CONVEX SURFACES IN CONTACT GEOMETRY: CLASS NOTES

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Abstract. These are notes covering part of a contact geometry course. They are in very preliminary form. In particular the last few sections have not really been proof read. Hopefully I will be able to come back to these notes in the near future to improve and extend them, but I hope they are useful as is. Section 8 have not been written yet. Section 7 currently contains no proofs and section 1 contains only the statements of necessary results.

Contents

1. Characteristic Foliations

Theorem 1.1. *Let Σ be an embedded surface in M^3 and ξ_i contact structures on $M, i = 0, 1$, such that $\Sigma_{\xi_0} = \Sigma_{\xi_1}$. Then for any neighborhood of Σ there is a smaller neighborhood U of Σ and an isotopy $\phi_t : M \to M$ such that*

1. $\phi_0 = id_M$,
2. ϕ_t *is fixed on* Σ *and*
3. $(\phi_1|_U)^*(\xi_0|_{\phi_1(U)}) = \xi_1$.

Theorem 1.2. *Let $M = \Sigma \times [0, 1]$ for a surface Σ. If two contact structures induce the same characteristic foliations on all the surfaces $\Sigma \times \{t\}$, for $t \in [0, 1]$, (and are the same in a neighborhood of $\partial\Sigma \times [0, 1]$ if $\partial\Sigma \neq \emptyset$) then they are isotopic rel. ∂M.*

Theorem 1.3. *Let $\mathcal{F}$ be a singular foliation on an oriented surface Σ. Then $\mathcal{F}$ is the characteristic foliation induced on Σ for some contact structure if and only if all the singularities of $\mathcal{F}$ have non-zero divergence.*

Theorem 1.4. *Let $\mathcal{P}$ be a C^∞ property of a singular foliation on a surface and let Σ be a surface embedded in a contact manifold (M, ξ). Then by a C^∞ small isotopy of Σ we can arrange Σ_ξ to satisfy $\mathcal{P}$.*

Lemma 1.5. *Let L be a Legendrian arc in a surface $\Sigma \subset (M, \xi)$ and x a point on L that is a singular point of Σ_ξ. If ξ crosses $T\Sigma$ along L at x in a right handed way (respectively, left handed way) then x is a sink (respectively, source) if it is a positive singularity and a source (respectively, sink) if it is a negative singularity.*

2. Introduction to Convex Surfaces

2.1. **Contact vector fields.**

Definition 2.1. *Given a contact manifold* (M, ξ), *a* contact vector field v *is a vector field whose flow* ϕ_t *preserves* ξ.

If α is a contact 1-form for ξ the v is a contact vector field for ξ if and only if

$$L_v\alpha = g\alpha,$$

where $g : M \to \mathbb{R}$ is a function on M. This is easily seen since

$$L_v\alpha = \frac{\partial}{\partial t}\phi_t^*\alpha|_{t=0}$$

and the flow of v preserves ξ if and only if $\phi_t^*\alpha = g_t\alpha$.

Given the contact form α there is a unique vector field X_α that satisfies

$$\alpha(X_\alpha) = 1, \quad \iota_{X_\alpha} d\alpha = 0.$$

This vector field is called the *Reeb vector field* of α. One easily computes

$$L_{X_\alpha}\alpha = d(\iota_{X_\alpha}\alpha) + \iota_{X_\alpha}(d\alpha) = 0$$

This the Reeb field X_α is a contact vector field. Moreover, the condition $\alpha(X_\alpha) = 1$ implies that the vector field is transverse to ξ.

Exercise 2.2. Show a contact vector field v is a Reeb field for some contact 1-form for ξ if and only if v is transverse to ξ.

Exercise 2.3. Show a contact vector field v is always tangent to ξ if and only if it is 0.

Lemma 2.4. *Given a contact manifold* (M, ξ), *let* α *be a contact 1-form for* ξ. *A vector field* v *is a contact vector field if and only if there is a function* $H : M \to \mathbb{R}$ *such that*

$$\alpha(v) = -H, \text{ and}$$
$$\iota_v d\alpha = dH - (dH(X_\alpha))\alpha.$$

Exercise 2.5. Show that given a function H the equation in the lemma uniquely define a contact vector field.
Hint: Let $v = w + fX_\alpha$, where f is a function and w is a vector field in ξ. Note the first equation determines f. Use the second equation to find w.

Remark 2.6. This lemma says that any locally defined contact vector field can always be extended to a globally defined vector field.

Proof. Assume v is a contact vector field. Set $H = -\alpha(v)$. We know

$$g\alpha = L_v\alpha = d\iota_v\alpha + \iota_v d\alpha = -dH + \iota_v d\alpha.$$

Thus

$$\iota_v d\alpha = dH + g\alpha.$$

Plug X_α into this equation to see that $dH(X_\alpha) = -g\alpha(X_\alpha) = -g$. Hence

$$\iota_v d\alpha = dH - (dH(X_\alpha))\alpha.$$

For the other implication one may easily compute that $L_v\alpha = -(dH(X_\alpha))\alpha$, and hence v is contact. □

Given a contact vector field v for a contact manifold (M, ξ) the *characteristic hypersurface* of v is

$$C = \{x \in M | v(x) \in \xi_x\}.$$

We will call a contact vector field generic if the function $\alpha(v)$ is transverse to 0.

Lemma 2.7. *For a generic contact vector field v the characteristic hypersurface is a nonsingular hypersurface. Moreover v is tangent to C and directs the characteristic foliation.*

Proof. With the notation above $C = H^{-1}(0)$. Genericity implies 0 is a regular value of H and hence C is a surface. Recall $L_v\alpha = g\alpha$ for some function g. Now if $x \in C, w \in T_xC \cap \xi_x$ and $v \neq 0$ then

$$d\alpha_x(v, w) = L_v\alpha(w) - d\iota_v\alpha(w) = g\alpha(w) + dH(w) = 0.$$

Thus $v, w \in \xi_x$, and $d\alpha(v, w) = 0$ implies that v is a multiple of w. Now if $v = 0$ then

$$g\alpha_x = (L_v\alpha)_x = (d\iota_v\alpha)_x = -dH_x = 0$$

for vectors tangent to C. Thus $\alpha = 0$ on vectors tangent to C. (Or $g(x) = 0$ but then $dH_x = 0$ contradicting genericity.) Therefor singularities in C_ξ occur at zeros of v. □

2.2. **Convex surfaces.**

Definition 2.8. *A surface Σ in a contact manifold (M, ξ) is a* convex *surface if there is a contact vector field v that is transverse to Σ.*

Lemma 2.9. *A surface Σ is convex if and only if there is an embedding $\phi : \Sigma \times \mathbb{R} \to M$ such that $\phi(\Sigma \times \{0\}) = \Sigma$ and $\phi_*^{-1}(\xi)$ is vertically invariant (that is invariant in the $\mathbb{R}$ direction).*

Proof. If Σ is convex let v be the transverse contact vector field. Set $H = -\alpha(v)$ for some contact 1-form for ξ. Cut off H so that it is zero outside a small tubular neighborhood of Σ (and nonzero in this neighborhood). Let v' be the vector field generated by this new function. The flow of v' gives ϕ. (Note, we needed to cut of H so that we did not have to worry about the flow not existing for all time or having periodic orbits.) Conversely, if t is the coordinate on $\mathbb{R}$ then $\phi_*\frac{\partial}{\partial t}$ is a contact vector field for ξ that is transverse to Σ. □

Using the coordinates coming from this lemma there is a contact 1-form α for ξ such that

$$\alpha = \beta + u\,dt,$$

where β is a 1-form on Σ and u is a function on Σ. Note:

(1) $\Sigma_\xi = \ker(\beta)$.

(2) $\Sigma \cap C = \{x \in \Sigma | u(x) = 0\}$.
(3) for α to be a contact form we need

$$\begin{aligned}\alpha \wedge d\alpha &= \beta \wedge (d\beta + du \wedge dt) + u\, dt \wedge d\beta \\ &= (\beta \wedge du + u\, d\beta) \wedge dt > 0.\end{aligned}$$

Lemma 2.10. *Let Σ be a surface in a contact manifold (M, ξ), i the inclusion map, and α a contact 1-form for ξ. Set $\beta = i^*\alpha$. The surface Σ is convex if and only if there is a function $u : \Sigma \to \mathbb{R}$ such that*

$$u\, d\beta + \beta \wedge du > 0. \tag{1}$$

Proof. If Σ is convex we know such a u exists from above. If such a u exists then on $\Sigma \times \mathbb{R}$ we can consider the contact structure $\ker(\beta + u\, dt)$. The characteristic foliation on $\Sigma \times \{0\}$ is the same as the one on $\Sigma \subset M$. Thus by Theorem 1.1 we know there is a contactomorphism from a neighborhood of $\Sigma \times \{0\}$ to a neighborhood of Σ. Push the vector field $\frac{\partial}{\partial t}$ to M using the contactomorphism. This is a contact vector field transverse to Σ. □

We will now "dualize" this construction. Let ω be a volume form on Σ define a vector field w on Σ as by

$$\iota_w \omega = \beta.$$

Note: w is in the kernel of β so w directs Σ_ξ. That is w is tangent to the foliation at non singular points and is zero at singular points. If Σ is convex we know

$$\beta \wedge du + u\, d\beta > 0.$$

$$\beta \wedge du + u(\mathrm{div}_\omega w)\omega > 0$$

$$-du(w)\omega + u(\mathrm{div}_\omega w)\omega > 0.$$

The last line follows because $du \wedge \omega = 0$ (since this is 3-form on a surface) and hence $0 = \iota_w(du \wedge \omega) = du(w)\omega - du \wedge (\iota_w \omega)$. Thus

$$-du(w) + u(\mathrm{div}_\omega w) > 0. \tag{2}$$

Remark 2.11. The set of functions u on a surface that satisfy Equation (1), or dually Equation (2), form a convex set. Different choices of u correspond to different choices of vertically invariant vector field transverse to Σ.

Shortly we will see that convex surface are very common. For the moment lets see that there are surface that are not convex.

Example 2.12. Let (r, θ, z) be cylindrical coordinates on $\mathbb{R}^3$ Consider the contact manifold $M = \mathbb{R}^3/\sim$ where $\sim$ is generated by $z \mapsto z + 1$ and $\xi = \ker(dz + r^2 d\theta)$. Let $T_c = \{(r, \theta, z) | r = c\}$. Then the characteristic foliation on T_c is linear. If β is as above it is easy to see that $d\beta = 0$. Thus if T_c is convex we need to find a function $u : T_c \to \mathbb{R}$ such that $-du(w) > 0$, where w is pointing along the foliation. Thus u must be decreasing along the flow lines. But this is not possible.

Definition 2.13. *Let Σ be a surface and $\mathcal{F}$ be a singular foliation on Σ. A multi-curve Γ is said to divide $\mathcal{F}$ if*

(1) $\Sigma \setminus \Gamma = \Sigma_+ \coprod \Sigma_-$,
(2) Γ *is transverse to* $\mathcal{F}$, *and*
(3) *there is a volume form ω on Σ and a vector field w on Σ so that*
   (a) $\pm L_w \omega > 0$ *on* $\Sigma_\pm$,
   (b) w *directs* $\mathcal{F}$, *and*
   (c) w *points transversely out of* Σ_+ *along* Γ.

Exercise 2.14. Show that if Γ_1 and Γ_2 both divide $\mathcal{F}$ then they are isotopic through dividing curves.

If Σ in (M, ξ) is convex then near Σ write a contact form for ξ as $\beta + u\, dt$. The multi-curve $\Gamma_\Sigma = \{x \in \Sigma | u(x) = 0\}$ (this is the intersection of Σ with the characteristic hypersurface) is called the *dividing set of* Σ.

Theorem 2.15 (Giroux 1991, [8])**.** *Let Σ be an orientable surface in (M, ξ) with Legendrian boundary (possibly empty). Then Σ is a convex surface if and only if Σ_ξ has dividing curves. Moreover, if Σ is convex Γ_Σ will divide Σ_ξ.*

Proof. Suppose Σ is convex. Let Γ_Σ be the dividing set of Σ. Let $\beta + u\, dt$ be a vertically invariant contact form for ξ in a neighborhood of Σ. We have $\Gamma_\Sigma = u^{-1}(0)$, thus it is clear that $\Sigma_+ = u^{-1}([0, \infty))$ and $\Sigma_- = u^{-1}(-\infty, 0])$ are the components of $\Sigma \setminus \Gamma_\Sigma$. Moreover, if Γ_Σ is not transverse to Σ_ξ then there would be a vector v tangent to Σ_ξ and Γ_Σ at some point. Thus $v \in \ker \beta$, $du(v) = 0$ and $\iota_v(\beta \wedge du + u\, d\beta) = 0$ at this point (recall $u = 0$ on Γ_Σ), contradicting the contact condition. (Note: we did not have to assume that the contact vector field is generic. So even though the characteristic hypersurface might not be cut out transversely, we still must have Γ_Σ cut out transversely.) We are now left to check condition (3) of the definition of dividing curve. The form $\omega' = \beta \wedge du + u d\beta$ is a volume form on Σ. Let w be the vector field determined by $\iota_w \omega' = \beta$. Then a simple computation yields $\mathrm{div}_{\omega'}(\frac{1}{u} w) = \frac{1}{u^2}$. Since $\mathrm{div}_\omega f v = f \mathrm{div}_{f\omega} v$ we see that $\pm \mathrm{div}_{\pm u^{-1}\omega'} w > 0$ on $\Sigma_\pm$. Now let A be a small tubular neighborhood of Γ_Σ so that the characteristic foliation on A is by arcs transverse to $\partial A \setminus (\partial \Sigma \cap A)$. Let $\Sigma'_\pm = \Sigma_\pm \cap (\Sigma \setminus A)$. On $\Sigma'_\pm$ let ω by $\pm \frac{1}{u} \omega'$. This defines a (properly oriented) volume form on $\Sigma \setminus A$ for which $\pm \mathrm{div}_\omega w > 0$ on $\Sigma'_\pm$. We are left to extend ω over A so that $\pm \mathrm{div}_\omega w > 0$ on $\Sigma_\pm$.

Exercise 2.16. Show ω can be so extended.
Hint: We have an explicit model for w in A. Write down any volume form on A with the desired properties and show that it can be patched into ω near the boundary of A preserving these properties.

Now assume Γ divides Σ_ξ. We want to show that Σ is convex. Let Σ_0 be a small tubular neighborhood of Γ in Σ so that the characteristic foliation on Σ_0 is by arcs running across Σ_0. (Note each component of Σ_0 is an annulus or strip and it's core is transverse to Σ_ξ thus it is possible to find Σ_0.) Let $\Sigma'_\pm = \Sigma_\pm \cap (\Sigma \setminus \Sigma_0)$. On $\Sigma'_+ \cup \Sigma'_-$ let $\beta = \iota_v\omega$. On $\Sigma'_\pm$ let $u = \pm 1$. Thus on $\Sigma'_+ \cup \Sigma'_-$ one easily checks that

$$u\,\mathrm{div}_\omega w - du(w) > 0. \tag{3}$$

Or in other words, $\beta + u\,dt$, is a contact form on $(\Sigma'_+ \cup \Sigma'_-) \times \mathbb{R}$. Of course $\beta = \iota_w\omega$ is well defined on all of Σ. We just need to extend u over Σ_0 so that Equation (3) is satisfied on all of Σ. To this end slightly enlarge Σ_0 to Σ'_0 so that Σ'_0 is still foliated by arcs transverse to the boundary and Σ_0 is on the interior of Σ'_0. Let $h = \mathrm{div}_\omega w$. Parameterize an arc A in the foliation of

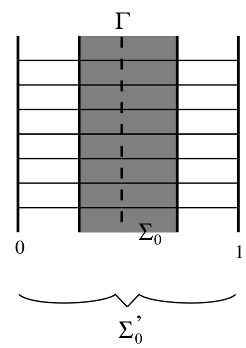


FIGURE 1. The region Σ'_0. Γ is the dotted line.

Σ'_0 as a flow line of w. Denote the parameterization as $f : [0,1] \to \Sigma_0$. Also denote $u|_A$ by u. Thinking of u as a function on $[0,1]$ we see $u(0) = 1$ and $u(1) = -1$. To extend u over Σ'_0 satisfying Equation(3) we need to solve the equation:

$$uh - \frac{du}{dt} > 0.$$

We can solve this with a function of the form

$$u(t) = g(t)e^{\int_0^t h(t)dt},$$

where $g(t)$ satisfies $g'(t) < 0$, and

$$g(t) = \begin{cases} \frac{1}{e^{\int_0^t h(t)}} & \text{near } t = 0 \\ -\frac{1}{e^{\int_0^1 h(t)}} & \text{near } t = 1. \end{cases}$$

It is easy to find such a g. Thus we have extended u over one arc in Σ'_0. One may easily see that we can consistently extend over all arc in this way. This extended u by construction satisfies Equation (3) and hence we have a vertically invariant contact form on $\Sigma \times \mathbb{R}$. In addition, the characteristic foliation on $\Sigma \times \{0\}$ is the same as Σ_ξ. Thus a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ is contactomorphic to Σ in M. Using this contactomorphism we see Σ is convex. $\square$

Example 2.17. The unit 2-sphere in $\mathbb{R}^3$ with the contact structure $\xi = \ker(dz + r^2 d\theta)$ is convex.

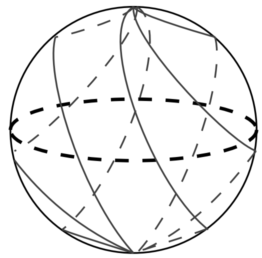


FIGURE 2. Convex 2-sphere.The thick dashed curve is the dividing curve.

Example 2.18. Let $M = \mathbb{R}^3/\sim$ where $(r, \theta, z) \sim (r, \theta, z+1)$ have the contact structure induced form Example 2.12. Consider $T_c = \{(r, \theta, z) : r = c\}$ for a fixed c so that the slope of the characteristic foliation is $\frac{p}{q}$. As discussed in Example 2.12, T_c is not a convex torus. Pick two orbits B and C in $(T_c)_\xi$. We have $T_c \setminus (B \cup C) = A_1 \cup A_2$, where A_i is an annulus. Push A_1 towards the z axis slightly and push A_2 slightly away from the z axis. One may easily see that the characteristic foliation is as shown in Figure 3. This foliation clearly has the dividing curves as shown in the figure. Note we easily could have arranged that our convex torus had $2n$ periodic orbits and $2n$ components to the dividing curve.

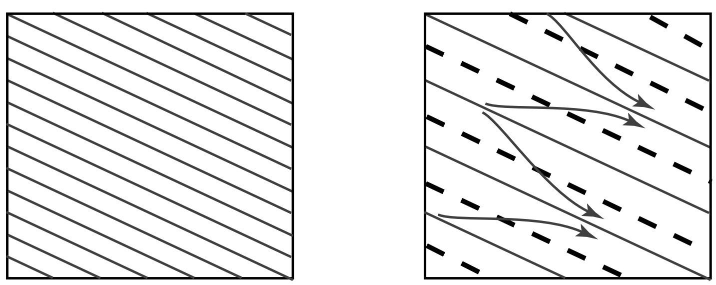


FIGURE 3. Non-convex torus, left, is perturbed into a convex torus right.

A singular foliation $\mathcal{F}$ is called *almost Morse-Smale* if

(1) the singularities are non degenerate,
(2) each close orbit is non degenerate (that is the Poincaré return map is not degenerate), and
(3) there are no flow lines running from a negative singularity to a positive singularity (this could only happen by having a connection between two saddle singularities).

Exercise 2.19. Show that if there is a flow line from a negative singularity to a positive singularity then the surface cannot be convex.
Hint: Assume the surface is convex. The contact form can be written $\beta + u\,dt$. At the negative singularity u must be negative at the positive singularity u must be positive.

Theorem 2.20 (Giroux 1991, [8])**.** *If Σ_ξ is almost Morse-Smale then Σ is convex.*

Proof. We will show that such a Σ_ξ has dividing curves. Along each close leaf put an annulus with boundary transverse to Σ_ξ. Around each elliptic singularity put a small disk with boundary transverse to Σ_ξ. Along the stable (unstable) manifolds of a positive (negative) hyperbolic singularity put a band. Let $\Sigma_\pm$ be the union of the regions just described associated to $\pm$ singularities and periodic orbits. (One should check that these unions define disjoint sets.) One may choose a volume form on $\Sigma_\pm$ that agrees with the orientation induced from Σ and so that a vector field w directing the characteristic foliation of $\Sigma_\pm$ has $\pm$ divergence. To do this note the divergence near the singularities and periodic orbits is independent of the volume form. Now on Σ_+, say, we know the divergence is positive near the singularities and periodic orbits. Note that if $\omega' = e^f\omega$ then $\text{div}_{\omega'} v = \text{div}_\omega v + df(v)$. Thus if one chooses a function f that is zero on the elliptic singularities and periodic orbits, K on the hyperbolic singularities and $2K$ near the boundary of Σ_+, then once can arrange $\text{div}_{\omega'} v > 0$.

Note $\Sigma' = \Sigma \setminus (\Sigma_+ \cup \Sigma_-)$ is a surface with a non singular foliation which is transverse to the boundary and without closed leaves (and sitting on an orientable surface). Thus the components of Σ' are annuli or strips foliated by arcs. The vector field directing the foliation on Σ' has $\pm$ divergence near the boundary components touching $\Sigma_\pm$. It is each to extend the volume form over Σ' so that the divergence is 0 only near the core of each annulus. □

Though this theorem suffices for most applications, there is a stronger version. An oriented singular foliation is said to satisfy the Poincaré Bendixson property if the limit set of each flow line (in either positive or negative time) is a singular point, periodic orbit or a union of singular points and connecting orbits.

Theorem 2.21 (Giroux 2000, [10])**.** *Let Σ be an oriented surface in (M, ξ) with Legendrian boundary such that Σ_ξ satisfies the Poincaré-Bendixson property. Then Σ is convex if and only if all the closed orbits are non-degenerate and there is no flow from a negative to a positive singularity.*

Exercise 2.22. Prove this theorem.
Hint: It is not too hard to show that the conditions in the theorem are necessary for convexity. The proof of sufficiency is almost identical to the proof of Theorem 2.20.

2.3. **Flexibility of the characteristic foliation.**

Theorem 2.23 (Giroux 1991, [8] for the closed case; Kanda 1998 [7] in general)**.** *Any closed surface Σ in a contact manifold (M, ξ) is C^∞-close to a* *convex surface. Any surface with Legendrian boundary satisfying $tw(\gamma, \Sigma) \leq 0$ for all boundary components γ may be C^0 small perturbed near the boundary and then C^∞ small perturbed on the interior so as to become convex.*

Proof. The closed case is clear by Theorem 1.4, since (almost) Morse-Smale foliations are generic. So consider Σ with boundary. Let γ be a boundary component of Σ and N a neighborhood of γ that is contactomorphic to a neighborhood of the y-axis in $\mathbb{R}^3/\sim$, where $(x, y, z) \sim (x, y+1, z)$, with the contact structure $\ker(dz - y dx)$. The intersection $\Sigma \cap N$ maps to an annulus A in N'. The curve $A \cap \partial N'$ wraps around $\partial N'$ some number of times. Let N'' be a smaller neighborhood of the y-axis and let A' be a "uniformly twisting" annulus in N'' that twists around $\partial N''$ the same number of times as A twists around $\partial N'$. Note the signs of the singularities of A' along γ alternate and can be assumed to be non-degenerate. Now connect $A' \cap \partial N''$ to $A \cap \partial N'$ to get an annulus A''. Note A'' is isotopic to A in N'. Replace A by A'' this is our C^0 isotopy of Σ near γ. Now repeat for the other boundary components.

Recall Lemma 1.5 says that if x is a singularity along the y-axis and ξ is twisting past A'' in a right handed way then x is a positive (negative) singularity of A''_ξ if and only if it is a sink (source) of the flow along γ. The opposite is true if ξ is twisting past A'' in a left handed manner. Thus if $tw(\gamma, \Sigma) > 0$ then there is no way we can make Σ convex! If the twisting is non positive then we can perturb the characteristic foliation to be almost Morse-Smale on the rest of Σ and thus make Σ convex. □

Example 2.24. One may achieve the following foliation near the boundary of a convex surface. Such a foliation is called the *standard foliation* near the boundary of a convex surface. Let $A = S^1 \times [0, 1]$ be a neighborhood of a boundary component γ in Σ. On $A \times \mathbb{R}$ with coordinates x, y, t, with $0 \leq x \leq 1$ the coordinate on S^1, consider the contact structure $\xi = \ker(\sin(2n\pi x)dy + \cos(2n\pi x)dt)$. The foliation on $A = A \times \{0\}$ is as shown in Figure 4. In particular it has $2n$ lines of singularities, at $x = \frac{k}{2n}$, called *Legendrian divides*. The other curves, lines with y constant, in the characteristic foliation are called *ruling curves*. We can perturb A so that the characteristic foliation is like this on $S^1 \times [0, \frac{1}{2}]$ and on the rest of A the Legendrian divides end in "half-elliptic" or "half-hyperbolic" points. See Figure 4.

