be a full twist in the newly attached handle).

Proof. We can Legendrian realize $K \cup \gamma$ on a page of the open book. The knot γ is a Legendrian unknot with $tb = -1$. Thus we can pick a disk D that γ bounds whose interior is disjoint from K and we can make this disk convex. In the standard contact structure on $\mathbb{R}^3$ we can take an unknot that is tangent to the x-axis and bounds a disk D' whose interior is disjoint from the x-axis. Since D and D' are convex with the same dividing set (since we can assume that D lies in the complement of the binding of the original open book we know it has a tight neighborhood) we can assume the characteristic foliations are the same. Now we can find a contactomorphism from a neighborhood of D union a segment of K to a neighborhood of D' union a segment of the x-axis (so that D goes to D' and the segment of K goes to the x-axis). We can now perform the desired operations in this local model to complete the proof. □

Returning to the proof of Theorem 4.1, suppose that Γ has bad vertices. Once again we can roll up Γ_1. When Γ_1 has bad vertices to which we wish to attach, say, Γ_2 we can use type 1 Legendrian Reidemeister moves as in the proof of Proposition 3.1 to construct a Legendrian link into which Γ_2 can be "hooked". For the open book we can stabilize as described in Lemma 4.5 to create an extra annulus in the page of the open book that will allow us to hook in the linear graph Γ_2. That is (using notation from the proof of Proposition 3.1) if Γ_2 is attached to Γ_1 at the unknot U_i then apply Lemma 4.5 to U_i' (and all the subsequent U_j'). This creates an annulus in the page of the open book that the U_j's each go over exactly once for each $j \geq i$. Now let the first unknot of Γ_2 be a Legendrian realization of the core of the new annulus. This core will link the U_j''s exactly once for all $j \geq i$. We may now proceed to attach the rest of the unknots in the rolled up version of Γ_2 as above. This is illustrated in Figure 4. Note that we can repeatedly apply Lemma 4.5

to hook in arbitrarily many branches to bad vertices of Γ_1 and only the first stabilization increases genus. Thus we see that if all the bad vertices are contained in Γ_1 the genus of the resulting open book is one. Repeating this argument for the other Γ_i's containing bad vertices we see that the genus of the resulting open book is precisely $g(\Gamma)$. We have now constructed an open book for the tight contact structure on S^3 with the link L from Proposition 3.1 on its pages. Thus Legendrian surgery on this link will yield an open book supporting the contact structure obtained by Legendrian surgery. □

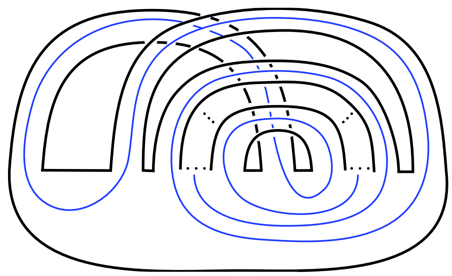

FIGURE 4. Positive stabilizations at a bad vertex where $n_i + d_i > 0$. The number of 1–handles in the figure is given by $1 + |n_i + d_i|$.

Proof of Proposition 4.4. The case with $e_0 \leq -3$ follows immediately from the Theorem 4.1 and the classification of tight contact structures on these manifolds from [11]. This result was originally proven in [9]. The case with $e_0 = 0$ follows from the classification given in [5]. In particular, all these contact structures can be obtained from the tight contact structure on $S^1 \times S^2$ by contact surgery of three Legendrian knots isotopic to $S^1 \times \{pt\}$. In [5] they show that all the tight contact structures are obtained from Legendrian surgery on Legendrian realizations of Figure 5. Thus we can start with an open book for the contact structure on $S^1 \times S^2$ with annular page and trivial monodromy. The contact framing on the components with surgery coefficient a_0^j's is zero and since the a_0^j's are all less than or equal to -2 we will need to stabilize the page of the open book to Legendrian realize these components with the appropriate framing. Now if we roll up the rest of the a_i^j's onto the a_0^j's, as in the proof of Proposition 3.1, we can easily modify the proof of Theorem 4.1 to construct a genus zero open book for these contact structures. The $e_0 > 0$ follows similarly, the only difference with the $e_0 = 0$ case is that a_0^1 will be -1. Thus we will not be able to stabilized the knot corresponding to a_0^1, however when one "roles up" the a_i^1's on a_0^1 we will still be able to realize them on the page of the open book. Thus we still get a genus zero open book. The sporadic examples with $e_0 = -1$ follow from the classification of tight contact structures given in [4] and using the methods in the proof of Theorem 4.1 to convert the contact structure diagrams in that paper to open books. □

