

TIGHT SURGERIES ON TORUS KNOTS

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ABSTRACT. We give a complete classification of tight contact structures on manifolds obtained from any surgery along any torus knot, which provides the first such classification on an infinite family of knots. The main tools are the classification of non-loose torus knots, bordered contact invariants via their immersed curve interpretations, and non-thickenable tori in knot complements. As a byproduct of this classification, we also give the first classification of contact structures on some surface bundles over the circle with fibers of genus larger than 1. The symplectic fillability of these contact structures is also studied and leads to, among other things, the disproof of a conjecture of Ghiggini and Van Horn-Morris on Stein fillability of some contact structures on a Brieskorn sphere.

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1. Introduction

A major outstanding open problem in 3-dimensional contact topology concerns the classification of tight contact structures on a given closed, oriented 3-manifold. Over the last three decades, extensive research has been dedicated towards the resolution of this fundamental problem [2, 3, 9, 25, 27, 28, 30, 35, 36, 51, 63, 67]. In this paper, we introduce a general framework for studying contact structures on manifolds obtained from another manifold via surgery. Using this framework, we give a complete classification of tight contact structures on every manifold obtained from surgery along (p, q) -torus knots. This is the first such classification on an infinite family of knots. The only previous classification of tight contact structures on all manifolds obtained from all surgeries on some knot type was for the unknot, which yields lens spaces [30, 35]. There are some results for manifolds

obtained from surgeries on other knots and links [7, 27, 53, 56, 67], but all previous results are missing (infinitely many) surgery coefficients.

1.1. General results. The exact formulation of the results is somewhat technical, so we list some general theorems that are simple corollaries of this work to give a flavor of the results. We begin by discussing some theorems concerning the classification of tight contact structures on manifolds obtained from surgery on a torus knot, and the symplectic fillability of these contact structures. We then consider what this says about the classification of tight contact structures on small Seifert fibered spaces and on surface bundles. Precise citations to prior work that is related to our results will be given in the following subsections where the detailed theorems are given.

We will denote smooth r surgery on the (p, q) -torus knot $T_{p,q}$ by $S_r^3(T_{p,q})$.

Theorem 1.1. *For any torus knot $T_{p,q}$, there is a closed-form formula to compute the number of tight contact structures on $S_r^3(T_{p,q})$ if*

$$r \notin \begin{cases} [pq - p - q, pq - 1) & \text{for } pq > 0 \\ [pq - |q|, pq - 1) & \text{for } pq < 0. \end{cases}$$

For r in these intervals, there is a simple algorithm to compute the number of tight contact structures.

The first part of this theorem is a direct corollary of Theorems 1.8 and 1.9. The second part of the theorem follows from Theorems 1.11 and 1.13. The technical classification results are detailed in Section 1.2.

Remark 1.2. We have implemented the classification algorithm in Rust and made the code available at [18]. We have also developed an interactive [web version](#) [19] for easy accessibility.

This classification also allows us to understand quite a bit about the symplectic fillability of contact structures on $S_r^3(T_{p,q})$.

Theorem 1.3. *If $r \geq pq - 1$, then all contact structures on $S_r^3(T_{p,q})$ are Stein fillable if $pq > 0$, and the same is true for $pq < 0$ except for some contact structures when r is in a union of well-defined intervals, and in those cases the contact structures are strongly symplectically fillable.*

If r is sufficiently negative, then all contact structures on $S_r^3(T_{p,q})$ coming from negative Legendrian surgery on non-loose representatives of $T_{p,q}$ are not weakly symplectically fillable.

This theorem is a direct corollary of Theorems 1.23 and 1.24.

We also use our understanding of the contact structures developed in this paper to disprove Conjecture 1.2 from [29] about Stein fillability of some contact structures on the Brieskorn sphere $-\Sigma(2, 3, 23)$. Details of the results on symplectic fillability can be found in Section 1.5.

The Heegaard-Floer contact invariant plays an important role in our study of contact structures on $S_r^3(T_{p,q})$, as the following result shows.