Exercise 2.25. Show how to perturb A so to achieve the above foliation.
Hint: consider the y-axis as a model of the Legendrian divide in $\mathbb{R}^3$ with the contact structure given by $dz - y dx$.

Theorem 2.26 (Giroux 1991, [8])**.** *Given a compact surface Σ and a contact manifold (M, ξ) let $i : \Sigma \to M$ be an embedding so that $i(\Sigma)$ is a convex surface in (M, ξ), if Σ is not closed then assume $i(\Sigma)$ has Legendrian boundary. Let $\mathcal{F}$ be a singular foliation on Σ such that $\mathcal{F}$ is divided by $\Gamma = i^{-1}(\Gamma_\Sigma)$. Then given any neighborhood U of $i(\Sigma)$ in M, there is an isotopy $\phi_s : \Sigma \to M, s \in [0, 1]$, such that*

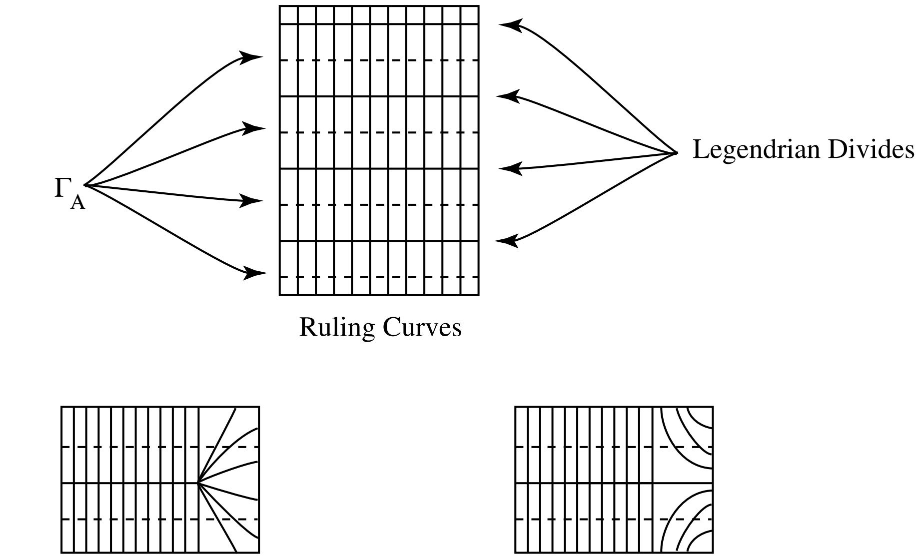


FIGURE 4. The standard foliation on A, top. In this figure the top and bottom are identified to form an annulus and the left edge is γ. The "half-elliptic" and "half-hyperbolic" point, bottom.

(1) $\phi_0 = i$,
(2) ϕ_s *is fixed on* $i^{-1}(\Gamma_\Sigma)$,
(3) $\phi_s(\Sigma) \subset U$ *for all* s,
(4) $\phi_s(\Sigma)$ *is convex with dividing set* Γ_Σ,
(5) $(\phi_1(\Sigma))_\xi = \phi_1(\mathcal{F})$.

Proof. Let $\Psi : \Sigma \times \mathbb{R} \to M$ be a vertically invariant neighborhood of $i(\Sigma)$ in $U \subset M$, so that $\Psi(\Sigma \times \{0\}) = i(\Sigma)$. We will construct the desired isotopy in $X = \Sigma \times \mathbb{R}$ then use Ψ to map it to M.

Since Γ divides both Σ_ξ and $\mathcal{F}$ there is a neighborhood A of Γ in Σ so that both Σ_ξ and $\mathcal{F}$ foliate A by arcs transverse to $\partial A \setminus (\partial\Sigma \cap A)$. Let $\Sigma'_\pm = \Sigma_\pm \cap (\Sigma \setminus A)$ and $X_\pm = \Sigma'_\pm \times \mathbb{R}$. By Remark 2.11 and the discussion in the proof of Theorem 2.15 we can find volume forms ω_i, vector fields w_i, and functions $u_i, i = 0, 1$, so that $\alpha_0 = \iota_{w_0}\omega_0 + u_0\, dt$ is a contact 1-form for $\xi_0 = \Phi^*(\xi)$, $\alpha_1 = \iota_{w_1}\omega_1 + u_1\, dt$ is a contact form on X that induces $\mathcal{F}$ as the characteristic foliation on $\Sigma \times \{0\}$, and so that $u_i = \pm 1$ on $\Sigma'_\pm$.

We concentrate on X_+ (the same arguments apply to X_-). Note $\omega_1 = f\,\omega_0$, for some positive function f. We have

$$f \operatorname{div}_{f\omega_0} w_1 = \operatorname{div}_{\omega_0}(f w_1),$$

Thus $w'_1 = f w_1$ dilates ω_0 on Σ'_+. Now set $w_s = (1-s)w'_1 + s w_0, s \in [0,1]$. All the vector fields w_s dilate ω_0 on Σ'_+. Thus the 1-forms $\alpha_s = \iota_{w_s}\omega_0 + dt$ are all contact forms on X'_+.

We have a family of 1-forms α_s on $X'_+ \cup X'_-$ which we will extend over $A \times \mathbb{R}$. Let B be a small neighborhood of A in Σ. On $X_0 = B \times \mathbb{R}$ we have functions u_s defined near $\partial B \setminus (\partial\Sigma \cap B)$. They are all equal to ± 1 according as the boundary component is in $\Sigma'_\pm$. We can also define the vector fields w_s on $B \times \mathbb{R}$ as a linear combination of w_0 and w_1 as above. Note w_s always generates a foliation by arcs transverse to $\partial B \setminus (\partial\Sigma \cap B)$. We may now extend the u_s's across B as in the proof of Theorem 2.15. We can in addition ensure that u_s is 0 on Γ for all s. We now have a one parameter family of contact forms α_s and vertically invariant contact structures $\xi_s = \ker \alpha_s$ on all of X. Moreover, $(\Sigma \times \{0\})_{\xi_0} = \Sigma_\xi$, $(\Sigma \times \{0\})_{\xi_1} = \mathcal{F}$, and $\Gamma \times \mathbb{R}$ is the characteristic hypersurface for all s.

We apply Moser's method, as in Theorem 1.1. One may check that since all the α_t are vertically invariant, the vector field that generates the flow in Moser's method is also vertically invariant. With this observation it is easy to see that the flow generated by this vector field satisfies all the collusions of the theorem. □

Example 2.27. Recall form Examples 2.18 we have convex tori with $2n$ dividing curves, and $2n$ periodic orbits of slope $\frac{p}{q}$. Using Theorem 2.26 we can arrange that the characteristic foliation on the torus is as shown in Figure 5. That is there are $2n$ lines of singularities (n lines of "sources" and n lines of "sinks") with slope $\frac{p}{q}$, called *Legendrian divides*. Between each two adjacent dividing curves there is a line of singularities. Moreover, the rest of the foliation can be assumes to be by lines of slope s where s is any rational number *not equal to* $\frac{p}{q}$. These curves are called *ruling curves*.

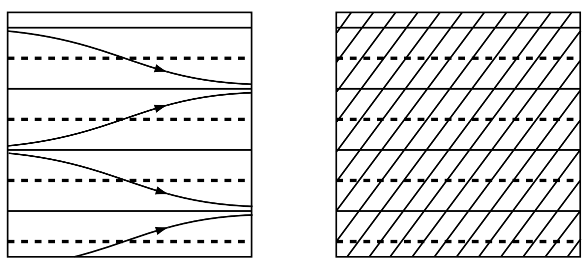


FIGURE 5. Two convex tori with the same dividing curves.

A particularly useful way of thinking about this theorem is as follows. Say a properly embedded graph G in a convex surface Σ with fixed dividing set is *non isolating* if $G \cap \Gamma_\Sigma$ transversely and every component of $\Sigma \setminus G$ intersects Γ_Σ.

Theorem 2.28 (Honda 2000, [3]). *Let Σ be an embedded convex surface in a contact manifold (M, ξ). Let G be a properly embedded graph which is non isolating. Then there is an isotopy of Σ, rel boundary, as in Theorem 2.26 to a surface Σ' so that G is contained in the characteristic foliation of Σ'.*

Proof. According to Theorem 2.26 we only need to construct a singular foliation on Σ that contains G and is also divided by Γ_Σ. To this end let Σ_0 be a component of $\Sigma \setminus (G \cup \Gamma_\Sigma)$. Assume Σ_0 is in Σ_+ so that all the elliptic points in a foliation must be positive. The boundary of Σ_0 contains simple closed curves and arcs (the arcs form circles too but not smooth circles). Each can either be in Γ_Σ, G or $\partial\Sigma$. Along the boundary components coming from Γ_Σ we have the foliation flowing out. Along an circle boundary component coming from G let the foliation be the circle with the flow near by flowing away from the circle. Along a boundary component consisting of arcs all coming from G let the foliation have positive elliptic points at the vertices and a positive hyperbolic point on the interior of each edge. Finally consider a boundary component c that is made up of arcs from G and Γ_Σ. At each vertex between arcs of G put a positive elliptic point and on the interior of an arc from G that has both end points on other arcs from G or both end points on arcs from Γ_Σ put a positive hyperbolic point (with the arc being the stable manifolds of the singularity). Any other arc from G just leave as a non singular Legendrian arc. See Figure 6. If you had any "elliptic vertices" use positive hyperbolic singularities to "shield" it from the inside of Σ_0 See Figure 6.

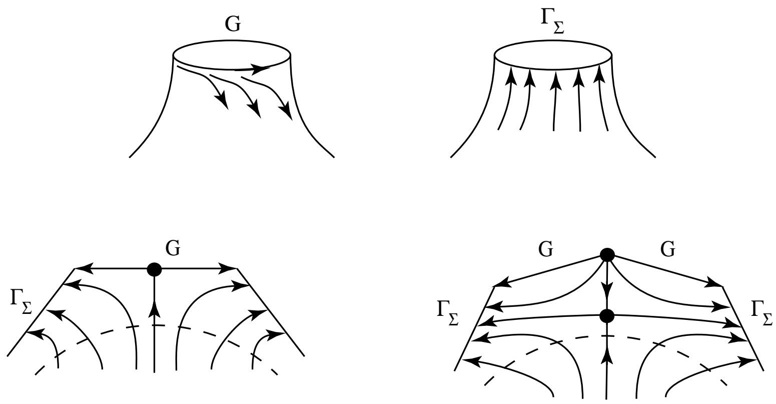


FIGURE 6. Possible foliations near the boundary of Σ_0. In the bottom right is an example with an elliptic vertex that needs to be "shielded" from the rest of Σ_0.

We have now defined the foliation in a tubular neighborhood A of $\partial\Sigma_0$. Moreover the flow is transverse to the boundary of $\Sigma_0' = \Sigma_0 \setminus A$ and along some boundary components is flowing out, we denote these ∂_-, and along others it is flowing in, we denote these ∂_+. The non isolating condition implies that ∂_- is non empty. Thus embed the surface Σ_0' in $\mathbb{R}^3$ between the hyperplanes $\{z = 1\}$ and $\{z = 0\}$ so that ∂_- is in $\{z = 0\}$ and ∂_+ is in $\{z = 1\}$. One can in addition insure that the z coordinate thought of as a hight function on Σ_0' is Morse and has no minima and only one maxima if ∂_+ is empty and no maxima if ∂_+ is not empty. The gradient flow of this hight function gives a foliation of Σ_0' that extends the foliation on A. Note there is at most a single elliptic point that is a source. The other singularities are all hyperbolic and one may always arrange them to have positive divergence. Thus we have constructed a foliation on Σ_0 that has strictly positive divergence. Continuing on the other pieces of $\Sigma \setminus (G \cup \Gamma_\Sigma)$ will eventually yield a foliation on Σ that is divided by Γ_Σ and has G as a union of leaves. □

Corollary 2.29 (Kanda 1997, [6])**.** *Let C be a simple closed curve in a convex surface Σ that intersects the dividing curves transversely and non trivially. Then Σ may be isotoped so that C is a closed leaf in the characteristic foliation.*

Now that we know we can have closed curves in the characteristic foliation of a convex surface we wish to see how the contact planes twist along the surface. Recall if L is a Legendrian curve on a surface Σ then $tw(L, \Sigma)$ denotes the twisting of the contact planes ξ along L measured with respect to the framing on L given by Σ.

Theorem 2.30. *Let L be a Legendrian curve on a convex surface Σ, then*

$$tw(L, \Sigma) = -\frac{1}{2}\#(L \cap \Gamma_\Sigma),$$

where $\#$ means the number of point in the set. If Σ is a Seifert surface, the the above formula computes the Thurston Bennequin number of L. Moreover, in this case one has

$$r(L) = \chi(\Sigma_+) - \chi(\Sigma_-)$$

Proof. Let v be a contact vector field for Σ such that the characteristic hypersurface C of v satisfies $\Gamma_\Sigma = C \cap \Sigma$. The twisting of ξ relative to Σ is the same as the twisting of ξ relative to v (since v is transverse to Σ). We claim that ξ always "twists past" v in a left handed manner. Indeed let N be a small tubular neighborhood of γ. So $N = \gamma \times D$ and we can fix the product structure so that the tips of v trace out the curve $\gamma \times \{p\}$ on ∂N where p is a point in ∂D. Now ξ can be represented by the 1-form $\beta + u\,dt$, where β is a 1-form on Σ and u is a function on Σ. Note in this set up v is $\frac{\partial}{\partial t}$. Let's say x is a point on Γ_Σ. Thus $u(x) = 0$. Let w_1 direct the flow of the characteristic foliation along γ and let w_2 be another vector in $T\Sigma$ such that $\{w_1, w_2\}$ is an oriented basis for Σ and $\beta(w_2) = 1$. Consider $\widetilde{w} = hw_2 + \frac{\partial}{\partial t}$ so that $\widetilde{w} \in \xi$. Near the dividing set, $\widetilde{w}$ gives the framing on N coming from ξ. Now

$$(\beta + u\,dt)(\widetilde{w}) = h + u = 0,$$

so $h = -u$. Thus at all intersection points of γ with Γ_Σ the curve on ∂N coming form v and the curve coming form ξ intersect. Moreover they intersect in a point with orientation $-$ (*i.e.* ξ twists past v in a left handed way). Thus each intersection of γ with Γ_Σ contributes a $-\frac{1}{2}$ to $tw(\gamma, \Sigma)$.

Now for the rotation number computation.

Exercise 2.31. Show that $r(\gamma)$ is the obstruction to extending an oriented tangent vector field along γ to a non zero section of $\xi|_\Sigma$. That is if s is a section of $\xi|_\Sigma$ that extends an oriented tangent vector field to γ then $r(\gamma)$ is the signed count of the image of s with the zero section in ξ.
Hint: Consider the situation when $r(\gamma) = 0$ first, then assume the characteristic foliation of Σ is generic and calculate the contribution of each singularity in Σ_ξ to $r(\gamma)$.

As a corollary to the above exercise it is easy to see that

$$r(\gamma) = r_+ - r_-,$$

where $r_\pm = e_\pm^{Int} - h_\pm^{Int} + \frac{1}{2}(e_\pm^\partial - h_\pm^\partial)$, and $e_\pm^{Int}$ is the number of interior $\pm$ elliptic points in Σ_ξ, $h_\pm^{Int}$ is the number of interior $\pm$ hyperbolic points and similarly $e_\pm^\partial$ and $h_\pm^\partial$ are the numbers of respective singularities along the boundary of Σ.

We first note that

$$\chi(\Sigma_\pm) = r_\pm + \frac{1}{2} tb(\gamma).$$

Indeed, double Σ_+ along $\partial\Sigma \cap \Sigma_+$, that is take two copies of Σ_+ and glue them along the $tb(\gamma)$ arcs in $\partial\Sigma_+$ coming from the boundary of Σ. One gets a surface Σ' such that $\chi(\Sigma') = 2\chi(\Sigma_+) - tb(\gamma)$. Now $\chi(\Sigma') = 2r_+$ (note $\partial\Sigma'$ is transverse to the characteristic foliation). Thus we have the desired equality. But now

$$r(\gamma) = r_+ - r_- = \chi(\Sigma_+) - \chi(\Sigma_-).$$

□

3. Tightness and Convex Surfaces

3.1. **Giroux's Criterion.**

Giroux has shown us how to tell if a vertically invariant neighborhood of a convex surface is tight or not.

Theorem 3.1 (Giroux's Criterion)**.** *Let Σ be a convex surface in (M, ξ). A vertically invariant neighborhood of Σ is tight if and only if $\Sigma \neq S^2$ and Γ_Σ contains no contractible curves or $\Sigma = S^2$ and Γ_Σ is connected.*

Proof. We first assume $\Sigma \neq S^2$ and there is a simple closed curve $\gamma \subset \Gamma_\Sigma$ that bounds a disk D in Σ. Let D' be a slightly larger disk containing D and not intersecting $\Gamma_\Sigma \setminus \gamma$. If $\Gamma_\Sigma \setminus \gamma \neq \emptyset$ then we may apply the Legendrian realization principle, Theorem 2.28, to isotop Σ inside a vertically invariant neighborhood to Σ' so that $\partial D'$ is a Legendrian curve in Σ'_ξ. By Theorem 2.30 we have that D' is an overtwisted disk.

If $\Gamma_\Sigma \setminus \gamma = \emptyset$ choose a homotopically non trivial curve c in Σ that is disjoint from Γ_Σ. We may apply the Legendrian realization principle to realize c as a circle of singularities in the characteristic foliation of Σ. We now study a neighborhood N of c. We can assume that N is contactomorphic to a neighborhood N' of the y-axis in $\mathbb{R}^3/\sim$, where $(x, y, z) \sim (x, y+1, z)$, with the contact structure $\xi = \ker(dz - y dx)$ with $\Sigma \cap N$ mapping to the $A = xy-\text{plane} \cap N'$. Now replace $A \subset \Sigma$ with A' shown in Figure 7. Denote

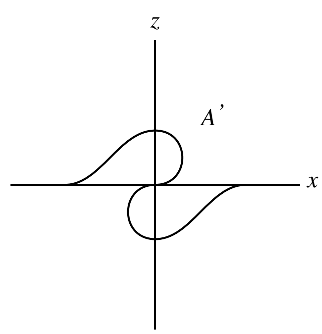


Figure 7. The annulus A' is obtained from the curve in the xz-plane shown here cross the y-axis.

the new surface Σ'. One may easily see that A' has three parallel circles of singularities and that there are two dividing curves in A'. Thus $\Gamma_{\Sigma'}$ has more than one component. Moreover Σ' is in a vertically invariant neighborhood of Σ. Thus if Σ' has an overtwisted vertically invariant neighborhood then so does Σ. But we may use the argument above to construct an overtwisted disk near Σ'.

If $\Sigma = S^2$ and Γ_Σ is disconnected then we can construct an overtwisted disk as above. If Γ_Σ is connected then the vertically invariant neighborhood of Σ can be identified with the neighborhood of the unit sphere in $\mathbb{R}^3$ with the standard tight contact structure. Thus the contact structure on the neighborhood is tight.

Now suppose $\Sigma \neq S^2$ and Γ_Σ has no component bounding a disk. Let U be a vertically invariant neighborhood of Σ. Let $\widetilde{\Sigma} = \mathbb{R}^2$ be the universal cover of Σ. Since no component of Γ_Σ bounds a disk in Σ, Γ_Σ will lift a union of properly embedded lines and arcs in $\widetilde{\Sigma}$. Let V be the universal cover of U. This is simply the vertically invariant "neighborhood" of $\widetilde{\Sigma}$. We claim that the pull back contact structure on V is tight. To see this let G be a graph in Σ that realizes the 1-skeleton of Σ. Clearly G is non-isolating so we may use Theorem 2.26 to isotope Σ to Σ' so that G as a Legendrian graph in Σ' and U is a vertically invariant neighborhood of Σ' too. (We now rename Σ', Σ.) Let $\widetilde{G}$ be the graph lifted to $\widetilde{\Sigma}$. If there is an overtwisted disk in V then it lies over some disk D in $\widetilde{\Sigma}$ and we can assume the disk D is a union of regions in $\widetilde{\Sigma} \setminus \widetilde{G}$. Thus we have ∂D is Legendrian. The disk D is convex and its dividing curves are a union of arcs (no closed disk since $\Gamma_{\widetilde{\Sigma}}$ is a union of lines and intervals).

Exercise 3.2. Show there is a disk D' in S^2 whose intersection with the equator has the same configuration as Γ_D. See for example Figure 8.

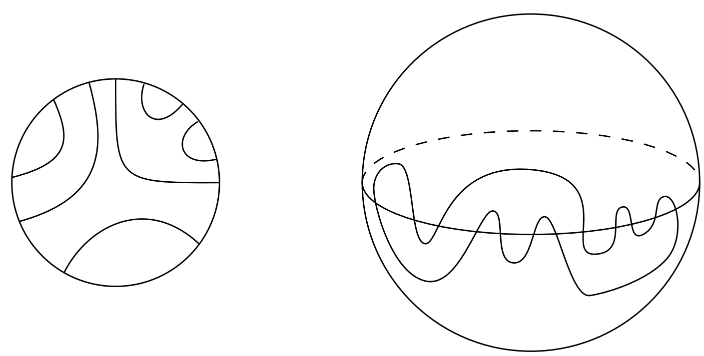


FIGURE 8. The dividing curves on D, left, and the disk $D' \subset S^2$, right.

Of course the unit S^2 in the standard tight contact structure on $\mathbb{R}^3$ has the equator as its dividing set so $D' \cap \Gamma_{S^2}$ is the same as $D \cap \Gamma_{\widetilde{\Sigma}}$. Now on S^2 we can draw the foliation $\mathcal{F}$ that on D' agrees with the characteristic foliation on D and on $S^2 \setminus D'$ is the foliation constructed in the proof of Theorem 2.28. Since this foliation is divided by Γ_{S^2} we can use Theorem 2.28 to realize it in a vertically invariant neighborhood $N = S^2 \times \mathbb{R}$ of S^2 in the standard tight contact structure on $\mathbb{R}^3$. Thus the contact structure on N is tight. Now the contact structure on $D' \times \mathbb{R}$ and $D \times \mathbb{R}$ are contactomorphic, but this contradicts the fact that in $D \times \mathbb{R}$ we assumed there was an overtwisted disk. □

3.2. **Bennequin type inequalities.** We are now ready to demonstrate the first connection between topology and tight contact structures.

Theorem 3.3. *Let (M,ξ) be a tight contact 3-manifold. Any embedded closed surface $\Sigma \neq S^2$ in M satisfies*

$$|e(\xi)([\Sigma])| \leq -\chi(\Sigma),$$

where $e(\xi)$ is the Euler class of the contact structure. If $\Sigma = S^2$ then

$$e(\xi)([\Sigma]) = 0.$$

Remark 3.4. There is a similar inequality for the Euler class of a taut foliation (actually one just needs a Reebless foliation), see [].

Remark 3.5. The inequality in this theorem implies that only finitely many homology classes in $H^2(M;\mathbb{Z})$ can be realized as the Euler class of a tight contact structure, in marked contrast to the situation for overtwisted contact structures where any (even) homology class is the Euler class of an overtwisted contact structure.

Exercise 3.6. Prove this Remark.
Hint: Recall that any element in $H_2(M,\mathbb{Z})$ can be realized by an embedded surface. Pick a basis of $H_2(M,\mathbb{Z})$, realize them by surfaces and then think about what the inequality says.

Proof. Given $\Sigma \neq S^2$ in M use Corollary 2.20 to make it convex. Since ξ is tight Γ_Σ is a union of S^1's that do not bound disks in Σ. As usual write $\Sigma \setminus \Gamma_\Sigma = \Sigma_+ \cap \Sigma_-$. Clearly

$$\chi(\Sigma) = \chi(\Sigma_+) + \chi(\Sigma_-)$$

since $\chi(\Gamma_\Sigma) = 0$. As in Exercise 2.31 one may easily check that

$$e(\xi)(\Sigma) = \chi(\Sigma_+) - \chi(\Sigma_-).$$

Subtracting the first equation form the second yields

$$e(\xi)(\Sigma) - \chi(\Sigma) = -2\chi(\Sigma_-) \geq 0.$$

Thus $-e(\xi)(\Sigma) \leq -\chi(\Sigma)$. By adding the equations together one may similarly show $e(\xi)(\Sigma) \leq -\chi(\Sigma)$, establishing the desired inequality.

If $\Sigma = S^2$ we know $\Sigma \setminus \Gamma_\Sigma$ is the disjoint union of two disks thus $e(\xi)(\Sigma) = 0$. □

Now for the famous Bennequin inequality.