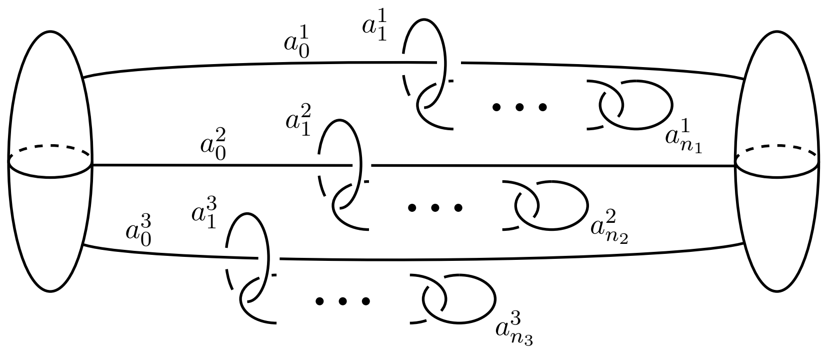

FIGURE 5. The surgery diagram for $M(0; r_1, r_2, r_3)$, here all the $a_i^j \leq -2$.

5. EXAMPLES

We now demonstrate how to use the above algorithm to construct open books for various plumbing diagrams.

Example 5.1. Consider the Poincaré homology sphere $\Sigma(2, 3, 5)$ which can be given by the negative definite E_8–plumbing of circle bundles over S^2 as in Figure 6.

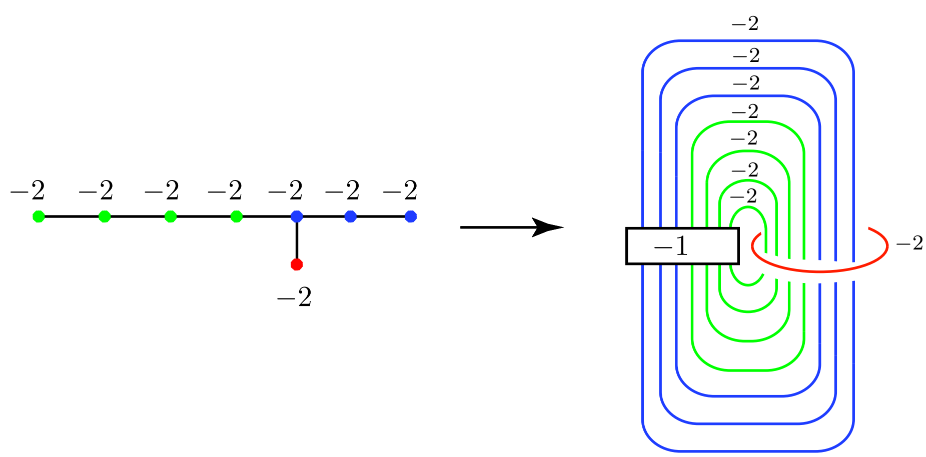

FIGURE 6. Negative definite E_8-plumbing on the left and its rolled-up version on the right.