Theorem 3.7. *Let (M,ξ) be a tight contact manifold and K a knot in M bounding the embedded surface Σ. If K is a transverse knot then*

$$sl(K) \leq -\chi(\Sigma).$$

If K is a Legendrian knot then

$$tb(K) + |r(K)| \leq -\chi(\Sigma).$$

Proof. Assume K be a Legendrian knot. We will prove $tb(K) + r(K) \leq -\chi(\Sigma)$ but a similar argument establishes $tb(K) - r(K) \leq -\chi(\Sigma)$. We may assume that $\text{tb}(K) \leq 0$. Indeed we may negatively stabilize K enough times so that $tb(K) \leq 0$ (recall each stabilization decreases $tb(K)$ by 1). Since we are using negative stabilizations $tb(K) + r(K)$ is unchanged. Thus proving the inequality for the stabilized knot will imply the inequality for the original knot.

Since $tb(K) \leq 0$ Theorem 2.23 says we can isotop Σ so that it is convex. By Theorem 2.30 we know $r(K) = \chi(\Sigma_+) - \chi(\Sigma_-)$. We claim that

$$tb(K) + \chi(\Sigma_+) + \chi(\Sigma_-) = \chi(\Sigma).$$

To see this note that there are precisely $tb(K)$ arcs in Γ_Σ, denote their union Γ_a, and so $\chi(\Sigma \setminus \Gamma_a) = \chi(\Sigma) - tb(K)$. But we also know that $\chi(\Sigma \setminus \Gamma_a) = \chi(\Sigma \setminus \Gamma_\Sigma) = \chi(\Sigma_+ \cup \Sigma_-) = \chi(\Sigma_+) + \chi(\Sigma_-)$, since all the components of Γ_Σ not in Γ_a are circles. Thus we have established the above inequality.

Note there are at most $-\text{tb}(K)$ disk components in $\Sigma \setminus \Gamma_\Sigma$. Thus there are at most $-\text{tb}(K)$ disk components in Σ_+. So $\chi(\Sigma_+)$ equals the number of disk components plus the Euler class of the other components. The Euler class of these other components in non-positive. Thus we have shown

$$tb(K) \leq -\chi(\Sigma_+).$$

Now we have

$$\begin{aligned} tb(K) + r(K) &\leq tb(K) + r(K) - 2tb(K) - 2\chi(\Sigma_+) \\ &= -tb(K) + r(K) - 2\chi(\Sigma_+) \\ &= -tb(K) + \chi(\Sigma_+) - \chi(\Sigma_-) - 2\chi(\Sigma_+) \\ &= -tb(K) - \chi(\Sigma_+) - \chi(\Sigma_-) = -\chi(\Sigma). \end{aligned}$$

To prove the inequality for transverse knots we need Lemma 3.8 below. With this lemma in hand let K be a transverse knot and L a Legendrian knot such that its positive transverse push off L_+ is transversely isotopic to K. From earlier we know $sl(K) = tb(L) - r(L)$. Thus we get $sl(K) \leq -\chi(\Sigma)$ from the previously established inequality for Legendrian knots □

Lemma 3.8. *If T is a transverse knot then there is a Legendrian knot L such that its positive transverse push off L_+ is transversely isotopic to T.*

Proof. Any transverse knot T has a neighborhood N contactomorphic to a neighborhood N' of the z-axis in $\mathbb{R}^3/\sim$, where $(x, y, z) \sim (x, y, z + 1)$, with the contact structure $\xi = \ker(dz + r^2 d\theta)$. Let $T_a = \{(r, \theta, z) : r = a\}$. There is a k such that for $a < k$ the torus T_a is in N'. Moreover there is an $a < k$ such that $(T_a)_\xi$ is a foliation by curves of slope $\frac{1}{n}$ for some large negative n. Let L be a leaf in this foliation. Clearly L is Legendrian. Let A be an annulus inside T_a that has one boundary component on L and the other on the z-axis. One may easily check that the foliation is as pictured in Figure 9 so, by definition, the z-axis, which we can think of as T, is the positive transverse push off of L. □

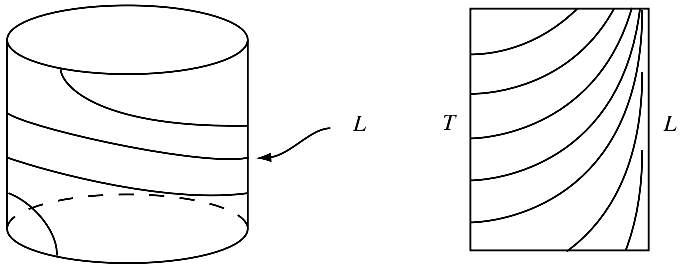


FIGURE 9. The torus T_a near the z axis, left, and the annulus A, right.

Theorem 3.9. *Let ξ_1 and ξ_2 be two tight vertically invariant contact structures on $\Sigma \times \mathbb{R}$. The contact structures are contactomorphic if and only if there is a diffeomorphism $\phi : \Sigma \to \Sigma$ taking the dividing curves Γ_{ξ_1} induced on $\Sigma = \Sigma \times \{0\}$ by ξ_1 to the dividing curves Γ_{ξ_2} induced by ξ_2.*

Theorem 3.10. *Let ξ_1 and ξ_2 be two overtwisted vertically invariant contact structures on $\Sigma \times \mathbb{R}$. The contact structures are contactomorphic if and only if*

$$\chi(\Sigma^1_+) - \chi(\Sigma^1_-) = \pm(\chi(\Sigma^2_+) - \chi(\Sigma^2_-)),$$

where $\Sigma \setminus \Gamma_{\xi_i} = \Sigma^i_+ \coprod \Sigma^i_-$.

4. Distinguishing contact structures and the first classification results

The power of convex surfaces is contained largely in Theorem 2.26 in conjunction with the ability to transfer information from one convex surface to another one meeting it along a Legendrian curve.

Lemma 4.1 (Kanda 1997, [6]; Honda 2000, [3])**.** *Suppose that Σ and Σ' are convex surfaces, with dividing curves Γ and Γ', and $\partial\Sigma' \subset \Sigma$ is Legendrian. Let $S = \Gamma \cap \partial\Sigma'$ and $S' = \Gamma' \cap \partial\Sigma'$. Then between each two adjacent points in S there is one point in S' and vice verse. See Figure 10. (Note the sets*

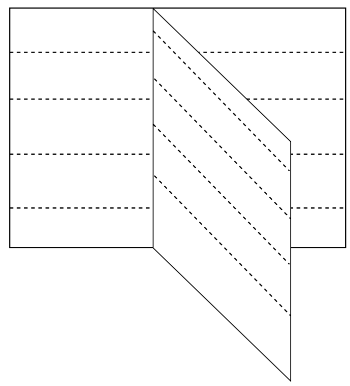


Figure 10. Transferring information about dividing curves from one surface to another. The top and bottom of the picture are identified.

S and S' are cyclically ordered since they sit on $\partial\Sigma'$)

To prove this lemma one just considers a "standard model". More specifically, consider $\mathbb{R}^3/\sim$, where $(x, y, z) \sim (x, y, z+1)$, with the contact structure $\xi = \ker(\sin(2n\pi z)dx + \cos(2n\pi z)dy$. Let $\Sigma = \{(x, y, z) : x = 0\}$ and $\Sigma' = \{(x, y, z) : y = 0, x \geq 0\}$. Note both these surface are convex and the boundary of Σ' is a Legendrian curve in Σ. In Figure 10 we see the situation for $n = 2$. The choice of n in this model is clearly determined by $tw(\partial\Sigma', \Sigma')$. Lemma 4.1 clearly follows form considering this model.

Exercise 4.2. Show that the situation described in Lemma 4.1 can always be modeled as described above.

Using this model it is also easy to see how to "round corners".

Lemma 4.3 (Honda 2000, [3])**.** *Suppose that Σ and Σ' are convex surfaces, with dividing curves Γ and Γ', and $\partial\Sigma' = \partial\Sigma$ is Legendrian. Suppose Σ and Σ' are modeled as above with $\Sigma = \{(x, y, z) : x = 0, y \geq 0\}$, then we may form a surface Σ'' from $S = \Sigma \cap \Sigma'$ by replacing S intersect a small neighborhood N of $\partial\Sigma$ (thought of as the z-axis) with the intersection of N with $\{(x, y, z) : (x-\delta)^2 + (y-\delta)^2 = \delta^2\}$ For a suitably chosen δ, Σ' will* *be a smooth surface (actually just C^1, but it can then be smoothed by a C^1 small isotopy which of course does not change the characteristic foliation) with dividing curve as shown in Figure 11.*

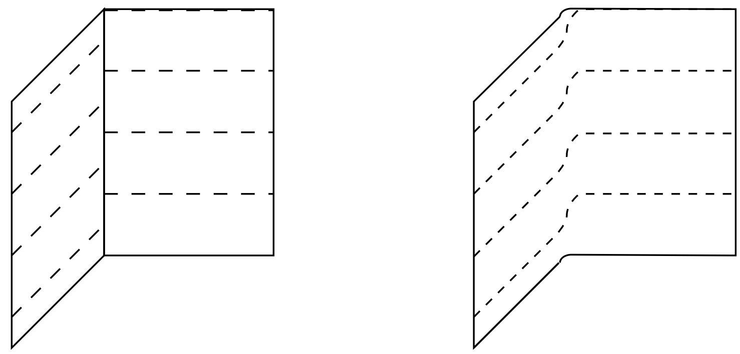


FIGURE 11. Rounding a corner between two convex surfaces.

Remark 4.4. *Note this lemma says that as you round a corner then the dividing curves on the two surfaces connect up as follows. Moving from Σ to Σ' the dividing curves move up (down) if Σ' is to the right (left) of Σ.*

4.1. **Neighborhoods of Legendrian curves.** We can now give a simple proof of the following result which is essentially due to Makar-Limanov [14], but for the form presented here see Kanda [6]. Though this theorem seems easy, it has vast generalizations which we indicate below.

Theorem 4.5 (Kanda 1997, [6])**.** *Suppose $M = D^2 \times S^1$ and $\mathcal{F}$ is a singular foliation on ∂M that is divided by two parallel curves with slope $\frac{1}{n}$ (here slope $\frac{1}{n}$ means that the curves are homotopic to $n[\partial D^2 \times \{p\}] + [\{q\} \times S^1]$ where $p \in S^1$ and $q \in \partial D^2$). Then there is a unique tight contact structure on M whose characteristic foliation on ∂M is $\mathcal{F}$.*

Proof. To see existence simply consider a standard neighborhood of a Legendrian knot or similarly consider the tori T_a in the proof of Lemma 3.8.

Suppose we have two tight contact structures ξ_0 and ξ_1 on M inducing $\mathcal{F}$ as the characteristic foliation on ∂M. We will find a contactomorphism from ξ_0 to ξ_1 (in fact this contactomorphism will be isotopic to the identity). Let $f\colon M \to M$ be the identity map. By Theorem 1.1 we can isotop f rel. ∂M to be a contactomorphism in a neighborhood N of ∂M. Now let T be a convex torus in N isotopic to ∂M. Moreover we can assume that the characteristic foliation on T is in standard form. We know the slope of the Legendrian divides is $\frac{1}{n}$ and we choose the slope of the ruling curves to be 0. Let D be a meridianal disk whose boundary is a ruling curve. We can perturb D so that it is convex and using Lemma 4.1 we know that the dividing curves for D intersect the boundary of D in two points. Moreover, since there are no closed dividing curves on D (since the contact structure is tight, see

Theorem 3.1) we know that Γ_D consists of one arc. We may isotop $f(D)$ (rel. boundary) to D' so that all of this is true for D' with respect to ξ_1. Now using Theorem 2.26 we can arrange that the characteristic foliations on D and D' agree; and further, we can isotop f (rel. N) so that f takes D to D' and preserves the characteristic foliation on D. Thus another application of Theorem 1.1 says we can isotop f so as to be a contactomorphism on $N' = N \cup U$, where U is a neighborhood of D. Note that $B = \overline{M \setminus N'}$ is a 3–ball, so Theorem **??** tells us that we can isotop f on B so that it is a contactomorphism on B too. Thus f is a contactomorphism on all of M and we are done with the proof. □

4.2. **The classification of tight contact structures on T^3.** Recall for each positive n we have the following tight contact structures on $T^3 = \mathbb{R}^3/\sim$, where $\sim$ is the equivalence relation generated by unit translation in each coordinate direction: $\xi_n = \ker(\cos(2\pi nz)\,dx + \sin(2\pi nz)\,dy)$. We will refer to "directions" in T^3 by their corresponding directions in $\mathbb{R}^3$.

Theorem 4.6 (Kanda Giroux)**.** *The contact structures ξ_n are distinct.*

This theorem follows immediately from the following lemma.

Lemma 4.7. *Identify $H_1(T^3)$ with $\mathbb{Z}^3$ by picking as a basis the loops in each of the coordinate directions. Any Legendrian knot L in (T^3, ξ_n) (with $tw(L) \leq 0$) isotopic to the linear simple closed curve in the homology class (a, b, c), where a, b, c are all prime to one another, satisfies*

$$tw(L) \leq -n|c|,$$

where the twisting of L is measured with respect to any incompressible torus containing L. Moreover, there is a linear Legendrian knot with $tw(L) = -n|c|$.

Remark 4.8. For $|c| \neq 0$ the assumption that $tw(L) \leq 0$ is not necessary because if there were a Legendrian knot as in the lemma with $tw(L) > 0$ then one could stabilize the knot until it had twisting 0. Moreover, one can show that the hypothesis that $tw(L) \leq 0$ is never necessary but it requires extra work and we do not need that result for the proof of Theorem 4.6.

Exercise 4.9. Show that the assumption that $tw(L) \leq 0$ is not needed.
Hint: If this is difficult read the rest of this section first.

Proof. We begin by considering the case where L is in the homology class $(0, 0, \pm 1)$. Assume L is a Legendrian knot isotopic to the linear simple closed curve in the homology class $(0, 0, 1)$ with $tw(L) = -n + 1$ (note if the inequality in the lemma is violated we can always find such a Legendrian knot by stabilization). Let A be the T^2 in T^3 corresponding to the xz-plane and let B be the torus corresponding to the yz-plane. There is a finite cover of T^3 that unwraps the xy directions (but not the z direction) in T^3 in which there are lifts $\widetilde{L}, \widetilde{A}$, and $\widetilde{B}$ of L, A, and B so that $\widetilde{L}$ is disjoint from $\widetilde{A} \cup \widetilde{B}$. Note $tw(\widetilde{L})$ (measured in the cover) is equal to $tw(L)$.

Furthermore $\widetilde{A}$ and $\widetilde{B}$ are convex each having $2n$ dividing curves running in the $(1,0,0)$ and $(0,1,0)$ direction, respectively. Let S be the manifold obtained by removing small vertically invariant neighborhoods of $\widetilde{A}$ and $\widetilde{B}$ and rounding the resulting corners. Clearly $S = D^2 \times S^1$ and using the edge rounding lemma (Lemma 4.3) we see that there are two dividing curves on S with slope $\frac{1}{n}$.

Exercise 4.10. Prove this.
Hint: consider how the dividing curves intersect a meridian and a longitude on S.

By Theorem 4.5 there is a unique tight contact structure on S. Note $\widetilde{L}$ is the core of S and with respect to the product structure on S the twisting of $\widetilde{L}$ is one greater than the slope of the dividing curves. In the standard tight contact structure on S^3 let S' be a standard neighborhood of the Legendrian unknot U with $tb(L) = -1$. Since there is a diffeomorphism from S to S' taking the dividing curves on S to the dividing curves on S' we know (again by Theorem 4.5) the contact structure on S is contactomorphic to the one on S'. Now $\widetilde{L}$ is a Legendrian knot in S' with twisting number equal to 0. But in S^3 this is an unknot with $tb = 0$ and hence violates the Bennequin inequality (Theorem 3.7). Thus our hypothesized L cannot exist.

For the case $(a, b, \pm 1)$ we need the following lemma, which we prove later.

Lemma 4.11. *Let $A \in SL(3,\mathbb{Z})$ and $\Psi_A : T^3 \to T^3$ be the induced diffeomorphism of T^3. Assume Ψ_A preserves the xy-plane in T^3 then Ψ_A is isotopic to a contactomorphism of ξ_n.*

Given this lemma we can clearly apply a contactomorphism to (T^3, ξ_n) taking $(a, b, \pm 1)$ to $(0, 0, \pm 1)$ and thus this case is reduced to the first case considered above.

Now consider the case with (a, b, c), and $|c| > 1$. Let L be a Legendrian knot isotopic to a linear simple closed curve in the homology class (a, b, c). Denote by $\Phi : T^3 \to T^3$ the $|c|$ fold covering map of T^3 that unwraps the z-direction. Clearly $\Phi^*\xi_n = \xi_{n|c|}$. Let $\widetilde{L}$ be a lift of L. It is easy to see that $tw(\widetilde{L}) = tw(L)$ (recall the twisting is measured with respect to any incompressible torus containing the Legendrian knot and the incompressible torus can also be lifted to the cover, see Figure 12). So we have

$$tw(L) = tw(\widetilde{L}) \leq -n|c|.$$

Exercise 4.12. In all the above cased find a Legendrian knot realizing the corresponding upper bound.

We are finally left to consider the case $(a, b, 0)$. It is easy to see that for any such homology class there is a constant k such that the torus $\{(x, y, z)|z = k\}$ in T^3 is foliated by (a, b) curves. Thus any leaf in this foliation is a Legendrian simple closed curve with $tb = 0$. □

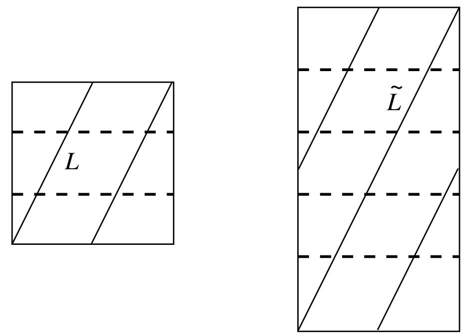


FIGURE 12. On the left is the knot L sitting in the xz-plane (or an isotopic copy of it). On the right is the two fold cover of the xz-plane. Note there are two lifts of L we have chosen one to be $\widetilde{L}$. As always, dashed lines are dividing curves.

Proof of Lemma 4.11. Under the hypothesis that Ψ_A preserves the xy-plane we can write

$$A = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix},$$

where $g(ad - bc) = 1$. Note this implies $g = \pm 1$ and we assume $g = 1$. The other case is handled similarly. Now let

$$\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in SL(2, \mathbb{R}), t \in [0, 1],$$

be a path a matrices from the identity matrix to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In addition let $e(t)$ and $f(t)$ be two functions on the unit interval starting at e and f, respectively and ending at 0. Finally set

$$A(t) = \begin{pmatrix} a(t) & b(t) & e(t) \\ c(t) & d(t) & f(t) \\ 0 & 0 & g \end{pmatrix},$$

and note that $A(t)$ is a path of matrices in $SL(3, \mathbb{R})$ from the identity to A. The matrices $A(t)$ do not necessarily induce diffeomorphisms of T^3 but we can pull back the 1-form on R^3 defining ξ_n with $A(t)$ to get

$$\begin{aligned} \alpha_t = {} & [a(t)\cos(2n\pi z) + c(t)\sin(2n\pi z)]\, dx \\ & + [b(t)\cos(2n\pi z) + d(t)\sin(2n\pi z)]\, dy \\ & + [e(t)\cos(2n\pi z) + f(t)\sin(2n\pi z)]\, dz. \end{aligned}$$

Note all these forms are invariant under the unit translations in the coordinate directions. So they all define 1-forms on T^3. Also, α_0 is the 1-form α defining ξ_n and $\alpha_1 = \Psi_A^* \alpha$. Thus we have a one parameter family if contact structure from ξ_n to $\Psi_A^*\xi_n$. Thus using Moser's theorem (Theorem **??**) we can find an isotopy of Ψ_A so that it preserves ξ_n. □

We are now ready for the classification of contact structures on T^3 up to contactomorphism.

Theorem 4.13. *If ξ is any tight contact structure on T^3 then it is contactomorphic to ξ_n for exactly one n.*

Proof. We are given a contact structure ξ on T^3. We would like to find two incompressible tori T_0 and T_1 such that

(1) T_i is convex for $i = 0, 1$,
(2) $T_0 \cap T_1 = L$ a Legendrian simple closed curve and
(3) $-2tw(L) = \#\Gamma_{T_i}$, for $i = 0, 1$.

If we find such tori the we can construct a contactomorphism to (T^3, ξ_n), where $n = tw(L)$. Indeed, from Theorem 2.30 we see that $tw(L) = -\frac{1}{2}\#(\Gamma_{T_i} \cap L)$ so each component of Γ_{T_i} intersects L once. Let h_i be the homology class of L on T_i and h_i' the homology class of a component of Γ_{T_i}, then h_i, h_i' form a basis for $H_1(T_i)$. Thus we can find a diffeomorphism f from T^3 to $T^3 = \mathbb{R}^3/\sim$ taking L to the z axis, h_0 to the homology class of the x-axis and h_1 to the homology class of the y-axis. By isotoping f we can assume f takes the dividing curves on T_0 to the dividing curves on the xz-plane (we are giving $T^3 = \mathbb{R}^3/\sim$ the contact structure ξ_n) and the dividing curves on T_1 to the dividing curves on the yz-plane. There is a further isotopy of f so that f takes the characteristic foliations of T_0 and T_1 to the characteristic foliations on the xz and yz-planes, respectively. Thus we can isotop f to be a contactomorphism on a neighborhood of $T_0 \cup T_1$. Now the complement of this neighborhood is a solid torus S.

Exercise 4.14. Show that there are two dividing curves on ∂S and the have slope $\frac{1}{m}$.
Hint: Work in $R^3/\sim$.

We can now use Theorem 4.5 to further isotop f so that it is a contactomorphism from (T^3, ξ) to (T^3, ξ_n).

To find the tori T_0 and T_1 let T be a convex incompressible torus in T^3 with the minimal number of dividing curves among all such tori. (T will have 2 dividing curves, but we only know this *a postiori*.) Let L' be a Legendrian knot isotopic to a linear simple closed curve and intersecting T transversely one time. We moreover assume that the lexicographically ordered pair $(\frac{1}{\#\Gamma_T}, tw(L'))$ is maximal (but $tw(L')$ non positive) among all convex tori isotopic to T and Legendrian knots isotopic to L'. Let γ and γ' be two linear non-homologous simple closed curves in T which each have homological intersection with a connected component of Γ_T equal to 1. (We of course choose γ and γ' to be non-homologous.) Using Theorem 2.28, we can Legendrian realize $\gamma \cup \gamma'$ on T.

Exercise 4.15. There is a convex torus S (respectively S') that contains L' and γ (respectively γ') as Legendrian arcs. Moreover, we can assume that $S \cap T = \gamma, S \cap T = \gamma'$ and $S \cap S' = L'$.
Hint: This argument is identical to the one used in Theorem 2.23. Basically find nice strips containing the arcs that could be convex. Then extend them to a torus in any way you can. Then C^∞ perturb this torus away from the strips to make it convex.

If the dividing curves of S are not homotopic to $S \cap T$ (or similarly for S' and T') then S and T will be the desired tori T_0 and T_1. Indeed, note that the number of dividing curves on S and T are the same, since

$$tw(\gamma) = -\frac{1}{2}\#(\Gamma_S \cap \gamma) \leq -\frac{1}{2}\#\Gamma_S,$$

and

$$tw(\gamma) = -\frac{1}{2}\#(\Gamma_T \cap \gamma) = -\frac{1}{2}\#\Gamma_T.$$

However, we chose T so that $\#\Gamma_T$ is minimal thus we must have $\#\Gamma_T = \#\Gamma_S$. Moreover the above inequality must be an equality and thus S and T satisfy all the conditions necessary for T_0 and T_1.

Since we are done unless the dividing curves of S are homotopic to $S \cap T$ (and similarly for S' and T') we will assume that is the case from this point on. We now claim that S and S' can be taken to be T_0 and T_1. To see this we only need to verity property (3). To this end we first note that homologically each dividing curve in Γ_S (and $\Gamma_{S'}$) intersects L' one time, because the dividing curves are homotopic to $S \cap T$. We now claim that the actual number of intersection points between each dividing curve in Γ_S (and $\Gamma_{S'}$) and L' is one, for if this were not the case then there would be and arc c in $\Gamma_S \setminus (\Gamma_S \cap L')$ that cobounded a disk D in $S \setminus L'$ with an arc on L'. See Figure 13. Thus there is a simple closed curve L''

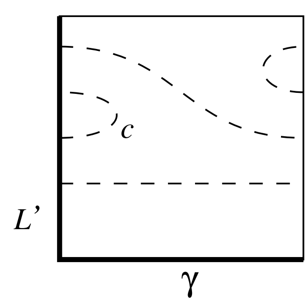


FIGURE 13. Dividing curves on S that intersect L' more than once.

on S that intersects Γ_S two fewer times than L' and is homotopic to L'. By the Legendrian realization principal (Theorem 2.28) we can assume L'' is a Legendrian simple closed curve on an isotoped copy of S. Moreover, Theorem 2.30 says that $tw(L'') = -\frac{1}{2}\#(\Gamma_S \cap L'') > -\frac{1}{2}\#(\Gamma_S \cap L') = tw(L')$ contradicting the choice of L', if L'' intersects T once. The only way for L''

to intersect T more than once is for D to intersect γ. If this happens let γ'' be γ pushed across the disk D. We can Legendrian realize $L'' \cup \gamma''$ and $tw(\gamma'') > tw(\gamma)$. Now if T' is a copy of T containing the Legendrian curves $\gamma' \cap \gamma''$ then the fact that $\#\Gamma_T$ is minimal among tori isotopic to T, each curve in Γ_T intersects γ' once and $tw(\gamma'') > tw(\gamma)$, implies $\Gamma_{T'}$ is parallel to γ''. In addition, since $tw(\gamma')$ is unchanged we must have $\#\Gamma_{T'} = \#\Gamma_T$. Thus the pair L'' and T' contradict the choice of L' and T. This contradiction implies that each curve in Γ_S intersects L' once, and hence one may easily verify property (3) of the tori T_0 and T_1 for S and S'. □

From the above proof one may easily obtain the classification of tight contact structures up to isotopy.

Theorem 4.16. *There is a one to one correspondence between tight contact structures on T^3 and*

$$\{(A,k) | A \in H_2(T) \text{ is primitive and } k \text{ is a negative integer}\}.$$

Exercise 4.17. Prove this theorem.
Hint: Given a contact structure ξ let $h : H_1(T^3) \to \{n | n \in \mathbb{Z}, n \leq 0\}$ be the map that records the maximal (non positive) twisting number of Legendrian knots isotopic to linear simple closed curves in a given homology class. From Theorem 4.13 we know there is a basis b_1, b_2, b_3 for $H_1(T^3)$ so that $h(b_1) = h(b_2) = 0$ and $h(b_3) = k$. There is a unique incompressible torus T associated to b_1 and b_2. The map in this theorem sends ξ to the homology class of T and $h(b_3)$. Show if ξ and ξ' have the same data then they are isotopic.

4.3. **Characteristic foliations on the boundary of a manifold.** Given a singular foliation $\mathcal{F}$ on a (possibly disconnected) surface Σ and a manifold M with $\partial M = \Sigma$ then denote by tight$(M, \mathcal{F})$ the *isotopy classes of tight contact structures* on M that induce $\mathcal{F}$ on the boundary.

Theorem 4.18. *Let $\mathcal{F}_i$ be a singular foliation on Σ that is divided by the multi-curve $\Gamma, i = 0, 1,$ and M a three manifold with boundary Σ. There is a one to one correspondence between tight$(M, \mathcal{F}_0)$ and tight$(M, \mathcal{F}_1)$*

This theorem makes precise the idea that the classification of tight contact structures on a manifold with convex boundary should only depend on the dividing curves and not the specific foliation on the boundary. From now on when classifying tight contact structures on a manifold with convex boundary we only specify the dividing curves on the boundary and the result will hold for any singular foliation on the boundary respecting these dividing curves. We denote by tight(M, Γ) the set tight$(M, \mathcal{F})$ for any singular foliation divided by Γ.

Proof. We define the map

$$\phi : \text{tight}(M, \mathcal{F}_0) \to \text{tight}(M, \mathcal{F}_1)$$

as follows: let $\Sigma \times (-\infty, 0] \subset M$ be a vertically invariant neighborhood of ∂M, defined using the contact vector field v. Take a surface $\Sigma_k = \Sigma \times \{k\} \subset$

$\Sigma \times (-\infty, 0)$. Use the Giroux flexibility theorem (Theorem 2.26) to isotop Σ_k to Σ' so that $\Sigma'_\xi = \mathcal{F}_1$. Let M' be the component of $M \setminus \Sigma$ not contained in $\Sigma \times (-\infty, 0]$. Finally we set $\phi(M, \xi) = (M', \xi|_{M'})$.

Exercise 4.19. Show this map is well defined.
Hint: To do this you must show the ϕ is independent of k and the contact vector field v. To show the first part prove that $(M' = M \setminus \Sigma \times (-\infty, l], \xi|_{M'})$ is contactomorphic to (M, ξ) for any l. For the second part consider Remark 2.11

Using the well definedness of ϕ it is easy to show the similarly defined map from tight$(M, \mathcal{F}_1)$ to tight$(M, \mathcal{F}_0)$ is the inverse of ϕ. □

4.4. **Relative Euler classes.** Let ξ be any contact structure (or even simply a plane field) on a manifold M with boundary. Assume $\xi|_{\partial M}$ is trivializable. Given any section s of $\xi|_{\partial M}$ we can define the relative Euler class $e(\xi, s) \in H^2(M, \partial M; \mathbb{Z})$ as follows: Let s' be any extension of s to a section of ξ of all of M. By genericity we can assume the image of s' intersects the zero section of ξ transversely. Denote this intersection by Z. The relative Euler class $e(\xi, s)$ is the Poincaré dual of $[Z]$. It is standard to show the relative Euler class depends only on the isotopy (thought non-zero sections) class of s.

Lemma 4.20. *Let (M, ξ) be a contact manifold with convex boundary and s a section of $\xi|_{\partial M}$. Let Σ be an oriented connected convex surface properly embedded in M. If a tangent vector field to $\partial\Sigma \subset \partial M$, inducing the correct orientation on $\partial\Sigma$, agrees with s along $\partial\Sigma$ then*

$$e(\xi, s)([\Sigma]) = \chi(\Sigma_+) - \chi(\Sigma_-).$$

The proof of this lemma is almost identical to part of the proof of Theorem 2.30. The one difference (which is irrelevant) is that $\partial\Sigma$ might have more than one boundary component.

Suppose ∂M has a convex torus boundary component T. Fix a section of ξ along the other boundary components. By Theorem 4.18 we can change the characteristic foliation on T without affecting the contact structure (up to isotopy). Thus we arrange that the characteristic foliation on T is in standard form (Example 2.27) with ruling slope r. Now let s be the section of $\xi|_{\partial M}$ that was given on $\partial M \setminus T$ and on T is tangent to the ruling curves.

Exercise 4.21. There are two seemingly different choices for this section along T, depending on the direction we traverse the ruling curves. Show these two choices are isotopic.

Exercise 4.22. Suppose we choose a different ruling slope r' and let s' be the corresponding section of ξ along ∂M. Show $e(\xi, s) = e(\xi, s')$.

4.5. **The basic slice.** In this section we consider the manifold $M = T^2 \times [0, 1]$. The contact manifold (M, ξ) is called a *basic slice* if

(1) ξ is tight,
(2) $\partial M = T_0 \cup T_1$ is convex with $\#\Gamma_{T_i} = 2$,

(3) v_0, v_1 form an oriented integral basis for $H_1(T^2 \times \{0\}; \mathbb{Z})$, where v_i is a minimal length integral vector with slope equal to the slope s_i of the dividing curves Γ_{T_i}.
(4) the slope of the dividing curves on any convex torus T in M parallel to the boundary is between s_0 and s_1, this condition is called *minimal twisting.*

By "between" in the definition above we mean that if we view slopes on the projectivized unit circle, then the slope is clockwise of s_0 and counterclockwise of s_1. Note given any basic slice there is a diffeomorphism of M taking s_0 to 0 and s_1 to -1. So the classification of tight contact structures follows from

Theorem 4.23 (Honda 2000, [3])**.** *There are exactly two basic slices with $s_0 = 0$ and $s_1 = -1$. Moreover their relative Euler classes are given by $(0, \pm 1) \in H^2(M, \partial M; \mathbb{Z}) = H_1(T^2; \mathbb{Z})$.*

The proof of this theorem is contained in the following lemmas.

Lemma 4.24. *There are at most two basic slices with $s_0 = 0$ and $s_1 = -1$.*

Proof. Using the ideas in Theorem 4.5, we will show that given a contact structure on M we can cut M along convex surfaces until we obtain a three ball with a unique tight contact structure on it. The number of possible tight contact structures will then be determined by the number of choices we had for the dividing curves on the surfaces we cut along.

Let (M, ξ) be a basic slice. So T_i is convex with two dividing curves, $i = 0, 1$, and the slope of the dividing curves on T_0, T_1 is $0, -1$, respectively. We can assume the characteristic foliation on $T_0 \cup T_1$ is standard with ruling slope infinite (that is the ruling curves are vertical). Now take a vertical annulus $A = S^1 \times [0, 1]$ the boundary component $S_i = S^1 \times \{i\}$ a ruling curve in T_i, and $A \cap \partial M = \partial A$. Note the twisting number of S_i with respect to A is -1, (since the twisting with respect to A is the same as with respect to T_i and on T_i the curve S_i intersects the dividing curves twice). Using Theorem 2.23 we can now perturb A to be convex with standard boundary.

Since ξ is tight and the dividing curves on A intersect each boundary component twice we know that Γ_A is either two boundary parallel arcs (*i.e.* each cobounds a disk with the boundary of A) or two arcs running from one boundary component of A to the other. See Figure 14 If the components of Γ_A were boundary parallel then we could Legendrian realize a vertical simple close curve L on A with twisting number 0. We could then find a torus T in M that contained L and was convex. Since the twisting of L with respect to T is 0, we know the dividing curves must also be vertical. This contradicts the fact that ξ has minimal twisting. Thus the dividing curves must run from one boundary component of A to the other.

We can fix an identification of the boundary components of A so that for each curve c in Γ_A, $c \cap S_0$ is taken to $c \cap S_1$. After identifying the boundary components of A, Γ_A with consist of two simple closed curves

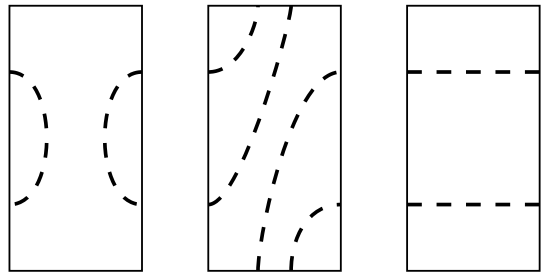


FIGURE 14. Three possible dividing curve configurations on A. (Top and bottom of each rectangle is identified to form A.)

of slope h for some integer h. We call $h = h(A)$ the holonomy of A. If Figure 14, with the natural identifications of the right and left side of the rectangles, the holonomy is 1 and 0 for the two rightmost pictures. (Note we never really identify the boundary components of A we are just using the identification to define h.) We can isotop A so as to increase or decrease $h(A)$ by 1 (and hence after further isotopy we can increase or decrease the holonomy by any integer). To see this let A' be a parallel copy of A in a vertically invariant neighborhood of A.. Consult Figure 15 throughout this construction. Denote the boundary components of A' by $S_i', i = 0, 1$. Further

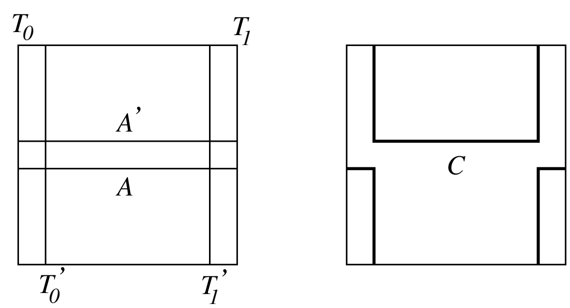


FIGURE 15. Both figures are $S^1 \times [0, 1]$ cross sections of the basic slice (identify the top and bottoms of the squares). On the left are the tori and annuli used in the isotopy of A. On the right is C before edge rounding.

let $T_i \times (-\infty, 0]$ be a small vertically invariant neighborhood of T_i in M, so that $S_i \times (-\infty, 0]$ is a standardly foliated annulus (we can of course assume that $S_i \times (-\infty, 0] = A \cap T_i \times (-\infty, 0]$) and similarly for S_i'. Let $T_i' = T_i \times \{k\}$ for any $k < 0$. Let B_i be the region on T_i' that is between A and A' and $B_i' = T_i' \setminus B_i$. Set A'' to be the sub-annulus of A' between T_0' and T_1' and B the union of the sub-annuli of A between T_0 and T_0' and between T_1 and T_1'. Now consider the annulus $C = A'' \cup B_0' \cup B_1' \cup A''$ with corners rounded.

Exercise 4.25. Show C is isotopic, rel boundary, to A and $h(C) = h(A) \pm 1$ where the ± 1 depends on which side of A the annulus A' sits.

Thus up to isotoping A rel boundary, there is only one possibility for Γ_A. Now if we cut M open along A and round the corners we get solid torus S with convex boundary having dividing curves of slope -2. (Note you should think of -2 as $-1+0+-1$ where the $-1+0$ comes from adding the slopes of the dividing curves on T_0 and T_1 and the other -1 comes from rounding the corners.)

Using Theorem 4.18 we can assume the characteristic foliation on ∂S is standard with ruling slope 0. Thus we can find a meridianal disk D with ∂D a ruling curve in ∂S. The twisting along ∂D is -2 so we can make D convex. There are two possibilities for the dividing curves on D. See Figure 16. After

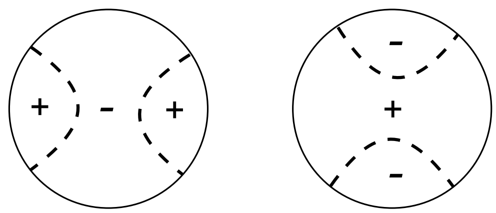


FIGURE 16. Possible dividing curves on D.

cutting S along D we get the three ball which has a unique tight contact structure. Thus there are only two possible basic slices corresponding to the different possibilities for the dividing curves on D. $\square$

Lemma 4.26. *There are two basic slices with* $s_0 = 0$ *and* $s_1 = -1$ *and they are distinguished by relative Euler classes which are given by* $(0, \pm 1) \in H^2(M; \mathbb{Z}) = H_1(T^2; \mathbb{Z})$.

Proof. Consider $T^2 \times \mathbb{R}$ with the contact structure $\xi = \ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy)$, where (x, y) are coordinates on T^2 and z is the coordinate on $\mathbb{R}$. Consider $T^2 \times [0, \frac{1}{8}] \subset T^2 \times \mathbb{R}$. The foliation on $T^2 \times \{0\}$ is linear of slope 0 and the foliation on $T^2 \times \{\frac{1}{8}\}$ is linear of slope -1. Thus we can perturb the boundary (as in Example 2.18) to be convex with two dividing curves on each boundary component of slope 0 and -1 respectively. Denote this contact manifold (M, ξ). To show (M, ξ) is a basic slice we only need to check that it is minimally twisting (the other properties are clear from construction).

For this we need a basic result about linear twisting. To this end let $M_{r,r'} = T^2 \times [a, b]$ with the contact structure $\xi = \ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy)$, where $0 < a < b < 1$ is such that the slope of the characteristic foliation on $T^2 \times \{a\}$ and $T^2 \times \{b\}$ is r and r', respectively. Note, the characteristic foliations on $T^2 \times \{t\}$ is linear and moving from r to r' in a left handed manner as t goes from a to b.

Lemma 4.27. *If s is a slope not in the interval $[r, r']$ then there is no convex torus in $(M_{r,r'}, \xi)$ with dividing slope s.*

Remark 4.28. *There is also no such linearly foliated torus*

Proof. If the lemma is not true then there is a convex torus T parallel to each boundary component of $M_{r,r'}$ with slope s not between r and r'. There is another (abstract) slope s' such that $s < s' < r' < r$ and the minimal integral vectors v and v' corresponding to slopes s and s' from an oriented basis of T^2. By "abstract" slope we just mean a slope of a curve on T^2 not necessarily related to dividing curves on a torus in $M_{r,r'}$ in any way.

Exercise 4.29. Prove such a slope s' exists
Hint: If you have trouble see Section 5.2 below. In fact, all statements you cannot figure out about slopes and bases below will be clarified there.

There is a linear diffeomorphism of T^2 taking s to 0 and s' to ∞. This diffeomorphism sends r and r' to some negative slope. Let ϕ be the corresponding diffeomorphism of $M_{r,r'}$. Note $\phi_*\xi$ is a subset of the contact structure $\ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy)$, on $T^2 \times (0, \frac{1}{4})$.

Exercise 4.30. Prove this assertion.
Hint: ϕ is a linear map on the torus and hence $\phi_*\xi$ is a foliation on torus times interval that induces linear foliations on $T^2 \times \{\text{point}\}$ whose slope rotates from $\phi(r)$ to $\phi(r')$. Now use Theorem 1.2.

Exercise 4.31. Show that $(T^2 \times (0, \frac{1}{4}), \ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy))$ embeds in S^3 with the standard tight contact structure such that the x-direction maps to a meridian of an unknot and the y-direction maps to a longitude for the same unknot.
Hint: The standard contact structure on S^3 comes from the complex tangencies to the unit S^3 in $\mathbb{C}^2$. Let $H = \{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}$ be the Hopf link in S^3. Check that H is transverse to the standard contact structure and that $S^3 \setminus H = T^2 \times (0, 1)$ with the characteristic foliation on $T^2 \times \{t\}$ linear for each t and their slopes running from 0 to $-\infty$ as t runs from 0 to 1 (all end points non-inclusive).

From the previous two exercises we see that $\phi(M_{r,r'})$ with $\phi_*\xi$ embeds in S^3 with the standard contact structure. Moreover the slope of the characteristic foliation on $\phi(T)$ is 0. But from the construction of this embedding $\phi(T)$ bounds a solid torus such that the curve of slope 0 is a meridian for the torus. Thus a meridianal disk for this solid torus with boundary a leaf in the characteristic foliation is an overtwisted disk, contradicting the tightness of the standard contact structure on S^3. □

We now return to the proof of Lemma 4.26. We left off trying to show that (M, ξ) is minimally twisting. Recall (M, ξ) is obtained from $(T^2 \times [0, \frac{1}{8}], \xi = \ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy))$ by perturbing the boundary to be convex. We discuss the perturbation of $T^2 \times \{0\}$ more carefully (a similar discussion holds for $T^2 \times \{\frac{1}{8}\}$). There is some function $f : T^2 \to [-\delta, \delta]$ such that the graph of $f \subset T^2 \times \mathbb{R}$ is the perturbation of $T^2 \times \{0\}$ that makes it convex. Note that $f_t(p) = tf(p)$ for any $t \in (0,1]$ is also a function whose graph is a perturbation of $T^2 \times \{0\}$ making it convex. Let (M_t, ξ_t) be the contact manifold obtained from $T^2 \times [0, \frac{1}{8}]$ by perturbing $T^2 \times \{0\}$ by f_t (and similarly for $T^2 \times \{\frac{1}{8}\}$). From Lemma 4.27 we see there are no convex tori (parallel to the boundary) in (M_t, ξ_t) with slope outside $[0+\epsilon_t, -1-\epsilon_t]$ where ϵ_t is some small number depending on t and going to zero as t goes to zero. Note there is a canonical way (up to topological isotopy) to identify M_t with M, we use t only to make precise the physical subset of $T^2 \times \mathbb{R}$ that we are talking about.

We now claim that for any $t \in (0,1]$ we can think of (M_1, ξ_1) as a subset of (M_t, ξ_t). More specifically there is a contactomorphism of (M_1, ξ_1) onto a subset of (M_t, ξ_t) that does not change the slopes of curves on the T^2 factor of $M = M_t = M_1$. Once this is proven we will see that $(M, \xi) = (M_1, \xi_1)$ is minimally twisting since if there were a convex torus T parallel to the boundary with slope s not in $[0, -1]$ then we could find such a torus (with slope s) inside (M_t, ξ_t) for any t contradicting our discussion in the previous paragraph.

To prove our claim we use a simple version of an important technique called "isotopy discretization" (see Section 7 for the full description). For each $t \in (0,1]$ there is a vertically invariant neighborhood U_t of the graph of f_t in $T^2 \times \mathbb{R}$. For each t let $O'_t = \{t' \in (0,1] : \text{the graph of } f_{t'} \subset U_t\}$ and O_t be an open connected interval in O'_t containing t. Fixing t_0 we want to show that (M_1, ξ_1) is a subset of (M_{t_0}, ξ_{t_0}). If t_0 is in O_1 then the graph of f_{t_0} is in U_1 the vertically invariant neighborhood of T_1, the graph of $f = f_1$. Thus we can use the invariant neighborhood structure to find two tori T'_1 and T''_1 that are translates of T_1 in the product neighborhood structure such that the graph of f_{t_0} separates T'_1 and T''_1. See Figure 17. As in the

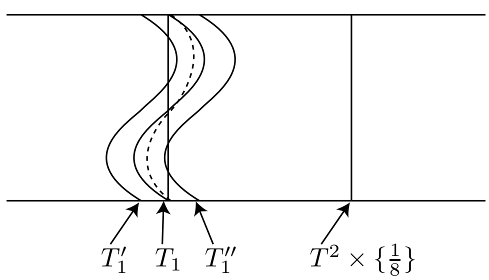


FIGURE 17. Part of $T^2 \times \mathbb{R}$. The graph of f_{t_0} is shown as a dotted line.

proof of Theorem 4.18 we see that whether we cut $T^2 \times \mathbb{R}$ along T_1, T'_1 or T''_1 we will get contact manifolds isotopic to (M_1, ξ_1). Since one of T'_1 or T''_1 is inside (M, ξ_{t_0}) we see that (M, ξ_1) can be thought of as a subset of (M_{t_0}, ξ_{t_0}). We can also think of (M_{t_0}, ξ_{t_0}) as a subset of (M_1, ξ_1) since one of T'_1 or T''_1 will be outside of M_{t_0}. (Recall, we are only discussing altering one boundary component but we must alter the other one in a similar fashion.)

Now suppose t_0 is not in O_1 but that $O_{t_0} \cap O_1 \neq \emptyset$. Let t_1 be a point in the intersection. As above (M_1, ξ_1) can be thought of as a subset of (M_{t_1}, ξ_{t_1}) and (M_{t_1}, ξ_{t_1}) can be though of as a subset of (M_{t_0}, ξ_{t_0}).

Exercise 4.32. Finish this line of argument to show that for any fixed $t_0 \in (0, 1]$ we may always think of (M, ξ) as a subset of (M_{t_0}, ξ_{t_0}).
Hint: For a fixed t_0 the interval $[t_0, 1]$ is compact so it can be covered by a finite number of the O_t's.

We now compute the Euler class of (M, ξ). Let A be a vertical annulus. That is A is a vertical curve in the torus T^2 times $[0, 1]$. We can assume the boundary of L are Legendrian ruling curves in ∂M. Perturb A to be convex. From the proof of Lemma 4.24 we know that there are two dividing curves on A that run from one boundary component to the other. Thus Lemma 4.20 implies that $e(\xi)(A) = 0$. Now if A' is a horizontal annulus we can assume that $A' \cap (T^2 \times \{0\})$ is a Legendrian divide and the other boundary component is a Legendrian ruling curve that intersects the dividing curves on ∂M twice. Now make A' convex. Again from the proof of Lemma 4.24 we see that A' has exactly one dividing curve beginning and ending on $A' \cap (T^2 \times \{1\})$. Thus from Lemma 4.20 we see that $e(\xi)(A') = \pm 1$. Thus $e(\xi) = (0, \pm 1) \in H_1(T^2; \mathbb{Z})$. We do not actually need to know the sign here. To see this consider the map $\Psi(x, y, t) = (-x, -y, t)$ of M to itself. The map Ψ preserves the dividing curves on ∂M and $\Psi_* \xi$ is minimally twisting, thus $\Psi_* \xi$ is a basic slice. Also note that acting on $H_1(T^2; \mathbb{Z})$ we see $\Psi_*(0, \pm 1) = (0, \mp 1)$. Thus (M, ξ) and $(M, \Psi_*(\xi))$ are distinct (up to isotopy) basic slices realizing the claimed relative Euler classes. □

Exercise 4.33. Identify the exact relative Euler class of (M, ξ).

Exercise 4.34. Show that the basic slice you get from perturbing the boundary of $(T^2 \times [\frac{1}{2}, \frac{5}{8}], \xi = \ker(\sin(2\pi z)\, dx + \cos(2\pi z)\, dy))$ to be convex will give the basic slice $(M, \Psi_* \xi)$ from the proof.

Examining our model (M, ξ) for a basic slice we have the following immediate corollary.

Corollary 4.35. *Inside a basic slice with dividing slopes s_0 and s_1 on its back and front, respectively, boundary components there is a torus T parallel to the boundary that is linearly foliated by curves of slope s for any s between s_0 and s_1.*

5. Bypasses and basic slices again

5.1. **Bypasses.** Let Σ be a convex surface and α a Legendrian arc in Σ that intersects the dividing curves Γ_Σ in 3 points p_1, p_2, p_3 (where p_1, p_3 are the end points of the arc). Then a *bypass for* Σ *(along* α*)*, see Figure 18, is a convex disk D with Legendrian boundary such that

(1) $D \cap \Sigma = \alpha$,
(2) $tb(\partial D) = -1$,
(3) $\partial D = \alpha \cup \beta$,
(4) $\alpha \cap \beta = \{p_1, p_3\}$ are corners of D and elliptic singularities of D_ξ.