Also consider the genus one surface $\Sigma_{1,1}$ with one boundary component as depicted in Figure 7. We claim that the open book with page $\Sigma_{1,1}$ and monodromy

$$\phi = t_a^2 t_c^3 t_b^5$$

supports the unique tight contact structure on $\Sigma(2,3,5)$, where t_γ denotes a right-handed Dehn twist along a curve γ on a surface. Note that $\Sigma(2,3,5)$ does not admit any planar open book supporting its unique tight contact structure (cf. [3]).

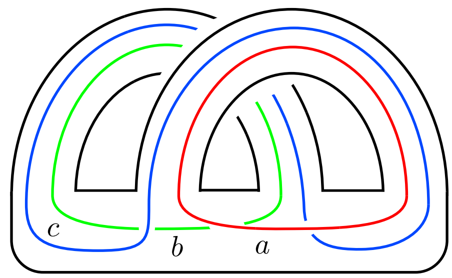

FIGURE 7. The curves a, b and c are embedded in *distinct* pages of an open book in S^3 as indicated above.

We now apply our algorithm to verify the claim about the existence of an elliptic open book on the Poincaré homology sphere $\Sigma(2,3,5)$, supporting its unique tight contact structure. The idea is to first construct an open book in S^3 and embed the surgery curves into the pages of this open book so that when we perform surgeries along each of these curves with framing one less than the surface framing we get $\Sigma(2,3,5)$ as the resulting 3-manifold with its associated open book. (Note that in terms of contact structures this corresponds a Legendrian surgery.) The monodromy of the open book in S^3 with page $\Sigma_{1,1}$ (as shown in Figure 7) is $t_b t_a$. This is obtained by stabilizing the annulus carrying the curve b in Figure 7 by attaching the 1–handle carrying the curve a. Now take the linear branch in the E_8-plumbing in Figure 6 with seven vertices. The first four vertices (which correspond to the innermost four curves in the rolled-up version in Figure 6) are represented by parallel copies of the curve b in Figure 7. The fifth vertex is a bad vertex and a branch splits off with only one vertex—the eighth vertex. The curve c represents this bad vertex. The sixth and the seventh vertices are represented by parallel copies of c. The eighth vertex is represented by a parallel copy of a. So we embedded all the curves in Figure 6 into distinct pages of the open book in S^3. By performing surgeries on these curves (and taking into account the right-handed Dehn twists we needed for the stabilizations) we get an open book in $\Sigma(2,3,5)$ with monodromy $\phi = t_a t_c^3 t_b^4 t_b t_a$ which is equivalent to

$$\phi = t_a^2 t_c^3 t_b^5.$$

Since the monodromy is a product of right-handed Dehn twists only, the supported contact structure is Stein fillable (and hence tight). Therefore this contact structure is isotopic to the unique tight contact structure on $\Sigma(2,3,5)$.

There is also another way of finding an elliptic open book supporting the unique tight contact structure on $\Sigma(2,3,5)$. The monodromy of the elliptic fibration $E(1) \to S^2$ can be given by $(t_b t_a)^6$, using the notation in Figure 7, except that we think of the curves a and b embedded on a non-punctured torus. By removing the union of a section and a cusp fiber from $E(1)$ we get a Lefschetz fibration on the 4–manifold W with punctured torus fibers whose monodromy is $(t_b t_a)^5$. One can check that ∂W is diffeomorphic to $\Sigma(2,3,5)$ by Kirby calculus (see, for example, [7]). Thus there is an induced open book on $\Sigma(2,3,5)$ with monodromy $(t_b t_a)^5$. Since the monodromy of this open book is a product of right-handed Dehn twists only, the contact structure supported by this open book is Stein fillable (cf. [6]) and in fact W is a Stein filling of its boundary. We conclude that the elliptic open book with monodromy $(t_b t_a)^5$ has to support the unique tight contact structure on $\Sigma(2,3,5)$. Finally we note that the two elliptic open books we described above are in fact isomorphic. In order to see the isomorphism we first observe that $t_c = t_a^{-1} t_b t_a$. Then we plug this relation into $t_a^2 t_c^3 t_b^5$ to get