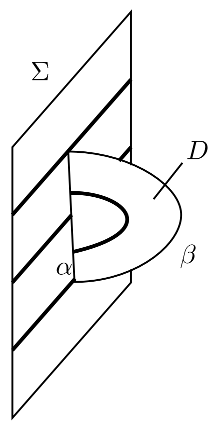


FIGURE 18. A piece of Σ and the bypass D.

After isotopy we can assume (by Giroux Flexibility Theorem 2.26) that the characteristic foliation on D is given if Figure 19.

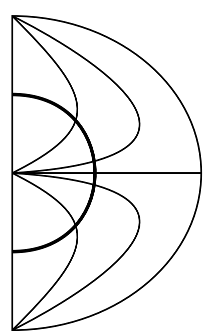


FIGURE 19. Standard foliation on a bypass. Light lines are leaves in characteristic foliation. Dark lines are dividing curves.

If there is a natural orientation on the disk D then the sign of the singularity on the interior of α is called the *sign of the bypass*.

Theorem 5.1 (Honda 2000, [3])**.** *Let Σ be a convex surface, D a bypass for Σ along $\alpha \subset \Sigma$. Inside any open neighborhood of $\Sigma \cup D$ there is a (one sided) neighborhood $N = \Sigma \times [0,1]$ of $\Sigma \cup D$ with $\Sigma = \Sigma \times \{0\}$ (and if Σ is oriented then orient N so that as oriented manifold $\Sigma = -\Sigma \times \{0\}$) such that Γ_Σ is related to $\Gamma_{\Sigma\times\{1\}}$ as shown if Figure 20.*

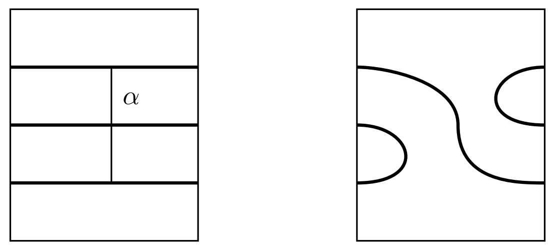


FIGURE 20. Result of a bypass attachment. Original surface Σ with attaching arc α, left. The surface Σ', right.

We say $\Sigma \times \{1\}$ is *obtained from Σ by a bypass attachment.*

Remark 5.2. If the end points of the attaching arc agree, that is $p_1 = p_3$, then the bypass is called *degenerate*.

Exercise 5.3. Show a degenerate bypass can be attached to a surface and have a similar effect on the dividing curves as a regular bypass. What exactly is the effect on the dividing curves? (Maybe reed the following proof first.)

Proof. There are many ways to try to prove this theorem and the reader is encouraged to think about their own way of showing this. The main idea is to use the Edge Rounding Lemma 4.3. With that said, we give an argument for the theorem. Though it might seem a bit convoluted the goal of this proof is to try to only round one edge at a time. (If the reader pursues the most obvious proof they will have to deal with three surfaces coming together in a triple point like three coordinate planes in $\mathbb{R}^3$.)

Using the transverse contact vector field v we get by definition of Σ being convex, we can construct a vertically invariant (one sided) neighborhood $\Sigma \times [0,\epsilon]$ of Σ with D attached to $\Sigma \times \epsilon$. To achieve this we push D forward using the flow of v. Now extend α in $\Sigma \times \{\epsilon\}$ to the simple closed curve in Figure 21 and let A be a neighborhood of α' in $\Sigma \times \{\epsilon\}$ that is standardly ruled by curves parallel to α'. Use v to flow α' forward to get an annulus A' and flowing D by v again we can assume that D is attached to the top of A' to obtain the annulus A'' seen in Figure 22.

Exercise 5.4. Show we can round the corners of A'' by a C^0 small isotopy to obtain the annulus B that has smooth Legendrian boundary. See Figure 22.
Hint: Show this situation can be modeled by the following as follows: Let Q be three quadrants in the xy-plane in $\mathbb{R}^3$. Consider the radially symmetric tight contact structure on $\mathbb{R}^3$. Show, after possibly perturbing A'' rel boundary, a neighborhood of the corner of A'' is contactomorphic to a neighborhood of the origin in $\mathbb{R}^3$ so

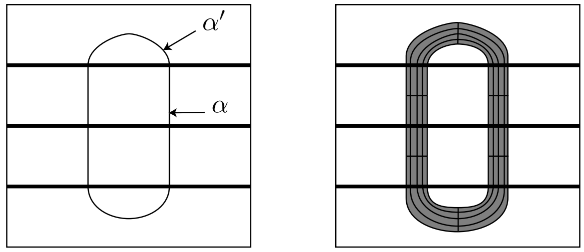


FIGURE 21. The arc α', left. The shaded region on the right is the annulus A.

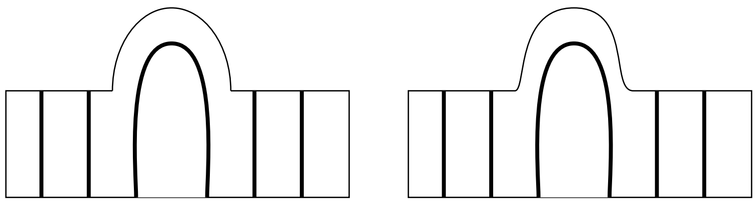


FIGURE 22. The annulus A'', left, and the annulus B, right. (The right and left sides are identified.)

that a neighborhood of the corner in A'' is taken to Q. Now in this explicit model prove the corner can be rounded.

The annulus B is convex so there is a contact vector field w for B. Moreover, we can assume that w is tangent to $\Sigma \times \{\epsilon\}$ along A and the flow of α' by w produces the Legendrian ruling of A.

Exercise 5.5. Show we can find a w with such properties.

Let $C = B \times [-\delta, \delta]$ be a vertically invariant neighborhood of B obtained from B by the flow of w. The boundary of C not identified with part of the boundary of $\Sigma \times [0, \epsilon]$ (this is called the *upper boundary*) is shown on the left in Figure 23. When the corners on these boundary components are rounded you get the picture on the right in Figure 23. Let $N' = \Sigma \times [0, \epsilon] \cup C$. Figure 24 shows the upper boundary of N'. Let N be N' with corners rounded. Then the upper boundary of N is shown in Figure 24 and N is a neighborhood of $\Sigma \cup D$. This completes the proof the Theorem □

Exercise 5.6. If Σ' be obtained from Σ by a bypass attachment then show

$$\chi(\Sigma'_+) - \chi(\Sigma'_-) = \chi(\Sigma_+) = \chi(\Sigma_-).$$

Theorem 5.7 (Honda 2002, [5]). *Let Σ be a convex surface and Σ' be obtained from Σ by a bypass attachment in a tight contact manifold. Then*

1. $\Gamma_{\Sigma'} = \Gamma_\Sigma$ *(this is called a* trivial bypass attachment*),*
2. $|\Gamma_{\Sigma'}| = |\Gamma_\Sigma| + 2$,
3. $|\Gamma_{\Sigma'}| = |\Gamma_\Sigma| - 2$,
4. $\Gamma_{\Sigma'}$ *is obtained from* Γ_Σ *by a positive Dehn twist about some curve in* Σ,

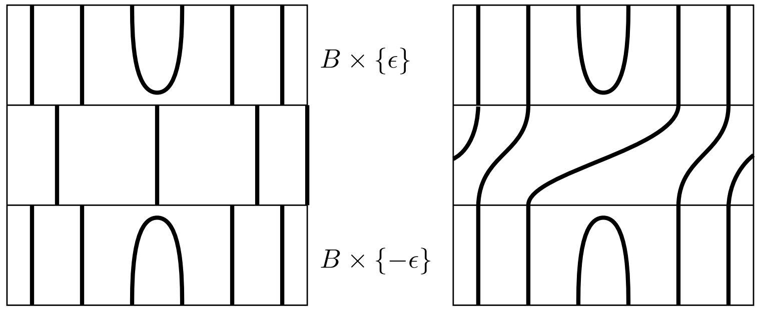


Figure 23. The upper boundary of ∂C, left. Upper boundary of C after corners are rounded, right. (The left and right sides of each picture are identified. The middle strip in both pictures is the upper boundary of ∂B times an interval.)

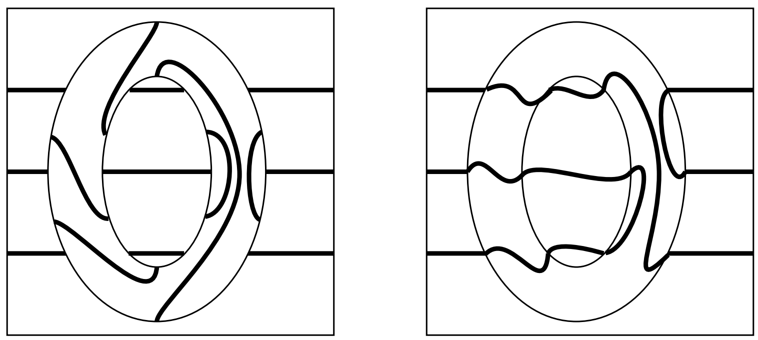


Figure 24. Top view of N', left. Top view of N, right.

(5) $\Gamma_{\Sigma'}$ *is obtained from* Γ_Σ *by a "mystery move", see Figure 25.*

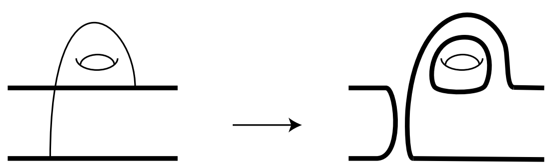


Figure 25. The Mystery Move.

Proof. Let α be the arc of attachment and p_1, p_2, p_3 the points where α intersects the dividing curves. As above we take the end points of α to be p_1 and p_3. Let γ_i be the component of the dividing set in which p_i sits.

Case (I): all the γ_i's are distinct. In this case it is easy to see that γ_1 is joined to γ_2, γ_2 is joined to γ_3 and γ_3 is joined to γ_1. So in Σ all the γ_i's are distinct but on Σ' they have all been joint. Thus conclusion (3) of the theorem holds.

Case (II): all the γ_i's are the same. There are four possible configurations for the bypass here. See Figure 26. It the first configuration we see a

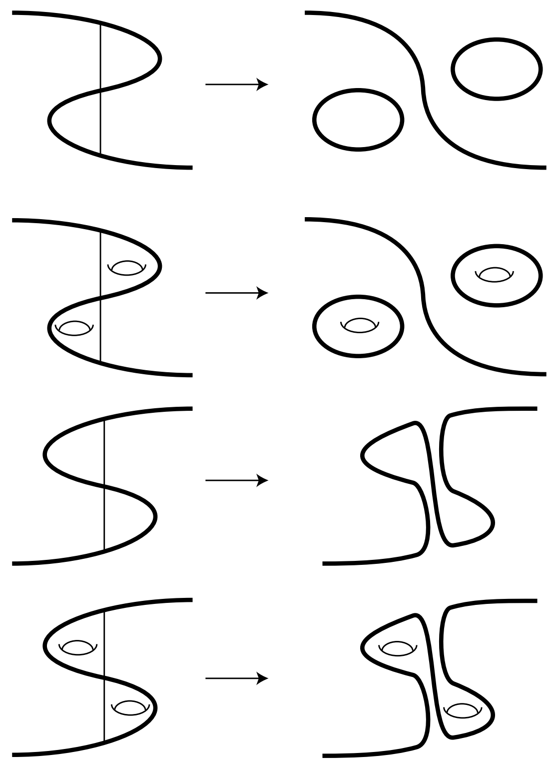


FIGURE 26. The four configurations in Case (II).

contractible dividing curve is formed. Thus this cannot happen in a tight contact structure. If the bypass is configured in the same way but "wraps around topology" (as in the second configuration) then the number of dividing curves increase by two so conclusion (2) of the theorem holds. In the third configuration the topology of the dividing set does not change so a trivial bypass has been attached and conclusion (1) of the theorem holds. In the last configuration conclusion (4) of the theorem holds

Exercise 5.8. Show conclusion (4) of the theorem holds in this case.
Hint: See Figure 27.

Case (III): $\gamma_1 = \gamma_3$ but γ_2 is distinct. In this situation one easily sees conclusion (4) of the theorem holds. See Figure 28. If the endpoints of the attaching arc are switched from what is shown in Figure 28 then we see conclusion (5).

Case (IV): If $\gamma_1 \neq \gamma_3$ but γ_2 agrees with γ_1 or γ_2. As similar analysis as to Case (II) show either conclusion (1), (4) or (5) happens. □

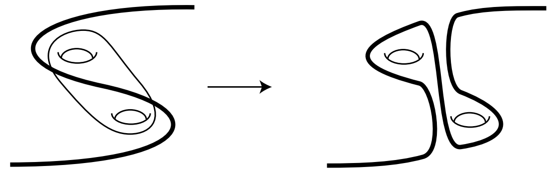


Figure 27. A right handed Dehn twist along the curve on the left changes the dividing curves as shown.

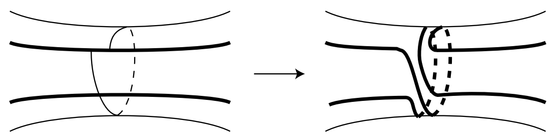


Figure 28. The configuration in Case (III).

Example 5.9. *If $\Sigma = S^2$ is a convex sphere in a tight contact manifold then any bypass attachment must be trivial! This is clear since Γ_Σ has only one component. See Figure 26.*

Example 5.10. *If $\Sigma = T^2$ then the situation in Theorem 5.7 also simplifies. Specifically we can have:*

(1) *Trivial bypass attachment (if the bypass attaching arc intersects only one dividing curve on Σ and does not wind around any topology).*
(2) *Attachment that increases the number of dividing curves (if the bypass attaching arc intersects only one dividing curve on Σ and winds around some topology).*
(3) *Attachment that decreases the number of dividing curves (if the bypass attaching arc intersects three different dividing curves on Σ)*
(4) *Attachment that performs a right handed Dehn twist (if the attaching arc involves only two dividing curves of Σ). NOTE: this can only happen when $|\Gamma_\Sigma| = 2$.*

Theorem 5.11 (Honda 2000, [3])**.** *Let T be a convex torus in a tight contact structure. Assume that T is standardly foliated with dividing slope 0 and ruling slope r with $-\infty < r \leq -1$. Assume there is a bypass D attached to T along a ruling curve. There is a (one sided) neighborhood $N = T^2 \times [0,1]$ of $T \cup D$ with $T \times \{0\} = T$ such that*

(1) *If $|\Gamma_T| > 2$ then $|\Gamma_{T^2\times\{1\}}| = |\Gamma_T| - 2$ and the dividing slope is unchanged.*
(2) *If $|\Gamma_T| = 2$ then the dividing slope on $T^2 \times \{1\}$ is -1 and $|\Gamma_{T^2\times\{1\}}| = 2$.*

Moreover in Case (2), N is a basic slice.

Proof. Part (1) is immediate from Theorem 5.7.
Part (2) is also obvious. See Figure 29.

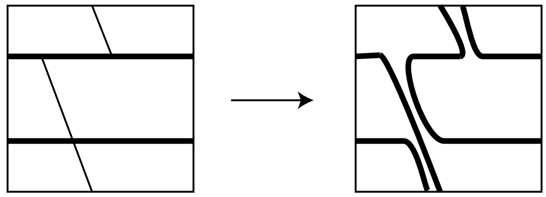


FIGURE 29. A bypass attached along a ruling curve slope less than -1.

Exercise 5.12. Show that if bypass is attached along a curve of positive slope then the slope of the dividing curves on $T^2 \times \{1\}$ is ∞.

Exercise 5.13. Try to figure out what happens if a bypass is attached with slope between 0 and -1.
Hint: If you have trouble see the next section.

All conditions for a basic slice are clearly satisfied except for minimal twisting. To show minimal twisting we embed N into a standard model where we know the twisting is minimal. To this end consider $T^2 \times \mathbb{R}$ with the contact structure $\xi = \ker(\sin(2\pi z)\,dx + \cos(2\pi z)\,dy)$. We can perturb $T_0 = T^2 \times \{0\}$ and $T_{\frac{1}{8}} = T^2 \times \{\frac{1}{2}\}$ to be convex with two dividing curves each, with slope 0 and -1, respectively.We further perturb these tori so that the characteristic foliations are standard with both having ruling slope $-\frac{p}{q}, p > q > 0$. Let A be a horizontal convex annulus between these two tori whose boundary is a ruling curve on each of the tori. So $\partial A \cap T_0$ is a $(-q, p)$ curve on T_0. The dividing curves on T_0 are $(1, 0)$ curves. So

$$\partial A \cap \Gamma_{T_0} = 2\det\begin{pmatrix} 1 & -q \\ 0 & p \end{pmatrix} = 2p.$$

Exercise 5.14. Show that a linear (p, q) curve and a linear (r, s) curve on a torus intersect

$$\det\begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

times.

Similarly

$$\partial A \cap \Gamma_{T_{\frac{1}{8}}} = 2\det\begin{pmatrix} -1 & -q \\ 1 & p \end{pmatrix} = 2(p - q).$$

Since $q \neq 0$, we see that $\partial A \cap T_0$ intersects Γ_A more times than $\partial A \cap T_{\frac{1}{8}}$. (Consult Lemma 4.1 if this is not obvious to you.) From the Imbalance Principle (see Lemma 5.22 below) we see there must be a bypass D' for T_0 along the ruling curve, when $p > 1$. We can arrange that the characteristic foliation on $T \cup D$ is the same as the characteristic foliation on $T_0 \cup D'$ so a they have contactomorphic neighborhoods. Thus N contact embeds into $T^2 \times [-\epsilon, \frac{1}{8} - \epsilon]$ with the standard contact structure. (Moreover, this embedding preserves slopes on T^2.) By Lemma 4.27 this is minimally twisting so N must also be minimally twisting.

We are left to consider the case when $r = -1$ (that is $p = q = 1$). In this case the argument is as above except the bypass is *degenerate*. As was shown in Exercise 5.3 this makes no difference and thus the conclusions of the proof still hold. □

5.2. **Bypasses and the Farey tessellation.** Theorem 5.11 tells us everything we need to know about attaching bypasses to tori along ruling curves because we can use a diffeomorphism of the torus to put any situation into the one considered in the theorem. To avoid having to do this all the time we translate this theorem into a more useful form. To state this more useful form we need some preliminaries on the Farey tessellation.

Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. We will think of this as the hyperbolic plane with the circle at ∞ adjoined. This interpretation is not so important. The main thing is when we say two points on the boundary of $\mathbb{D}$ are connected by a geodesic, we mean the are connected by a segment of a circle (or line) that is orthogonal to $\partial\mathbb{D}$. With this understood, we can ignore the fact that $\mathbb{D}$ is the hyperbolic plane.

Label the point $(1, 0)$ on $\partial\mathbb{D}$ by $0 = \frac{0}{1}$ and the point $(-1, 0)$ with $\infty = \frac{1}{0}$. Now joint them by a geodesic. If two points $\frac{p}{q}, \frac{p'}{q'}$ on $\partial\mathbb{D}$ with non-negative y-coordinate have been labeled then label the point on $\partial\mathbb{D}$ half way between them (with non-negative y-coordinate) by $\frac{p+p'}{q+q'}$. Then connect this point to $\frac{p}{q}$ by a geodesic and also connect this point to $\frac{p'}{q'}$ by a geodesic. Continue this until all positive fractions have been assigned to points on $\partial\mathbb{D}$ with non-negative y-coordinates. Now repeat this process for the points on $\partial\mathbb{D}$ with non-positive y-coordinate except start with $\infty = \frac{-1}{0}$. See Figure 30.

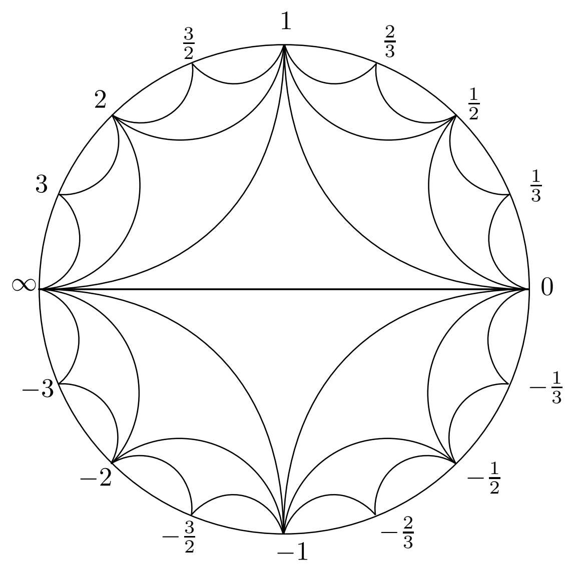


FIGURE 30. The Farey Tessellation.

Exercise 5.15. Show that two points on $\partial\mathbb{D}$ correspond to an integral basis of $\mathbb{Z}^2$ if and only if there is an edge in the Farey tessellation connecting them. (A point $\frac{p}{q}$ corresponds the vector $\begin{pmatrix} q \\ p \end{pmatrix}$ in $\mathbb{Z}^2$.)

Theorem 5.16 (Honda 2000, [3])**.** *Let T be a convex torus in standard form with $|\Gamma_T| = 2$, dividing slope s and ruling slope $r \neq s$. Let D be a bypass for T attached along a ruling curve. Let T' be the torus obtained from T by attaching the bypass D. Then $|\Gamma_{T'}| = 2$ and the dividing slope s' of $\Gamma_{T'}$ is determined as follows: let $[r, s]$ be the arc on $\partial\mathbb{D}$ running from r counterclockwise to s, then s' is the point in $[r, s]$ closest to r with an edge to s.*

Proof. Let s' be as described in the theorem. Let s'' be chosen as s' but in the arc $\overline{\partial\mathbb{D} \setminus [r, s]}$. By the choice of s' there is an edge between s and s' in the Farey tessellation. Similarly for s'' and s. We claim there is also an edge between s'' and s'. Indeed since there is an edge between s and s' their corresponding integral vectors form a basis for $\mathbb{Z}^2$ thus we can find an element B of $SL(2, \mathbb{Z})$ that sends s to 0 and s' to ∞.

Exercise 5.17. Show applying an element of $SL(2, \mathbb{Z})$ to points in the Farey tessellation does not change properties like their ordering on $\partial\mathbb{D}$ and whether or not they are connected by an edge.

The map B will send r and s'' to some positive numbers. Since we know s'' has an edge to s it must go to $1, \frac{1}{2}, \frac{1}{3}, \ldots$. If s'' is send to say $\frac{1}{2}$ then r will be strictly between 1 and $\frac{1}{2}$ on $\partial\mathbb{D}$. This is by the way s'' is defined. But the contradicts how s' is chosen since 1 is closer to the image of r than ∞ is and it has an edge to 0 (the image of s). Thus s'' cannot go to $\frac{1}{2}$ similarly it cannot go to $\frac{1}{n}$ for any $n \neq 1$. Thus s'' is sent to 1. Since 1 and ∞ are connected by an edge so are s' and s'' as claimed.

Since s, s' and s'' are all connected by edges we may find an element C of $SL(2, \mathbb{Z})$ that sends s to 0, s' to -1 and s'' to ∞. Thus r must go to some number less than or equal to -1. Now applying Theorem 5.11 to this transformed problem we see attaching a bypass along a ruling curve with slope the image of r will produce a torus with dividing slope -1. Transforming all this back to the original situation using C^{-1} we see that attaching bypass along part of a ruling curve of slope r will produce a torus with dividing slope s'. □

5.3. **Finding bypasses.** We have seen that bypasses allow you to understand how convex surfaces change. We now discuss how to find bypasses in particular situations.

Lemma 5.18. *Let Σ be a convex surface and Σ' be a convex surface with Legendrian boundary such that one component of $\partial\Sigma'$ a subset of Σ. Moreover, assume $\Sigma' \cap \Sigma \subset \partial\Sigma'$. If $\Gamma_{\Sigma'}$ has an outermost "boundary parallel" dividing curve γ and $|\Gamma_{\Sigma'}| \neq 1$ then Σ' may be isotoped rel boundary so that*

γ is isotopic in Σ' to a Legendrian curve β such that the disk β cuts off from Σ' is a bypass for Σ along part of $\partial\Sigma'$.

An arc properly embedded in a surface with boundary is called *boundary parallel* if one of the components of the complement of the arc is a disk. If the arc is part of a dividing set then we say it is outermost if the disk it separates off contains no other components of the dividing set.

Remark 5.19. Because of this lemma, boundary parallel dividing curves are frequently called bypasses. This is an abuse of language, but should not cause confusion if one is careful.

Proof. This is an easy application of the Legendrian Realization Principle (Theorem 2.28).

Exercise 5.20. Use the Legendrian Realization Principle to prove this theorem.

□

Lemma 5.21. *Suppose Σ and Σ' are as above. Assume further that Σ' is a disk. If $tb(\partial\Sigma') < -1$ then there is a bypass for Σ.*

Proof. If $\text{tb}(\partial\Sigma') = -n$ then there will be n arcs in $\Gamma_{\Sigma'}$. At least one of them must be boundary parallel and outermost. So if $n > 1$ Lemma 5.18 finishes the proof. □

Lemma 5.22 (Imbalance Principle: Honda 2000, [3])**.** *Suppose Σ and Σ' are as in Lemma 5.18. Moreover, assume that $\Sigma' = S^1 \times [0, 1]$ with $S^1 \times \{0\} \subset \Sigma$ (and of course $\partial\Sigma'$ Legendrian). If $tw(S^1 \times \{0\}, \Sigma') < tw(S^1 \times \{1\}, \Sigma') < 0$ then there is a bypass for Σ.*

Proof. It is clear that $\Gamma_{\Sigma'}$ will have more than one component. The dividing set $\Gamma_{\Sigma'}$ will intersect $S^1 \times \{i\}$, $-2tw(S^1 \times \{i\}, \Sigma')$ times. Thus all arc starting at $S^1 \times \{0\}$ cannot run to $S^1 \times \{1\}$. So at least one must be boundary parallel and outermost. Thus Lemma 5.18 finishes the proof. □

6. Classification on $T^2 \times [0,1]$, solid tori and lens spaces

We first discuss contact structures on $T^2 \times [0,1]$. As usual, we think of T^2 as $\mathbb{R}^2/\mathbb{Z}^2$ so that we can discuss slopes of curves in T^2. Recall a contact structure on $T^2 \times [0,1]$ with convex boundary is called minimally twisting if the slope of the dividing curves on any convex torus parallel to the boundary is between s_0 and s_1, the slopes of the dividing curves on $T_0 = T \times \{0\}$ and $T_1 = T \times \{1\}$, respectively. Also recall $Tight^{min}(T^2 \times [0,1], \Gamma_1 \cup \Gamma_1)$ is the set of minimally twisting tight contact structures on $T^2 \times [0,1]$ with convex boundary having dividing curves $\Gamma_{T_0} = \Gamma_0$ and $\Gamma_{T_1} = \Gamma_1$. We have the following two cases for minimally twisting contact structures

Theorem 6.1 (Giroux 2000, [10]; Honda 2000, [3])**.** *Let Γ_i be a multi curve with two components and slope $s_i = -1, i = 0, 1$. There is a bijection*

$$h : \pi_0(Tight^{min}(T^2 \times [0,1], \Gamma_0 \cup \Gamma_1) \to \mathbb{Z}.$$

The map h is called the "holonomy map". Moreover, up to contactomorphism there is a unique tight contact structure on $T^2 \times [0,1]$ with the given boundary data.

We use the following notation for continued fractions

$$(r_0, r_1, \dots, r_k) = r_0 - \cfrac{1}{r_1 - \frac{1}{r_2 \cdots - \frac{1}{r_k}}}.$$

Theorem 6.2 (Giroux 2000, [10]; Honda 2000, [3])**.** *Let Γ_i be a multi curve with two components and slope $s_0 = -1$ and $s_1 = -\frac{p}{q}, p > q > 0$. Then*

$$|\pi_0(Tight^{min}(T^2 \times [0,1], \Gamma_0 \cup \Gamma_1)| = |(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1) r_k|,$$

where $-\frac{p}{q} = (r_0, r_1, \dots, r_k)$ and $r_i < -1$. That is there are $|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1}+1) r_k|$ contact structures up to isotopy in $Tight^{min}(T^2 \times [0,1], \Gamma_0 \cup \Gamma_1)$. Moreover, these contact structures are also distinct up to contactomorphism and are even distinguished by their relative Euler classes.

Remark 6.3. Note that given any two sets of curves Γ_0 and Γ_1 with distinct slopes we can find a linear diffeomorphism of T^2 inducing a diffeomorphism of $T^2 \times [0,1]$ that arranges their slopes to satisfy the hypothesis of the theorem. Thus the previous two theorems classify all minimally twisting contact structures on $T^2 \times [0,1]$. We will address the non-minimally twisting contact structures at the end of this section.

We prove Theorem 6.1 towards the end of this section, but first we first prove Theorem 6.2. In the course of the proof we will also classify tight contact structures on the solid torus and on lens spaces. In particular we will show the following.

Theorem 6.4 (Giroux 2000, [10]; Honda 2000, [3])**.** *Let Γ be a multi curve with two components with slope $s = -\frac{p}{q}, p > q > 1$. Then*

$$|\pi_0(Tight(S^1 \times D^2, \Gamma)| = |(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1) r_k|$$

where $-\frac{p}{q} = (r_0, r_1, \ldots, r_k)$ *and* $r_i < -1$*. That is there are* $|(r_0+1)(r_1+1)\cdots(r_{k-1}+1)r_k|$ *contact structures up to isotopy in* $Tight(S^1 \times D^2, \Gamma)$*. Moreover, these contact structures are also distinct up to contactomorphism and are even distinguished by their relative Euler classes.*

Recall a *lens space* $L(p,q)$ is the 3-manifold obtained from gluing two solid tori together via the map

$$A = \begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix}$$

where $qp' - qq' = -1$. Now we have

Theorem 6.5 (Giroux 2000, [10]; Honda 2000, [3])**.**

$$|\pi_0(Tight(L(p,q))| = |(r_0+1)(r_1+1)\cdots(r_k+1)|$$

where $-\frac{p}{q} = (r_0, r_1, \ldots, r_k)$ *and* $r_i < -1$*. That is there are* $|(r_0+1)(r_1+1)\cdots(r_k+1)|$ *tight contact structures on* $L(p,q)$ *up to isotopy. Moreover, all these structures are distinguished by their (half) Euler classes.*

Theorems 6.2, 6.4, and 6.5 will clearly follow from the following lemmas.

Lemma 6.6. *With the notation from Theorems 6.2 we have*

$$|\pi_0(Tight^{min}(T^2 \times [0,1], \Gamma_1 \cup \Gamma_1)| \leq |(r_0+1)(r_1+1)\cdots(r_{k-1}+1)r_k|.$$

Lemma 6.7. *With the notation from Theorems 6.2 and 6.4 we have*

$$|\pi_0(Tight(S^1 \times D^2, \Gamma)| \leq |\pi_0(Tight^{min}(T^2 \times [0,1], \Gamma_1 \cup \Gamma_1)|.$$

Lemma 6.8. *With the notation from Theorems 6.4 and 6.5 we have*

$$|\pi_0(Tight(L(p',q'))| \leq |\pi_0(Tight(S^1 \times D^2, \Gamma)|,$$

where $-\frac{p'}{q'} = (r_0, r_1, \ldots, r_{k_1}, r_k - 1)$ *and* Γ *has slope* $-\frac{p}{q} = (r_0, \ldots, r_k)$*.*

Lemma 6.9. *With the notation from Lemma 6.8 we have*

$$|(r_0+1)(r_1+1)\cdots(r_k)| \leq |\pi_0(Tight(L(p',q'))|$$

In the first two subsection we will prove these lemmas and hence establish Theorems 6.2, 6.4 and 6.5. Specifically, in the next subsection we prove the upper bounds, that is prove Lemmas 6.6, 6.7 and 6.8, and in Subsection 6.2 we prove Lemma 6.9. We then study the contact structures on the various manifolds under consideration more carefully, examining the realizable Euler classes for the contact structures and study their behavior under coverings. In the subsequent two subsections we consider zero twisting contact structures on $T^2 \times [0,1]$ (that is prove Theorems 6.1) and non-minimally twisting contact structures.

6.1. **Upper bounds on the number of contact structures.** We begin with

Proof of Lemma 6.6. Consider the manifold $N = T^2 \times [0,1]$. We wish to get an upper bound on the number of tight minimally twisting contact structures on N such that $T_i = T^2 \times \{i\}, i = 0,1$, is convex with two dividing curves Γ_i of slope $s_0 = -1$ and slope $s_1 = -\frac{p}{q}, p > q > 0$, respectively. Given such a contact structure on N the strategy will be to break N into pieces on which any contact structure under consideration will restrict to basic slices. From this we get an upper bound since each basic slice admits only two possible contact structures. This first upper bound is not good enough, so we introduce "slice shuffling" which will give the desired upper bound.

We begin by taking the characteristic foliation on ∂N to be standard with dividing slope s_i on the respective boundary components and ruling slope 0. Let $A = S^1 \times [0,1]$ where S^1 is a circle on T^2 with slope 0. We also take ∂A to be Legendrian ruling curves on ∂N. Note

$$tw(S^1 \times \{0\}, A) = -\frac{1}{2}(S^1 \times \{0\} \cap \Gamma_0) = -\det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = -1$$
$$tw(S^1 \times \{1\}, A) = -\frac{1}{2}(S^1 \times \{1\} \cap \Gamma_1) = -\det \begin{pmatrix} 1 & -q \\ 0 & p \end{pmatrix} = -p.$$

Thus the Imbalance Principle gives a bypass D for T_1 along $S^1 \times \{0\}$. Attaching D to T_1 we get a neighborhood of $T_1 \cup D$ that we denote $T^2 \times [\frac{1}{2}, 1]$. Thus we can write N as

$$N = T^2 \times [0, \frac{1}{2}] \cup T^2 \times [\frac{1}{2}, 1],$$

where $T^2 \times [\frac{1}{2}, 1]$ is a basic slice.

Claim 6.10. *The contact structure restricted to $T^2 \times [\frac{1}{2}, a]$ is a basic slice with $s_1 = -\frac{p}{q}$ and*

$$s_{\frac{1}{2}} = -\frac{p'}{q'} = (r_0, r_1, \ldots, r_k + 1),$$

where if $r_k = -2$ then $(r_0, r_1, \ldots, r_k + 1) = (r_0, r_1, \ldots, r_{k-1} + 1)$. Recall, $-\frac{p}{q} = (r_0, r_1, \ldots, r_k)$.

Proof of Claim. We begin by showing

$$s_{\frac{1}{2}} = -\frac{p'}{q'} \quad \text{where } -pq' + qp' = 1 \tag{4}$$

and

$$p > p' > 0, \quad q \geq q' > 0. \tag{5}$$

In order to see this we use Theorem 5.16 concerning attaching bypasses to a torus. Recall though that in that theorem we attach a bypass to a torus and get a one sided neighborhood of the torus where the original torus is the back face of the neighborhood, but in our situation our original torus

T_1 is the front face of the neighborhood $T^2 \times [\frac{1}{2}, 1]$. To fix this we reflect T^2 and the interval direction, *i.e.* apply the map $(x, y) \mapsto (-x, y)$ and flip the interval direction. This makes T_1 the back face of the basic slice. (Note the diffeomorphism is orientation preserving on the neighborhood.) Thus we should think of the slope s_1 as $\frac{p}{q} > 1$. According to Theorem 5.16 when attaching a bypass along slope 0 the slope on $T_{\frac{1}{2}}$ is determined by taking the positive rational number $\frac{p'}{q'}$ that is closest to 0 but has an edge to $\frac{p}{q}$. Since $\frac{p'}{q'}$ has an edge to $\frac{p}{q}$ we know $pq' - pq' = \pm 1$.

Exercise 6.11. Since $\frac{p'}{q'}$ is clockwise of $\frac{p}{q}$ show $-pq' + qp' = 1$.

Let $\frac{p''}{q''}$ be the point counterclockwise of $\frac{p}{q}$ that is closest to 0 but has an edge to $\frac{p}{q}$. As in the proof of Theorem 5.16 there is not only an edge from $\frac{p}{q}$ to $\frac{p'}{q'}$ and $\frac{p''}{q''}$ there is also an edge between $\frac{p'}{q'}$ and $\frac{p''}{q''}$. Thus

$$\frac{p}{q} = \frac{p' + p''}{q' + q''}.$$

Since $p'' > 0$ and $q'' \geq 0$ (since $\frac{p''}{q''} \leq \infty$) we see that $p > p' > 0$ and $q \geq q' > 0$.

Exercise 6.12. Show $p' + p''$ and $q' + q''$ have no common factors.
HINT: Recall the relation between the Farey tessellation and integral basis vectors for Z^2.

Thus we have established Equations (4) and 5.

We would now like to identify the continued fraction expansion of $-\frac{p'}{q'}$. To establish the claimed expansion set $-\frac{a}{b} = (r_0, r_1, \ldots, r_k + 1)$ and we will show that $a = p'$ and $b = q'$. We begin by showing that $-pb + qa = 1$ using induction on k. If $k = 0$ then

$$\det \begin{pmatrix} r_0 + 1 & r_0 \\ 1 & 1 \end{pmatrix} = r_0 + 1 - r_0 = 1.$$

For the inductive step notice that if $-\frac{c}{d}$ and $-\frac{c'}{d'}$ from an oriented integral basis for any integral r so does $r - \frac{1}{c/d} = \frac{cr-d}{c}$ and $r - \frac{1}{c'/d'} = \frac{c'r-d'}{c'}$. Indeed

$$\det \begin{pmatrix} c & c' \\ rc - d & rc' - d' \end{pmatrix} = rcc' - d'c - rcc' + c'd = c'd - c'd = 1.$$

Thus since r_k and $r_k{+}1$ form an integral basis so do $r_{k-1} - \frac{1}{r_k}$ and $r_{k-1} - \frac{1}{r_k+1}$. Continuing in this manner we see that $(r_0, \ldots, r_k)$ and $(r_0, \ldots, r_k + 1)$ form an integral basis. Thus we have $-pb + qa = 1$ (you should check the sign). Next, we claim

$$p > a > 0, \quad q \geq b > 0. \tag{6}$$

To see this we again work by induction. If $k = 0$, then $p = -r_0 > -r_1 - 1 = a$ and $q = 1 \geq 1 = b$. Now assume that $-\frac{c}{d}, -\frac{c'}{d'}$ are both less than -1,

$c > c', d \geq d'$ and $r < -1$ is an integer, then $r - \frac{1}{\frac{c}{d}} = \frac{rc-d}{c}$ and $r - \frac{1}{\frac{c'}{d'}} = \frac{rc'-d'}{c'}$ are both less than -1. Moreover, the denominator c is greater than the denominator c' and one may check that the absolute value of the numerator $-rc+d$ is greater than the numerator $-rc'+d'$. Thus by induction we have shown Equation (6).

Exercise 6.13. Show there is a unique fraction $\frac{a}{b}$ such that $\frac{a}{b}$ and $\frac{p}{q}$ form an integral basis and $p > a > 0$ and $q \geq b > 0$.
HINT: Any fraction $\frac{a}{b}$ satisfying the given conditions will be the point on the Farey tessellation farthest clockwise from $\frac{p}{q}$ that has an edge to $\frac{p}{q}$. To see this recall the construction of the Farey tessellation and look back at the proof of Theorem 5.16.

□

Now continuing as above we can break $T^2 \times [0,1]$ into

$$n = |r_k + 1| + |r_{k-1} + 2| + \ldots + |r_0 + 2|$$

basic slices denoted $B_i = T^2 \times [\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, \ldots n$. (Note what we called $T^2 \times [\frac{1}{2}, 1]$, we now are calling $T^2 \times [\frac{n-1}{n}, 1]$.)

Exercise 6.14. How did we get the expression for n?
HINT: Think about the rule about how to get $\frac{p'}{q'}$ from $\frac{p}{q}$.

Moreover, if the dividing slope on the front face of B_i has continued fraction expansion $(t_0, \ldots, t_l)$ then on its back face the dividing curves have expansion $(t_0, \ldots, t_l + 1)$. Since each basic slice has only two possible contact structures we have proven the upper bound of 2^n for the number of possible tight minimally twisting contact structures on N. This number is much larger than the upper bound for which we are looking. To improve the upper bound we notice that the B_i's can be grouped into natural "continued fraction blocks". That is we can break N up into $k+1$ pieces N_i such that the dividing slope on the font face and back face of N_i is $(r_0, \ldots, r_i + 1)$ and respectively $(r_0, \ldots, r_{i+1} + 1)$, for $i = 1, \ldots, k-1$ and for $i = 0$ we have $r_0 + 1$ and -1, respectively, and for $i = k$ we have $-\frac{p}{q} = (r_0, \ldots, r_k)$ and $(r_0, \ldots, r_{k-1} + 1)$, respectively. The N_i are called *continued fraction blocks.*

Exercise 6.15. Suppose $p_1, p_2, \ldots, p_l$ is a sequence of points moving clockwise on the boundary of the Farey tessellation. Then show the points correspond to a continued fraction block if $l = 2$ and p_1 and p_2 are connected by an edge or $l > 2$ and p_1 and p_2 are connected by an edge and after p_2 each successive p_i is reached by choosing the second farthest point from p_{i-1} that is connected to p_{i-1} by an edge.

Claim 6.16. *The continued fraction block N_i has at most $r_i + 1$ tight minimally twisting contact structures if $i < k$ and N_k has at most r_k.*

Note once this claim is proven we will be done with the proof of the lemma since we will have the upper bound of

$$|r_0 + 1| \cdots |r_{k-1} + 1||r_k|$$

for the number of possible tight minimally twisting contact structures on N. $\square$

Proof of Claim. Fist note that we can apply an element of $SL(2,\mathbb{Z})$ to N_i so that the dividing slope on the front and back face of N_i are respectively r_i+1 and -1 for $i<k$ and for $i=k$ we have slopes r_k and -1. Indeed this is obviously true for $i=0$ since its original slopes are r_0+1 and -1. Now consider the matrix

$$A_0 = \begin{pmatrix} -r_0 & 1 \\ -1 & 0 \end{pmatrix}$$

This matrix sends

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} -r_0 - 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q \\ -p \end{pmatrix} \mapsto \begin{pmatrix} -r_0 q - p \\ -q \end{pmatrix}.$$

So the back face of N has dividing slope $\frac{1}{r_0+1}$ and the front face has dividing slope $\frac{q}{r_0 q+p}$. Moreover the front face of N_0 has dividing slope r_0+1, so after applying A_0 we see this slope becomes -1. Thus if we remove N_0 from N we have a $T^2\times[0,1]$ with back face having dividing slope -1 and front face having slope $\frac{q}{r_0q+p}$. Now observe that

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 \cdots \frac{1}{r_k}}}.$$

So

$$-p = r_0 q - \frac{q}{r_1 - \frac{1}{r_2 \cdots \frac{1}{r_k}}}.$$

Thus

$$\frac{q}{p+r_0 q} = r_1 - \frac{1}{r_2 \cdots \frac{1}{r_k}}.$$

In other words when we remover N_0 from N we get a torical annulus with back face having dividing slope -1 and front face having dividing slope $(r_1,\ldots,r_k)$. Continuing in this manner we see that N_i has the claimed dividing slopes.

The proof of the claim is now completed by the following lemma. $\square$

Lemma 6.17. *Given $T^2\times[0,1]$ and multi-curves Γ_i on $T_i, i=0,1,$ with two components and slope $s_0=-1$ and $s_1=-m$ where $m>0$ is an integer. Then*

$$|\pi_0(Tight^{min}(T^2\times[0,1],\Gamma_1\cup\Gamma_1)| \leq -m.$$

Proof. As in the previous proof we can break $T^2\times[0,1]$ into $m-1$ pieces $B_i = T^2\times[\frac{i-1}{m-1},\frac{i}{m-1}], i=1,\ldots,m-1$, each of which is a basic slice. Moreover the dividing slope on $T_{\frac{i-1}{m-1}}$ is $-i$. Each B_i has two tight contact structures on it which we denote ξ_+ and ξ_-. We immediately get the upper bound of 2^{m-1}. To improve this upper bound we notice that we can "shuffle the layers". That is any contact structure on $T^2\times[0,1]$ is isotopic to one in which all the ξ_+'s, say, come first, followed by all the ξ_-'s. Thus there is an integer $k = 0, \dots, m-1$ that counts the number of say ξ_+ regions and the contact structure on $T^2 \times [0,1]$ is completely determined by this integer.

It is clearly sufficient to show that adjacent layers can be shuffled. If the adjacent layers have the same sign there is nothing to prove. So we assume B_i and B_{i+1} have signs $-$ and $+$ respectively. Let A be a horizontal annulus with Legendrian boundary running from $T^2 \times \{i-1\}$ to $T^2 \times \{i+1\}$ such that $T^2 \times \{i\} \cap A$ is also a Legendrian ruling curve. Make A convex. There are two essentially distinct possibilities for the dividing curves on A, either the bypasses can be nested or not. See Figure 31.

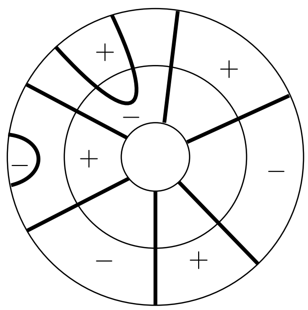


FIGURE 31. Two possible configurations of dividing curves. Nested bypasses on the right non-nested on the left.

If the bypasses are not nested then we can clearly attach them in any order. That is we can attach the $+$ one first then the $-$ one. This would make B_i have sign $+$ and B_{i+1} have sign $-$. So the only difficulty is shuffling the layer is when the bypasses are nested. Supposing the bypasses are nested we isotop A to A' by "adding copies of T_{i-1} and T_i". See Figure 32. If it is

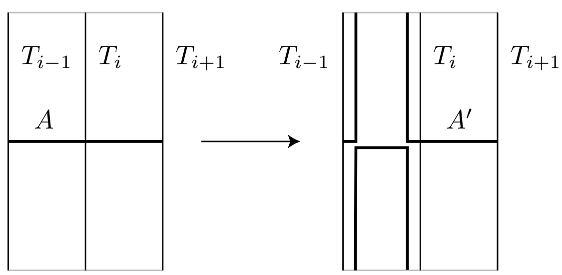


FIGURE 32. Isotoping A, left, to A', right. This is a cross section of $T^2 \times [0,1] = B_i \cup B_{i+1}$, so the top and bottom of the figures should be identified.

not obvious what is going on from the figure see the proof of Lemma 4.24 where this procedure was carefully described. The slope of the dividing curves on T_{i-1} is $-\frac{1}{i-1}$ and the slope of the dividing curves on T_i is $-\frac{1}{i}$. Thus the dividing curves on A' are as shown in Figure 33. In particular they are no longer nested and we can shuffle the layers. $\square$

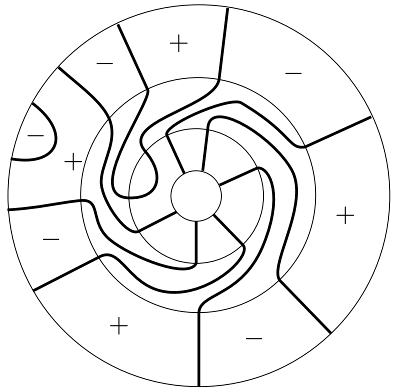


FIGURE 33. Dividing curves on A'. The middle annulus corresponds to T_i. We did not draw T_{i-1} in the figure since it does not affect the nesting of the bypasses.

Remark 6.18. We cannot shuffle layers between different continued fraction blocks. This will follow from the computation of the relative Euler class below. You should think about what goes wrong with the above proof.

We have the immediate corollary.

Corollary 6.19. *Let ξ be a minimally twisting tight contact structure on $T^2 \times [0,1]$ with dividing slopes on the boundary s_0 and s_1. Then for any s in $[s_0, s_1]$ there is a convex torus parallel to the boundary of $T^2 \times [0,1]$ with slope s.*

Proof. This is clear from the proof of Lemma 6.6 and Corollary 4.35 concerning the slopes of curves on tori inside a basic slice. □

Proof of Lemma 6.7. Let $M = S^1 \times D^2$ and on the boundary of M we have a multi curve Γ with slope $-\frac{p}{q}, p > q > 0$. In M let L be a Legendrian curve topologically isotopic to $S^1 \times \{pt\}$ with twisting number $-m$ (measured with respect to the product structure on M). Note we may assume that $-m$ is very negative by stabilizing the Legendrian knot if necessary. Let N' be a standard neighborhood of L. Set $N = M \setminus N' = T^2 \times [0,1]$. Clearly we have the boundary of N convex with dividing slope $s_0 = -\frac{1}{m}$ and $s_1 = -\frac{p}{q}$.

We claim that any tight contact structure on M is minimally twisting when restricted to N. Indeed if not there is a convex torus T in N parallel to the boundary of N with dividing slope s lying outside $[-\frac{p}{q}, -\frac{1}{m}]$. Thus by Corollary 6.19 we can realize any slope in $[s, -\frac{1}{m}]$ and in $[-\frac{p}{q}, s]$, that is we can realize any slope. In particular we can realize the 0 slope on some torus T in N. The Legendrian divides on T bound meridianal disks in M. These disks are overtwisted disks, contradicting the tightness of the contact structure. Thus the contact structure on N is minimally twisting.