$$\begin{aligned} t_a^2 t_c^3 t_b^5 &= t_a t_b^3 t_a t_b^5 \\ &= (t_b t_a t_b) t_b (t_b t_a t_b) t_b^3 \\ &= t_a t_b (t_a t_b t_a)(t_b t_a t_b) t_b^2 \\ &= t_a t_b t_b t_a t_b t_a t_b t_a t_b^2 \\ &= (t_b t_a t_b) t_b t_a t_b t_a t_b t_a t_b \\ &= t_a t_b t_a t_b t_a t_b t_a t_b t_a t_b \\ &= (t_b t_a)^5 . \end{aligned}$$

Note that we used the "braid" relation $t_a t_b t_a = t_b t_a t_b$ repeatedly and cyclically permuted the words in the calculation above.

Example 5.2. Consider the plumbing diagram of circle bundles and its rolled-up version shown in Figure 8.

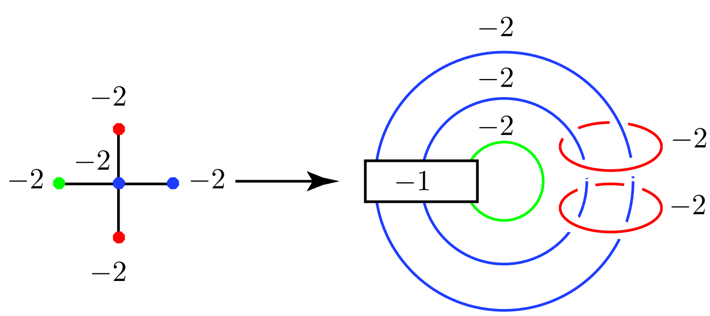

FIGURE 8. A plumbing diagram on the left and its rolled-up version on the right.

Then applying our algorithm we can construct an open book supporting the contact structure obtained by the (unique) Legendrian realization of this plumbing diagram. The page

$\Sigma_{1,2}$ of the open book is a torus with two boundary components and the monodromy is given by

$$\phi = t_{a_1} t_{a_2} t_c^2 t_b t_b t_{a_2} t_{a_1} \ ,$$

which is equivalent to the more symmetric form

$$\phi = t_{a_1}^2 t_{a_2}^2 t_c^2 t_b^2 \ ,$$

where we depicted the curves a_1, a_2, b and c on $\Sigma_{1,2}$ in Figure 9. Note that Dehn twists along two disjoint curves commute. Moreover, by plugging in $t_c = t_{a_2}^{-1} t_{a_1}^{-1} t_b t_{a_1} t_{a_2}$ we can also express the monodromy as

$$\phi = (t_{a_1} t_{a_2} t_b^2)^2.$$

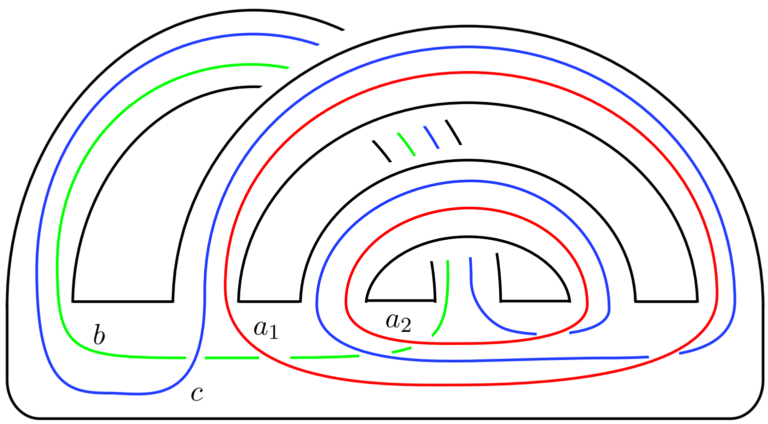

FIGURE 9. The curves a_1, a_2, b and c are embedded in *distinct* pages of the open book in S^3 as indicated above.