Now since -1 lies in $[-\frac{p}{q}, -\frac{1}{m}]$ we can find a torus T in N parallel to the boundary with dividing slope -1. We can split M along T into two pieces, S

a solid torus with dividing slope -1 and A a torical annulus with boundary slopes -1 and $-\frac{p}{q}$. From Theorem 4.5 we know S has a unique tight contact structure on it thus

$$|\pi_0(Tight(S^1 \times D^2, \Gamma)| \leq |\pi_0(Tight^{min}(T^2 \times [0,1], \Gamma_0 \cup \Gamma_1)|.$$

□

Again we have the corollary.

Corollary 6.20. *Let ξ be a tight contact structure on $S^1 \times D^2$ with dividing slopes on the boundary s. Then for any s' in $(0, s]$ there is a convex torus parallel to the boundary of $S^1 \times D^2$ with slope s'.*

Proof. Given any slope s' with $s' \in (0, s]$ there is some m such that $-\frac{1}{m} > s' > s$. Now there is a Legendrian curve L topologically isotopic to the core of $S^1 \times D^2$ that has twisting number $-m$ (for -1 this comes from the above proof and for any other m simply stabilize). Let N' be a standard neighborhood of L. Now $S^1 \times D^2 \setminus N' = T^2 \times [0,1]$ and the dividing slopes on the boundary are $-\frac{1}{m}$ and s. Thus this corollary follows from Corollary 6.19.

□

Proof of Lemma 6.8. Recall a *lens space* $L(p, q)$ is the 3-manifold obtained from gluing two solid tori V_0 and V_1 together via the map

$$A = \begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix}$$

where $qp' - pq' = -1$. Let C_0 be a Legendrian curve in V_0 topologically isotopic to the core of V_0. We can assume its twisting number is $n < 0$ (stabilize if necessary). We can think of V_0 as being a standard neighborhood of C_0. Thus ∂V_0 is convex with two dividing curves of slope $-\frac{1}{n}$. To see what ∂V_1 looks like we map ∂V_0 to ∂V_1 by A. Thus ∂V_1 is convex with dividing slope $\frac{pn-p'}{-qn+q'}$.

Exercise 6.21. Show that

$$-\frac{p}{q} < \frac{pn - q'}{-qn + q''} < -\frac{p'}{q'}.$$

HINT: Consider the graph of $f(n) = \frac{pn-q'}{-qn+q'}$.

From Corollary 6.20 there is a torus T in V_1 isotopic to ∂V_1 that is convex with dividing slope -1. Split V_0 into $M = S^1 \times D^2$ and $N = T^2 \times [0,1]$ using T. The boundary of N is convex with dividing slope $\frac{pn-p'}{-qn+q'}$ on the front face and -1 on the back face. Since $-\frac{p'}{q'}$ lies in $[-1, \frac{pn-p'}{-qn+q'}]$ we can find another convex torus T' in N that is isotopic to ∂V_1 with dividing slope $-\frac{p'}{q'}$. Use T' to split $L(p, q)$ into V_0 and V_1. So ∂V_1 is convex with dividing slope $-\frac{p'}{q'}$ while ∂V_0 is convex with dividing slope $-\infty$. (Check this using A^{-1}.) Thus from Theorem 4.5, V_0 has a unique tight contact structure with this boundary data and hence all the different contact structure on $L(p,q)$ come from the different ones on V_1 the given boundary data. In other words

$$|\pi_0(Tight(L(p,q))| \leq |\pi_0(Tight(S^1 \times D^2, \Gamma)|,$$

where Γ is a multi curve on $\partial S^2 \times D^2$ with slope $-\frac{p'}{q'}$. The lemma now follows by starting with $-\frac{p}{q} = (r_0, r_1, \ldots, r_{k_1}, r_k - 1)$. □

6.2. **Construction of contact structures.** The goal of this subsection is to prove Lemma 6.9 and thus complete the proofs of Theorems 6.2,6.4 and 6.5. We begin by recalling the "slam dunk". This is a way to alter a surgery description of a 3-manifold, see [11]. Let r, p, q be integers. Let K be an unknoted meridianal curve to a knot L. Then $r - \frac{q}{p}$ surgery on L is equivalent to r surgery on L and $\frac{p}{q}$ surgery on K. See Figure 34. Thus if we

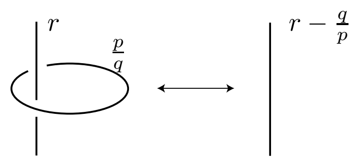


Figure 34. The slam dunk move.

write the continued fraction $-\frac{p}{q} = (r_0, r_1, \ldots, r_k)$, with all $r_i \leq -2$ then the lens space $L(p,q)$ is described as shown in Figure 35.

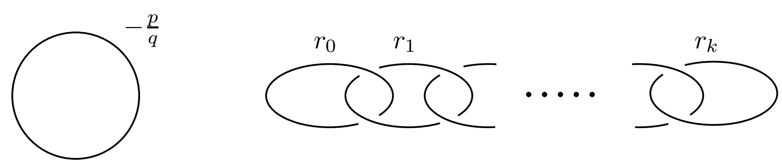


Figure 35. Two descriptions of the lens space $L(p,q)$.

Recall, given a Legendrian knot L in S^3 with the standard contact structure, Legendrian surgery on L is topologically the same as $(tb(L)-1)$-Dehn surgery on L and produces a fillable, hence tight, contact structure. See Section **??**. Given any integer $r < 0$ there are $|r+1|$ Legendrian unknots on which Legendrian surgery yield r-Dehn surgery. See Figure 36. Note these

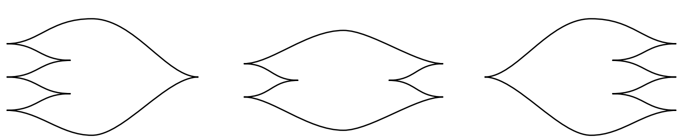
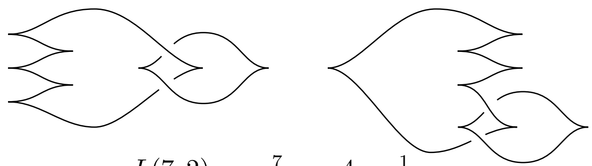


Figure 36. The 3 unknots when $r = -4$.

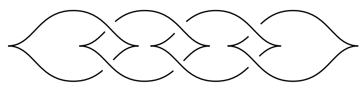
Legendrian unknots are all distinguished by their rotations numbers (orient all the unknots counterclockwise). Thus using the description of $L(p,q)$ in Figure 35 we have

$$|r_0+1||r_1+1|\dots|r_k+1|$$

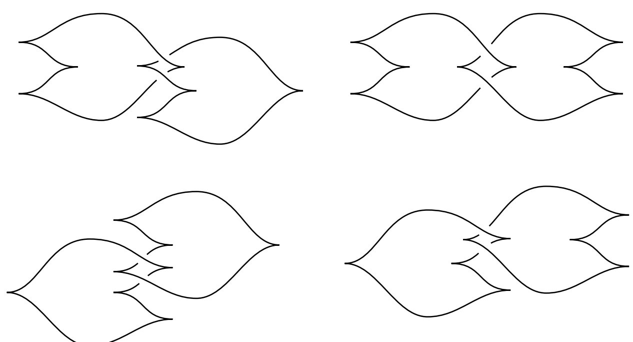
Legendrian surgery diagrams representing $L(p,q)$. For a few examples see Figure 37. Lemma 6.9 follows from the following lemma.



$L(7,2), \quad -\frac{7}{2} = -4 - \frac{1}{-2}$



$L(5,4), \quad -\frac{5}{4} = -2 - \frac{1}{-2-\frac{1}{-2-\frac{1}{-2}}}$



$L(8,3), \quad -\frac{8}{3} = -3 - \frac{1}{-3}$

FIGURE 37. Legendrian diagrams for various Lens spaces.

Lemma 6.22. *The contact structures on $L(p,q)$ coming from different Legendrian surgery diagrams as shown if Figure 37 are all distinct. In particular when p is odd they have distinct Euler classes and when p is even they have distinct "$\frac{1}{2}$-Euler classes".*

Proof. Recall, if M is obtained from Legendrian surgery on a Legendrian link $L_0 \cup \ldots \cup L_k$ in S^3 with its standard cotnact structure, then M is Stein fillable by a Stein manifold X topologically described by adding 2-handles to D^4 along the components of the Legnedrian link with framing one less than the contact framing. Moreover, the first Chern class of the Stein manifold is Poincaré dual to

$$\sum_{i=0}^{k} r(L_i)[D_i],$$

where the D_i's are the co-cores of the corresponding two handles. (See [11] or Section **??**.) The contact structures on the Lens spaces are clearly distinguish by the following theorem.

Theorem 6.23 (Lisca and Matic 1997; [12])**.** *Let X be a smooth 4-manifold with boundary. Suppose J_1 , J_2 are two Stein structures with boundary on X. If the induced contact structures ξ_1 and ξ_2 on ∂X are isotopic, then $c_1(J_1) = c_1(J_2)$.*

This completes the proof of the lemma except for the claims about the Euler class distinguishing the contact structures. It is somewhat more difficult to prove this. We need to recall that when performing Legendrian surgery on the link $L_0 \cup \ldots \cup L_k$ in the standard contact structure on S^3 the Euler class of the resulting contact structure is Poincaré dual to

$$\sum_{i=0}^{k} r(L_i)[\mu_i],$$

where $[\mu_i]$ is the homology class of the meridian to L_i in the surgered manifold. (See [11] or Section **??**).

Exercise 6.24. Write out some specific cases when p is odd and check the Euler class distinguishes the contact structures.

Now suppose $L_0 \cup \ldots \cup L_k$ is the topological link in Figure 35. Note the 1^{st} homology of $L(p, q)$ is generated by μ_0.

Exercise 6.25. Show that $[\mu_i] = |r_0||r_1| \ldots |r_{i-1}|[\mu_0]$

Exercise 6.26. After reading the next section see if you can show the Euler class distinguishes contact structures on Lens spaces when p is odd. Do the same for p even using the $\frac{1}{2}$-Euler class.

$\square$

6.3. **Euler classes, gluings and covering spaces.** We have finished the classification of minimally twisting tight contact structures on $T^2 \times [0, 1]$ and tight contact structures on $S^1 \times D^2$ up to isotopy. In this subsection we explore the relative Euler classes of these contact structures (the relative Euler classes determine the contact structures on these manifolds). We then study gluing two tight contact structures on $T^2 \times [0, 1]$ together. In general it is very difficult to determine when you get a tight contact structure when you glue two tight contact structures, but on torical annuli we will be able to completely understand gluing. Lastly we will determine which tight contact structures on $T^2 \times [0, 1], S^1 \times D^2$ and $L(p, q)$ are universally tight.

We begin with an "invariant" description of minimally twisting contact structures on torical annuli and their relative Euler classes. Before stating the theorem we need a definition. A clockwise path in the Farey tessellation is a sequence of vertices $s_0, \ldots, s_l$ in the tessellation such that s_{i+1} is clockwise of s_i and connected to it by an edge in the tessellation. We also want all the s_i to be between s_0 and s_l (that is clockwise of s_0 and counterclockwise of s_l). The path is minimal if s_i and s_j are connected by an edge in the tessellation if and only if $|i-j|=1$. The path is signed if to each edge in the path we attach a sign.

Theorem 6.27 (Honda 2000, [3]; *cf* Giroux 2000, [10])**.** *The minimally twisting contact structures on $T^2\times[0,1]$ with $|\Gamma_i|=2, i=0,1,$ and dividing slopes s_0 and s_1 are*

(1) *in one to one correspondence with minimal signed clockwise paths in the Farey tessellation from s_0 to s_1 modulo shuffling sings in continued fraction blocks.*
(2) *all determined by their relative Euler classes which are computed in the following way: let $s_0 < s_{\frac{1}{k}} < \ldots < s_{\frac{1}{k-1}} < s_1$ be a minimal signed clockwise paths in the Farey tessellation from s_0 to s_1 with the sign from $s_{\frac{i}{k}}$ to $s_{\frac{i+1}{k}}$ denoted ϵ_i. Let v_i be the minimal integral vector with slope s_i and negative x coordinate. Then*
$$\textit{Poincaré Dual of } e(\xi,s)=\sum_{i=0}^{k-1}\epsilon_i(v_{i+1}-v_i).$$
(Recall s is simply a section of the contact planes along the boundary as described in Subsection 4.4.)

Proof. To prove (1) simply find an element of $SL(2,\mathbb{Z})$ that send s_0 to -1 and s_1 to a number less than -1. Then (1) is precisely the content of Theorem 6.4, or more precisely the proof of Lemma 6.6. (One should check that applying an element of $SL(2,\mathbb{Z})$ takes minimal paths to minimal paths.)

For statement (2) we first compute the Euler class. Recall that for a basic slice B with $s_0=0$ and $s_1=-1$ we computed in Theorem 4.23 that
$$P.D.e(\xi,s)=\pm(0,1)=\pm((-1,1)-(-1,0)).$$
We will use $P.D.$ to mean Poincaré dual. Given any basic slice B', if the front face has dividing slope $s_0=\frac{p}{q}$ and the back face has slope $s_1=\frac{r}{s}$ then we know that $\det\begin{pmatrix}q & s\\ p & r\end{pmatrix}=\pm1$. If we choose p,q,r,s so that q and s are both negative then the determinant will be 1. The matrix
$$\begin{pmatrix}q & s\\ p & r\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}=\begin{pmatrix}-s & q-s\\ -r & p-r\end{pmatrix}$$
is an element of $SL(2,\mathbb{Z})$ that sends our standard basic slice B to B'. Thus it send the Euler class for B to the Euler class for B'. So the Euler class for B' is
$$P.D.e(\xi,s)=\pm(q-s,p-r)=\pm((q,p)-(s,r)).$$
So the computation of the relative Euler class is correct for paths of length one (*i.e.* for basic slices). The general computation follows from the additivity of relative Euler class.

Exercise 6.28. Show that if two basic slices are glued together their relative Euler classes add.

We now want to show that the relative Euler classes distinguish minimally twisting tight contact structures on $T^2 \times [0,1]$. Using elements of $SL(2,\mathbb{Z})$ it suffices to prove this for a torical annulus with $s_0 = -1$ and $s_1 = -\frac{p}{q}, p > q > 0$. Let $(r_0, r_1, \ldots, r_k)$ be a continued fraction expansion of $-\frac{p}{q}$ with $r_i \leq -2$. We now prove that the relative Euler class distinguishes contact structures by induction on k. Assume $k = 0$ then the minimal path from s_0 toe s_1 is $-1, -2, \ldots, r_0$. So the possible Euler classes are

$$\sum_{i=1}^{|r_0+1|} \pm((-1,i) - (-1,i+1)) = \sum_{i=1}^{|r_0+1|} \pm(0,-1).$$

In other words the Euler classes realized by minimally twisting tight contact structures on $T^2 \times [0,1]$ with these boundary conditions are

$$(0, |r_0+1|), (0, |r_0+1|-2), \ldots, -(0, |r_0+1|-2), -(0, |r_0+1|).$$

One may easily check there are $|r_0|$ distinct Euler classes which is equal to the number of contact structures. Thus all the contact structures are distinguished. Moreover, note the contact structures are distinguished by the way the Euler class evaluates on a horizontal annulus A and $|e(\xi, s)(A)| < |r_0|$ (which is p in this case).

Now we suppose that all the contact structures are distinguished by the value of the Euler class on the horizontal annulus for $-\frac{p}{q}$ with continued fraction expansions of length k or less, and moreover $|e(\xi, s)(A)| < p$ for all contact structures under consideration. Suppose $-\frac{p}{q}$ has the length $k+1$ expansion $(r_0, \ldots, r_k)$. Let $-\frac{a}{b} = (r_0, \ldots, r_{k-1})$ and let $-\frac{a'}{b'} = (r_0, \ldots, r_{k-1} + 1)$. From the classification of contact structures we know we can split $N = T^2 \times [0,1]$ into two torical annuli N' and N'' where the dividing slopes on the front and back of N' are -1 and $-\frac{a'}{b'}$ and the dividing slopes on N'' are $-\frac{p}{q}$ and $-\frac{a'}{b'}$. Note N'' is a continued fractions block. Also by induction we know the contact structures on N' are distinguished by the value of the Euler class on the horizontal annulus A' and the values of the Euler class on A' are less than a' in absolute value.

Exercise 6.29. Show that $(b, -a)$ and $(-b', a')$ form an oriented integral basis for $\mathbb{Z}^2$.

Let

$$A = \begin{pmatrix} b & -b' \\ -a & a' \end{pmatrix}.$$

Note that A is in $SL(2,\mathbb{Z})$ and takes $(1,0)$ to $(b,-a)$ and $(0,1)$ to $(-b', a')$.

Exercise 6.30. Show A takes the continued fraction block B with back face having dividing slope 0 and front face having slope $r_k + 1$ to N''.
HINT: Consider the Farey tessellation.

Note the Euler class for contact structures on B are of the form $(0, m)$ where $|m| < |r_k + 1|$. (Note not all such m are realized.) Thus the Euler classes on N'' are of the form $(-mb', ma')$. So its values on the horizontal annulus A'' in N'' are multiples of a'.

Exercise 6.31. Show the possible values of the relative Euler class of N on the annulus $A = A' \cup A''$ distinguish all contact structures on N.
HINT: The values on A'' are all multiples of a strict upper bound on the values on A'.

Exercise 6.32. Show $a' < a$.

Finally since

$$\begin{pmatrix} b & -b' \\ -a & a' \end{pmatrix} \begin{pmatrix} 1 \\ r_k + 1 \end{pmatrix} = \begin{pmatrix} q \\ -p \end{pmatrix}$$

we see that $p = a'(|r_k| - 1) + a > a'(|r_k| - 1) + a'$. Note the fist term in the sum bounds the relative Euler class on A'' and the second term bounds the Euler class on A'. Thus we have that $|e(\xi, s)(A)| < p$. □

Now for the Euler classes of solid tori.

Theorem 6.33 (Honda 2000, [3]; *cf* Giroux 2000, [10])**.** *Tight contact structures on $S^1 \times D^2$ with convex boundary having two dividing curves with slope $-\frac{p}{q}, p > q > 0$, are determined by their relative Euler classes. Moreover,*

$$e(\xi, s)(D) = |D_+| - |D_-| = r(\partial D),$$

where D is a convex meridianal disk with Legendrian boundary, $D_\pm$ are the positive and negative regions of $D \setminus \Gamma_D$ and $r(\partial D)$ is the rotation number.

Exercise 6.34. Show the assumption on the slope of the dividing curves is unnecessary. It is just used to make the proof easier.

Proof. The fact that the relative Euler numbers distinguish the contact structures follows from Theorem 6.27. Indeed, recall we can split $S^1 \times D^2 = M \cup N$, where M is a solid torus and N is a torical annulus, such that any contact structure under consideration can be isotoped so that when restricted to M it gives the unique contact structure with dividing slope -1 on the boundary and on N gives one of the possible contact structures on $T^2 \times [0, 1]$ with dividing slopes $s_0 = -1$ and $s_1 = -\frac{p}{q}$. Thus the contact structure on the original solid torus is determined by its restriction to N which in turn is determined by its relative Euler class. The above proof shows it is determined by how the Euler class evaluates on a horizontal annulus. It is precisely this portion of the Euler class that survives when we glue M back into N to get our solid torus.

The formula for the Euler class is simply restating Theorem 4.20 in this case (note that all components of $D_\pm$ are disks and have Euler characteristic 1). □

We now consider gluing minimally twisting torical annuli.

Theorem 6.35 (Honda 2000, [3])**.** *Let ξ be a contact structure on $T^2 \times [0, n]$ such that on each $N_i = T^2 \times [i-1, i], i = 1, \ldots, n$, it is a basic slice, the dividing slope on $T^2 \times \{i\}$ is s_i, each s_i lies in $[s_0, s_n]$ with $s_0 < s_1 < \ldots < s_n$ and to the arc connecting s_{i-1} to s_i assign a sign ϵ_i depending on the relative Euler class of the basic slice N_i. Then ξ is tight if and only if*

(1) *$s_0, \ldots, s_n$ is a minimal clockwise path in the Farey tessellation from s_0 to s_n, or*

(2) *$s_0, \ldots, s_n$ is not a minimal clockwise path in the Farey tessellation but a sequence of the following moves will get you to case (1): replace a triple s_{i-1}, s_i, s_{i+1} where there is an arc in the Farey tessellation from s_{i-1} to s_{i+1} and $\epsilon_i \epsilon_{i+1} > 1$ by the pair s_{i-1}, s_{i+1} and assign the sing $\epsilon_i = \epsilon_{i+1}$ to the path connecting these points.*

Remark 6.36. Note condition (2) says that if you can shorten the path while making a coherent choice of signs then to so and check if you are in case (1).

Exercise 6.37. Recall you can shuffle layers in a continued fraction block. In condition (2) why are you not allowed to do this before trying to shorten the path? Would it make any difference if you were allowed to do this?

Proof. From Theorem 6.27 it is clear that (1) implies tightness. To see that (2) implies tightness we just need to see that if s_{i-1}, s_i, s_{i+1} satisfy the conditions in (2) then N_i and N_{i+1} can be glued to get a basic slice. We fist consider a model case. Let N be a basic slice with dividing slopes 0 and -1. Make the ruling slope on N equal to $-\frac{2}{3}$ and let A be a convex annulus in N with slope $-\frac{2}{3}$ and Legendrian boundary. The imbalance principle gives a bypass for the back face of N and using Theorem 5.16 if we attach this bypass we get a convex torus T in N with dividing slope $-\frac{1}{2}$. We can split N along T into two basic slices N' and N'' with dividing slopes 0 and $-\frac{1}{2}$ for N' and $-\frac{1}{2}$ and -1 for N''. Since the possible relative Euler classes for N' are $\pm(-1, 1)$ and for N'' are $\pm(1, 0)$ we see that if the relative Euler class of N was $(0, 1)$ the relative Euler classes for N' and N'' must be $(-1, 1)$ and $(1, 0)$ respectively. Similarly if the relative Euler class of N was $-(0, 1)$ we must have the relative Euler classes for N' and N'' being $-(-1, 1)$ and $-(1, 0)$ respectively. Now given N_i and N_{i+1} as above. Choose the sign of the basic slice N to be $\epsilon_i = \epsilon_{i+1}$, then we can find an element of $SL(2, \mathbb{Z})$ that will take N_i to N' and N_{i+1} to N''. Thus N_i and N_{i+1} glue together to give a basic slice.

We now wish to show the other implication. To this end assume that (1) and (2) are not true about ξ then we will show that ξ is overtwisted. Since (1) and (2) are not true there must be a triple s_{i-1}, s_i, s_{i+1} connected by an edge in the Farey tessellation such that $\epsilon_i \neq \epsilon_{i+1}$. We can find a diffeomorphism in $SL(2, \mathbb{Z})$ that sends s_{i-1} to 0, s_i to -1 and s_{i+1} to $-\infty$. Since N_i is a basic slice its relative Euler class must be $\pm(0, 1)$. Similarly the relative Euler class of N_{i+1} must be $\pm(\pm 1, 0)$. We are assuming $\epsilon_i \neq \epsilon_{i+1}$, so the relative Euler class on $N_i \cup N_{i+1}$ is $\pm(1,-1)$. The contact structure on $N_i \cup N_{i+1}$ is overtwisted or not minimally twisting since if it were tight and minimally twisting then it would be a basic slice, but it does not have the appropriate relative Euler number for a basic slice (which must be $\pm(1,1)$ for the given boundary data).

Exercise 6.38. Show $N_i \cup N_{i+1}$ also does not have the appropriate Euler class for a tight non-minimally twisting contact structure.
HINT: See Subsection 6.5 below. Show the contact structure on $N_i \cup N_{i+1}$ can be split into a minimally twisting part with dividing slopes 0 and ∞ and a non-minimally twisting part with both dividing slopes 0. Show, using the notation of that section, the non minimally twisting part must be $\xi^{\pm}_{2m}$ for some sign and some m.

□

Finally we consider coverings.

Theorem 6.39. *We have the following*

(1) *There are exactly two tight contact structures on $T^2 \times [0,1]$ with minimally twisting, minimal number of dividing curves and dividing slopes $s_0 = -1$ and $s_1 = -\frac{p}{q}, p > q > 0$, which are universally tight. These contact structures satisfy*
$$P.D.(e(\xi, s)) = \pm((-q,p) - (-1,1)).$$
Said another way the minimal path in the Farey tessellation corresponding to these contact structures has no mixing of signs.
(2) *There are exactly two tight contact structures on $S^1 \times D^2$ with minimal number of dividing curves and dividing slope $-\frac{p}{q} < -1$ which are universally tight. There is only one if the dividing slope is -1.*
(3) *There are exactly two tight contact structures on $L(p,q), q \neq p-1$, which are universally tight. There is only one if $q = p-1$.*

Proof. To prove (1) we consider $A = S^1 \times \{0\} \times [0,1]$ in $N = T^2 \times [0,1]$. Clearly $(S^1 \times \{0\} \times \{0\}) \cap \Gamma_0 = 2$ and $(S^1 \times \{0\} \times \{1\}) \cap \Gamma_0 = 2p$. Thus if $P.D.(e(\xi,s)) = \pm(-q+1, p-1)$ then the dividing curves on A can be seen in Figure 38.