Example 5.3. Consider the plumbing tree Γ of circle bundles and its rolled-up version shown in Figure 10.

In this final example we will illustrate how open books corresponding to two subtrees that meet at a bad vertex are placed on an open book. As the first subtree Γ_1 take the linear tree on top with five vertices with a bad vertex in the second and fourth place, and as the second subtree take Γ_2 the subtree of $\Gamma \setminus \Gamma_1$ branching from the left most bad vertex on Γ_1. Notice that in the rolled-up version the part corresponding to Γ_2 is "linked" to the part corresponding to Γ_1. So we start with the open book for Γ_1 and make sure the open book has been stabilized twice so that Γ_2 and the top most vertex can be linked into Γ_1. Then Γ_2 is put on the pages of the open book. To this end we must stabilize again to accommodate the bad vertex in Γ_2. The resulting open book has page as shown in Figure 11. In particular, it is a surface of genus two with one boundary component. The monodromy of the open

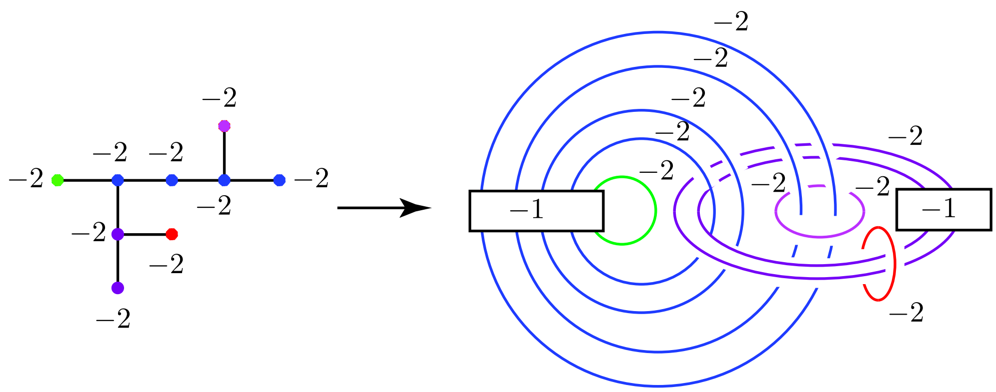

FIGURE 10. A plumbing diagram on the left and its rolled-up version on the right.

book supporting the contact structure obtained by Legendrian surgery on the Legendrian realization of Γ is then given by

$$\phi = t_{a_4} t_{a_3} t_{b_2}^2 t_{c_1}^2 t_{b_1}^2 t_{a_1} t_{a_4} t_{a_3} t_{a_2} t_{a_1}.$$

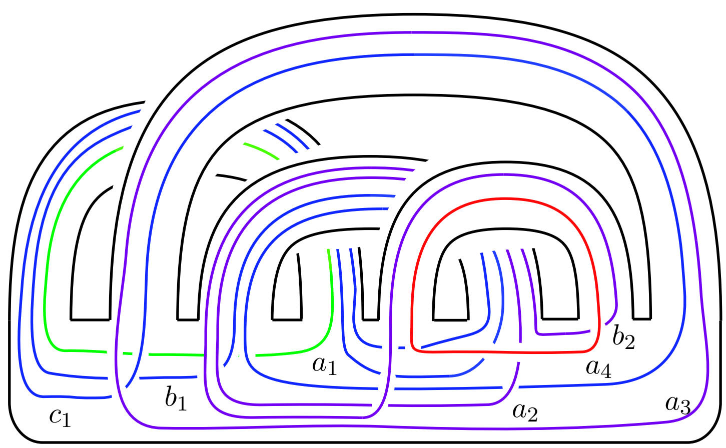

FIGURE 11. The curves $a_1, a_2, a_3, a_4, b_1, b_2$ and c_1 are embedded in *distinct* pages of the open book in S^3 as indicated above.

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