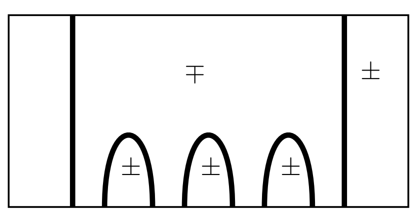


FIGURE 38. The dividing curves on A in the case when $P.D.(e(\xi, s)) = \pm(-q+1, p-1)$. (Right and left sides of the rectangle are identified to form the annulus A.)

Exercise 6.40. In this case show that the contact structure contact embeds in a basic slice and hence is universally tight.

From the proof of Theorem 6.27 we know that if $P.D.(e(\xi, s)) \neq \pm(-q + 1, q-1)$ then the Euler class evaluates to something other than $p-1$, so there must be bypasses of both signs on A. There are two situations: case (A) is when there are two outermost boundary parallel dividing curves on A separating off disks D_1 and D_2 of opposite signs and case (B) where this is not the case. See Figure 39 for an illustration of these cases. Let

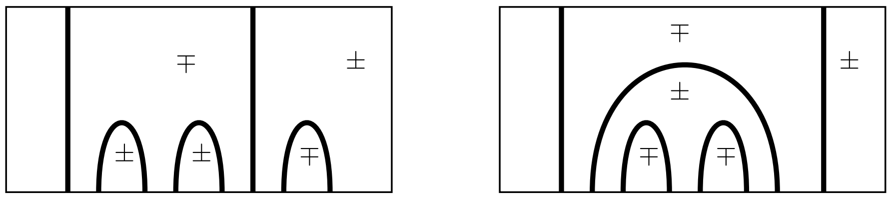


FIGURE 39. The two cases: (A), left, with boundary parallel bypasses bounding disks of opposite sign and (B), right, with no such disks.

$\widetilde{N} = S^1 \times \mathbb{R} \times [0,1]$ be the indicated $\mathbb{Z}$ cover of N. Suppose we are in case (A). Let A' and A'' be two adjacent lifts of A and let B_1 and B_2 be the annuli in $\partial\widetilde{N}$ lying between $\partial A'$ and $\partial A''$. Note $A' \cup A'' \cup B_1 \cup B_2$ bound a solid torus M.

Exercise 6.41. Show we can choose the lifts A' and A'' so that the lift of D_1 on A' and the lift of D_2 on A'' combine to give a simple closed curve bounding a disk on ∂M when the corners are rounded. Thus Giroux's criterion for tightness implies this contact structure is overtwisted near ∂M. Moreover, show that we can find this overtwisted disk in an n-fold cover of N where $n < p$.
HINT: Note that, as the boundary of M, one of the orientations of A' or A'' must switch from its lifted orientation.

In case (B) we have a similar situation. Let A', A'' and A''' be three lifts of A to $\widetilde{N}$ and let $B1$ and B_2 be the annuli in $\partial\widetilde{N}$ between $\partial A'$ and $\partial A'''$ such that $\partial A''$ is inside B_1 and B_2. In case (B) we have a disk D in A cut off by the dividing curves that contains k, say, other boundary parallel dividing curves.

Exercise 6.42. Show there are lifts of D to the three annuli as shown in Figure 40. Moreover, show such a lift exists in an n-fold cover of N where $n < p$.

Exercise 6.43. Show in the lift from the previous exercise the Legendrian arcs in Figure 40 can be realized. Show the union of these arcs is a Legendrian unknot with corners and Thurston-Bennequin invariant 0. Finally, show the corners can be smoothed without changing the Thurston-Bennequin invariant (thus we have an overtwisted disk).

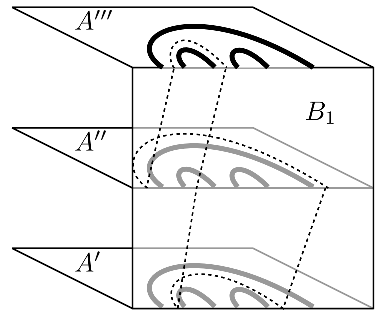


FIGURE 40. The configurations of the dividing curves on appropriate lifts of A. The dotted lines represent Legendrian curves.

The proof of (2) is now obvious since we can split the solid torus into a torical annulus and a solid torus with unique contact structure on it and the coverings considered in case (1) were "longitudinal" and so correspond to coverings of the solid torus as well. We are done by the following exercise.

Exercise 6.44. If there is no mixing of signs then we may embed the contact structure on the solid torus into a standard neighborhood of a Legendrian knot (which is universally tight).

The proof of (3) is almost obvious. We think of $L(p, q)$ as the union of two solid tori V_0 and V_1. As we observed in the proof of Lemma 6.8 we can assume V_0 has dividing slope ∞ and V_1 has dividing slope $-\frac{p'}{q'}$ where $-\frac{p'}{q'} = (r_0, \dots, r_k+1)$ if $-\frac{p}{q} = (r_0, \dots, r_k)$. The fundamental group of $L(p, q)$ is generated by the core of V_1 and thus when going to the universal cover of $L(p, q)$ we will be taking the p fold cover of V_1.

Exercise 6.45. Show that if V_1 does not have a universally tight contact structure on then it becomes overtwisted by the $p' < p$ fold cover.

□

6.4. **Zero twisting contact structures on $T^2 \times [0, 1]$.** In this section we prove Theorem 6.1. Recall the theorem says: Let Γ_i be a multi curve with two components and slope $s_i = -1, i = 0, 1$. There is a bijection

$$h : \pi_0(Tight^{min}(T^2 \times [0, 1], \Gamma_1 \cup \Gamma_1) \to \mathbb{Z}.$$

The map h is called the "holonomy map". Moreover, up to contactomorphism there is a unique tight contact structure on $T^2 \times [0, 1]$ with the given boundary data.

Proof of Theorem 6.1. Set $N = T^2 \times [0, 1]$. We assume that ∂N has a standard foliation with ruling slope $r_0 = r_1 = 0$. Let A be a horizontal annulus (that is $S^1 \times [0, 1]$ with S^1 a curve of slope 0 on T^2) with Legendrian boundary. Make A convex. First note that all the dividing curves on A run from one boundary component to the other. In other words there are no boundary parallel dividing curves, because if there were we would have a bypass on A which when attached to ∂N would produce a convex torus with dividing slope not equal to -1.

Exercise 6.46. Show that if we cut N along A and round the corners we get $S^1 \times D^2$ with boundary having dividing curves of slope -1.

There is a unique such tight contact structure on $S^1 \times D^2$. Thus the contact structure on N is determined by the dividing curves on A.

Now pick an identification $\phi : S^1 \times \{0\} \to S^1 \times \{1\}$ of the boundary components of A (for example use the product structure on A). Given a contact structure on N we can always isotop A so that $\phi(\Gamma_A \cap (S^1 \times \{0\})) = \Gamma_A \cap (S^1 \times \{1\})$. So a component of Γ_A becomes a $(k, 1)$ curve on $A/\sim = T^2$, where $\sim$ is induced by ϕ. Define $h(A) = k$. From the above discussion if two contact structures on N have the same $h(A)$ then they are isotopic.

You might worry that as you isotop A there might be more than one k associated to A. This is not actually possible as we will see below. But first we show that any integer can be $h(A)$ for some tight contact structure. Indeed let ξ_0 be an $[0, 1]$-invariant neighborhood of a convex torus with dividing slope -1. Then $k(A)$ for this contact structure is clearly 0. Now set $\phi_k(x, y, t) = (x, y + tk, t)$. This is a diffeomorphism of N ((x, y) are the obvious coordinates on T^2 and t is the coordinate on $[0, 1]$). Moreover, it is clear that $h(A) = k$ for the contact structure $\xi_k = (\phi_k)_*(\xi_0)$.

Exercise 6.47. Show that ξ_0 is minimally twisting since it is $[0, 1]$-invariant. Thus conclude that all the ξ_k are minimally twisting.

We have model contact structures realizing all possible $k \in \mathbb{Z}$ and any tight minimally twisting contact structure is isotopic to one of these. Note that all these contact structures are contactomorphic by definition.

We now show that ξ_k is not isotopic to $\xi_{k'}$ if $k \neq k'$. We do this by showing that if A' is any annulus in (N, ξ_k) isotopic to A then $h(A') = k$. This shows that h is a map on contact structures not just on the pairs (A, ξ). Assume in ξ_0 there is an annulus A' isotopic to A, rel boundary, such that $h(A') = k \neq 0$. Let $\widetilde{N} = S^1 \times \mathbb{R} \times [0, 1]$ be a covering space of N. There is a lift of $\widetilde{A'}$ of A' to $\widetilde{N}$. The annulus $\widetilde{A'}$ splits $\widetilde{N}$ into two pieces N_1 and N_2. Form one of N_1 or N_2 the slope on $\widetilde{A'}$ looks negative, suppose it is N_1. Let $\widetilde{A}$ be a lift of A to N_1 (note this implies $\widetilde{A'}$ and $\widetilde{A}$ are disjoint). Let M be the region between $\widetilde{A'}$ and $\widetilde{A}$. It is easy to see M is a solid torus. Let C_1 and C_2 be the two components of $\partial\widetilde{N} = S^1 \times \mathbb{R} \times \{0, 1\}$ between $\widetilde{A'}$ and $\widetilde{A}$ The dividing slope on ∂M is

$$\text{slope } (\partial M) = -1 + |h(A')| + 0 - \text{slope } (C_1) + \text{slope } (C_2) = |h(A')| - 1 \geq 0,$$

where the -1 comes from edge rounding, the 0 is the slope of the dividing curves on A and the slopes of the dividing curves on C_1 and C_2 are opposite since the orientations on C_1 and C_2 are opposite as the boundary of M. Thus

M is the standard neighborhood of a Legendrian curve of twisting greater that or equal to 0.

Let $\widetilde{A}_2$ be a lift of A to N_2 and set M_2 equal to the region between $\widetilde{A}$ and $\widetilde{A}_2$. This is a solid torus containing M. Moreover, the slope of the dividing curves on M_2 is -1. There is a contactomorphism of M_2 to a standard neighborhood of a maximal Thurston-Bennequin unknot in S^3 with the standard contact structure that sends the product framing on M_2 to the 0 framing on the unknot (since the Thurston-Bennequin of the unknot is -1). But from the above construction inside this neighborhood of the Legendrian unknot there is a neighborhood M of a Legendrian knot (in the same knot type) with twisting greater than or equal to 0. This contradicts the Bennequin inequality. Thus the annulus A' could not have existed in the first place. □

6.5. **Non-minimally twisting contact structures on $T^2 \times [0,1]$.** We would now like to classify non-minimally twisting contact structures on $T^2 \times [0,1]$. To this end we construct various model contact structures. Let $(N_0 = T^2 \times [0,1], \overline{\xi})$ be the basic slice with dividing slopes $s_0 = 0$ and $s_1 = -\infty$ and having relative Euler class Poincaré dual to $(1,1)$. Let $N_{\frac{n\pi}{2}}$ be $(N_0, \overline{\xi})$ rotated counterclockwise by $\frac{n\pi}{2}$. (Note we are denoting the manifold and the contact structure by $N_{\frac{n\pi}{2}}$. In this section we will frequently blur the line between contact manifold and contact structure. This should not cause any confusion since all manifolds in this section are $T^2 \times [0,1]$.) Now set

$$\xi_1^- = N_\pi \cup N_{\frac{3\pi}{2}},$$

$$\xi_2^- = N_\pi \cup N_{\frac{3\pi}{2}} \cup N_{2\pi} \cup N_{\frac{5\pi}{2}}$$

and similarly define ξ_k^- for all positive integers k. Also define

$$\xi_1^+ = N_0 \cup N_{\frac{\pi}{2}}$$

and for each $k > 1$ set $\xi_k^+ = \xi_1^+ \cup \xi_{k-1}^-$. Intuitively $\xi_k^\pm$ goes through k half twists as you go from the back fact of $T^2 \times [0,1]$ to the front face. The structures ξ_k^+ and ξ_k^- differ in that the orientation on the contact planes along the back face of $T^2 \times [0,1]$ is opposite. Note that all the contact structures $\xi_k^\pm$ contact embed in $T^2 \times \mathbb{R}$ with the contact structure $\ker(\sin(2\pi z)\,dx + \cos(2\pi z)\,dy)$ and thus are universally tight.

Theorem 6.48 (Honda 2000, [3])**.** *A complete non-repeating list of non-minimally twisting tight contact structures on $T^2 \times [0,1]$ with convex boundary, each boundary component having two dividing curves of slope 0, is given by $\xi_k^\pm$ where k runs through all positive integers.*

This theorem clearly follows from the following two lemmas.

Lemma 6.49. *Let ξ be a non-minimally twisting tight contact structure on $T^2 \times [0,1]$ satisfying the hypothesis of Theorem 6.48. Then ξ is isotopic to $\xi_k^\pm$ for some choice of sign and some k.*

Lemma 6.50. *The contact structures $\xi_k^\pm$ are all distinct.*

Proof of Lemma 6.49. Set $N = T^2 \times [0,1]$ and assume ∂N is standard with ruling slope ∞. Let A be a vertical annulus in N (*i.e.* $A = S^1 \times [0,1]$ with S^1 a curve on T^2 of slope ∞) with Legendrian boundary. Assume $|\Gamma_A|$ is minimal among all annuli isotopic rel boundary to A. Orient A so that $S^1 \times \{0\}$ is oriented up (that is $\frac{\partial}{\partial y}$ is positively tangent to $S^1 \times \{0\}$). Note A must have boundary parallel dividing curves.

Exercise 6.51. Prove that is A did not have boundary parallel dividing curves then the contact structure is minimally twisting.

Cut N open along A and round corners to get M a solid torus. Let D be a meridianal disk to the solid torus. Note the boundary of D is broken into four pieces; two α_0 and α_1 that run along A, one that runs along T_0 and one that runs along T_1. We can choose D so that the parts of ∂D that run along T_0 and T_1 do not intersect any dividing curves. In this case all the dividing curves on D run from α_0 to α_1.

Exercise 6.52. Show that if a dividing curve on D began and ended on α_0, say, then you could find a bypass for A and decrease $|\Gamma_A|$.
HINT: You need to be careful if the outermost boundary parallel dividing curve is adjacent to the boundary of one of the α_i's.

Thus there is only one possible configuration for the dividing curves on D. Since cutting M along D yields the unique tight contact structure on the 3-ball we see that the contact structure on N is determined by the dividing curves on A. The topological type of the dividing curves on A is determined by the number of simple closed curves k in Γ_A and the signs of the bypasses on the front and back face.

Exercise 6.53. Show that the sign of the bypass on the front face of N is determined by k and the sign of the bypass on back face of N.

Exercise 6.54. Given a fixed k show that ξ_k^+ and ξ_k^- have a vertical annulus with k simple closed dividing curves.

Exercise 6.55. Show that no vertical annulus in ξ_k^+ has fewer that k simple closed curves.
HINT: Recall all the ξ_k's embed in various tight contact structures on T^3. Use the classification of contact structures on T^3. If you are stuck read the next proof.

These homeworks and the above discussion clearly finish the proof of the lemma. □

Proof of Lemma 6.50. We begin by observing that all the ξ_{2m}^+'s are distinct up to isotopy. Indeed note that

$$(T^2 \times [0,1], \xi_{2m}^+)/\sim$$

is contactomorphic to (T^3, ξ_m), where $\sim$ glues the front and back face by the identity. Since the ξ_m's on T^3 are distinct for distinct m's so are the ξ_{2m}^+'s.

Note ξ_{2m}^- is contactomorphic to ξ_{2m}^+ via a diffeomorphism that rotates the T^2 by π. Thus all the ξ_{2m}^- are distinct up to isotopy too. If we glue $N_{\frac{\pi}{2}}$ to the front of ξ_{2m-1}^+ we get a contact manifold contactomorphic to ξ_{2m}^+. Thus all the ξ_{2m-1}^+ are distinct up to isotopy. We can similarly see that all the ξ_{2m-1}^- are distinct up to isotopy.

We are left to show the four sets of contact structures $\mathcal{S}_{o+}, \mathcal{S}_{o-}, \mathcal{S}_{e+}$ and $\mathcal{S}_{e-}$ are non-overlapping, where $\mathcal{S}_{o+}$ is the set of ξ_{2m-1}^+'s, $\mathcal{S}_{o-}$ is the set of all ξ_{2m-1}^-'s and similarly for the $\mathcal{S}_{e+}$ and $\mathcal{S}_{e-}$. The key to this is to observe that the annulus A from the previous proof will have a positive bypass on the back face of any element in $\mathcal{S}_{e+}$ or $\mathcal{S}_{o+}$ and a negative bypass otherwise. Similarly A will have a positive bypass on the front face of $\mathcal{S}_{o+}$ and $\mathcal{S}_{e-}$ and a negative one otherwise.

Exercise 6.56. Check these assertions.
HINT: Recall the relative Euler class of N_0 is $(1, -1)$.

We are not done yet since we have only shown the bypasses on the obvious annulus in contact structures form the various sets have different signs. To really show these sets are disjoint we do the following: if we glue $N_{-\frac{\pi}{2}}$ to the back of any element in $\mathcal{S}_{e+}$ or $\mathcal{S}_{o+}$ we see the resulting contact structure is contactomorphic to an element in $\mathcal{S}_{o-}$ or $\mathcal{S}_{e-}$, respectively. In particular the contact structure is tight after gluing. However if we glue $N_{-\frac{\pi}{2}}$ to the back of any element of $\mathcal{S}_{e-}$ or $\mathcal{S}_{o-}$ then we get overtwisted contact structures.

Exercise 6.57. Find the overtwisted disk
HINT: Look on the annulus A.

Thus the sets $\mathcal{S}_{e+} \cup \mathcal{S}_{o+}$ and $\mathcal{S}_{e-} \cup \mathcal{S}_{o-}$ are disjoint. Similarly by gluing N_0 to the font of various contact manifold you see the sets $\mathcal{S}_{o+} \cup \mathcal{S}_{e-}$ and $\mathcal{S}_{o-} \cup \mathcal{S}_{e+}$ are disjoint. This completes the proof. □

Exercise 6.58. Justify the following relative Euler class computations

$$P.D.(e(\xi_{2m}^{\pm}, s)) = (0, 0)$$
$$P.D.(e(\xi_{2m-1}^{\pm}, s)) = \pm(2, 0)$$

Theorem 6.59. *Let ξ be a tight contact structure on $T^2 \times [0,1]$ with convex boundary having minimal number of dividing curves with dividing slopes s_0 and s_1, then we may isotop ξ so that ξ restricted to $T^2 \times [\frac{1}{2}, 1]$ is minimally twisting with dividing slopes $s_{\frac{1}{2}} = s_0$ and s_1 and ξ restricted to $T^2 \times [0, \frac{1}{2}]$ is non-minimally twisting.*

Exercise 6.60. Prove this theorem.

7. Families of surfaces, bypasses, and basic gluing results

Proofs will be added later.

Lemma 7.1 (Giroux 1991, [8]). *Let ξ_0 and ξ_1 be two contact structures on $N = \Sigma \times [0,1]$ inducing the same characteristic foliation on ∂N. If there is a multi-curve $\Gamma_t \subset \Sigma_t = \Sigma \times \{t\}$ continuously varying with t that divides $(\Sigma_t)_{\xi_i}, i = 0, 1$ for all t then ξ_0 and ξ_1 are isotopic rel boundary.*

Theorem 7.2. *Given ξ_0 a contact structure on $N = \Sigma \times [-1,1]$, if Σ_{-1} and Σ_1 are convex then ξ_0 is isotopic rel boundary to a contact structure ξ such that there exist $t_i, i = 1, \ldots, k$ with $-1 < t_1 < \ldots < t_k < 1$, such that*

1. *Σ_t is convex if $t \neq t_i$ for any i,*
2. *Σ_{t_i} is not convex for any i,*
3. *for very small ϵ, $\Sigma_{t_i - \epsilon}$ and $\Sigma_{t_i+\epsilon}$ are related by a bypass attachment,*
4. *On each Σ_{t_i} there is either*
   - (a) *a birth-death of periodic orbits or*
   - (b) *a "retrograde" saddle-saddle connection.*

We need the following two lemmas to prove this Theorem, but first some terminology. Suppose ξ is a contact structure on $N = \Sigma \times [-1.1]$ such that C_0 is a degenerate periodic orbit in $(\Sigma_0)_\xi$. Call C_0 *positive (negative)* if the second derivative of Poincaré return map is positive (negative).

Lemma 7.3 (Giroux 2000, [10]). *Suppose C_0 is a positive (negative) degenerate orbit of $(\Sigma_0)_\xi$ then there is a neighborhood U of C_0 in N such that for small $t, (\Sigma_t)_\xi \cap U$ has*

- *two non-degenerate orbits if $t > 0$ ($t < 0$) and*
- *no closed orbits if $t < 0$ ($t > 0$).*

Lemma 7.4 (Giroux 2000, [10]). *Suppose C_0 is a retrograde connection of $(\Sigma_0)_\xi$ for small $t, (\Sigma_t)_\xi$ has two separatrices which are close to C_0. The stable separatrix passes above the unstable one when $t > 0$ and below it when $t < 0$.*

Proof of Theorem 7.2. □

Theorem 7.5 (Discritization of Isotopy: Colin 1997, [1]). *Let ξ be a tight contact structure on M, Σ and Σ' convex surfaces in M (with $\partial\Sigma = \partial\Sigma'$ Legendrian if non empty) that are isotopic (rel boundary). Then there is a sequence of convex surfaces $\Sigma_0 = \Sigma, \Sigma_1, \ldots, \Sigma_n = \Sigma'$ such that Σ_i and Σ_{i+1} co-bound a $\Sigma \times I$ in M and Σ_i and Σ_{i+1} are related by a single bypass attachment.*

Remark 7.6. If Σ_0 and Σ_1 have boundary then to make sense of the conclusion of the theorem we do the following. Let A be an convex standardly foliated annulus that contains $\partial\Sigma_i$ and is transverse to $\partial\Sigma_i, i = 0, 1$. Then we can arrange that the $\Sigma \times I$ in the theorems has $(\partial\Sigma) \times I subset A$.

A simple form of this theorem was considered in the proof of Lemma 4.26

Proof. □

In order to use the isotopy discritization it will be useful to understand bypasses that are always present and don't really "do anything", these are the trivial bypasses. Recall a trivial bypass is a bypass that does not change the dividing set.

Lemma 7.7 (Honda 2000, [3])**.** *Let ξ be vertically invariant tight contact structure on $N = \Sigma \times [-1, 1]$. Let δ be the Legendrian arc in $\Sigma \times \{0\}$ shown in Figure 41. Then there is a bypass in $\Sigma \times [0, 1]$ attached to $\Sigma \times \{0\}$ along*

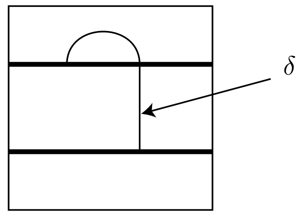


FIGURE 41. The arc of attachment for a trivial bypass.

δ.

Lemma 7.8 (Honda 2000, [3])**.** *Let Σ be a convex surface. If a bypass D is attached to Σ along a curve δ as shown in Figure 41 then the contact structure on a (one sided) neighborhood N of $\Sigma \cup D$ is isotopic to and to a vertically invariant contact structure. That is, there is a product structure $\Sigma \times [0, 1]$ on N such that the contact structure is invariant in the $[0, 1]$ direction and in particular $\Sigma \times \{t\}$ is convex for all $t \in [0, 1]$.*

We now prove our first real "gluing results". That is we will see how to take two tight contact manifolds, glue them along their boundary components and see that the result is still tight. Recall that Theorem 6.35 was called a gluing theorem for torical annuli. This is true, but we proved tightness on the glued up manifolds by having a *complete* classification of contact structures on torical annuli. When we say we are now considering the first "real gluing result" we mean that we will prove tightness of glued up contact structures without having a complete classification on the pieces. This is considerably more difficult.

Theorem 7.9 (Colin 1997, [1]; Makar-Liminov, [13])**.** *Let $M = M_1 \# M_2$, where M_i is irreducible for $i = 1, 2$. There is a bijection*

$$\mathcal{T}(M_1) \times \mathcal{T}(M_2) \to \mathcal{T}(M),$$

where $\mathcal{T}(M)$ is the isotopy classes of tight contact structures on M.

Theorem 7.10 (Colin 1999, [2])**.** *Let (M_i, ξ_i) be tight contact manifolds with boundary, $i = 1, 2$. Suppose $D_i \subset \partial M_i$ is a disk with Legendrian boundary such that the dividing curves are all boundary parallel (and outermost). If $|\Gamma_{D_1}| = |\Gamma_{D_2}|$ then we can glue M_1 and M_2 along these disk and the induced contact structure is tight.*

8. Constructing tight contact structures

To be added later.

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