Theorem 1.4. *A contact structure on $S_r^3(T_{p,q})$ is tight if and only if its Heegaard Floer contact invariant is non-zero except when $(p, q) = (2, \pm 3)$ and $r = 0$.*

Moreover, two tight contact structures on $S_r^3(T_{p,q})$ obtained by contact surgery on a Legendrian representative of $T_{p,q}$ in some (tight or overtwisted) contact structure on S^3 are isotopic if and only if they have the same Heegaard Floer contact invariant.

This theorem is essentially equivalent to Theorem 1.28.

Remark 1.5. For most values of r , all the tight contact structures on $S_r^3(T_{p,q})$ come from some contact surgery on a Legendrian knot in some (tight or overtwisted) contact structure on S^3 , and so we see the Heegaard Floer contact invariant is a complete invariant of contact structure on these manifolds.

Since the manifold $S_r^3(T_{p,q})$, for most values of r , is a small Seifert fibered space, the results above also illuminate the classification of tight contact structures on many small Seifert fibered spaces. We recall that small Seifert fibered spaces are arranged in families by an invariant, denoted by e_0 . While there is a complete understanding of tight contact structures and their fillability on small Seifert fibered spaces when $e_0 \neq -1$ or -2 , it is precisely these two families where the most interesting phenomena happen and where we have learned the most about the nature of contact structures in dimension 3. For example, it is these families where we see manifolds that do not admit tight contact structures and admit tight but not symplectically fillable contact structures. We also note that we do have a better understanding of tight contact structures on Seifert fibered spaces when they are Heegaard Floer L -spaces.

Theorem 1.6. *There is a classification of tight contact structures on many new families of $e_0 = -1$ and -2 small Seifert fibered spaces, including infinite families that are not L -spaces.*

This theorem is justified in Section 1.3. In future work, the authors will give an algorithm to classify contact structures on any small Seifert fibered space by applying our approach to torus knots in lens spaces. This will complete a line of research that has been going on for at least 25 years and has already revolutionized our understanding of contact structures in dimension 3.

Our final results regards surface bundles over the circle. Recall that 0 surgery on any torus knot yields a surface bundle over the circle. So as a byproduct of our work, we provide the first complete classification results for contact structures on surface bundles over the circle with fiber genus (arbitrarily) larger than 1.

Theorem 1.7. *The result of 0 surgery on the (p, q) -torus knots $S_0^3(T_{p,q})$ is a surface bundle over the circle with fiber genus*

$$\frac{(|p| - 1)(|q| - 1)}{2}.$$

If $(p, q) \neq (2, \pm 3)$ then $S_0^3(T_{p,q})$ has a finite number of contact structures given by an explicit formula in Theorem 1.21.

We discuss these results in Section 1.4.

1.2. Surgeries on torus knots. In this section, we discuss how to classify tight contact structures on $S_r^3(T_{p,q})$. We denote the set of all tight contact structures up to isotopy on $S_r^3(T_{p,q})$ by $\text{Tight}(S_r^3(T_{p,q}))$. Our goal is to compute the number $|\text{Tight}(S_r^3(T_{p,q}))|$ for all r and all (p, q) .

We will achieve this goal by giving simple formulas for $|\text{Tight}(S_r^3(T_{p,q}))|$ when $r \geq pq - 1$ and for "sufficiently small" surgeries. For the remaining values of r , we give an algorithm to determine $|\text{Tight}(S_r^3(T_{p,q}))|$ and then consider special families of torus knots that illustrate how the algorithm is carried out.

We start by introducing some notation that will be essential in all our theorems. Throughout we will consider (p, q) -torus knots with

$$|q| > p > 0.$$

Thus, negative (p, q) -torus knots correspond to $pq < 0$, and positive (p, q) -torus knots correspond to $pq > 0$. Any (p, q) -torus knot is isotopic to a unique torus knot in the form above.

When $pq < 0$ we set

$$\frac{q}{p} = [a_1, \dots, a_m] = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_m}}}$$

where $a_i \leq -2$ for $1 \leq i \leq m$.

When $pq > 0$ we set

$$\left(\frac{p}{q} - 1\right)^{-1} = [a_1, \dots, a_m]$$

where $a_i \leq -2$ for $1 \leq i \leq m$. In either case, we set

$$\left(\frac{q}{p} - \left\lceil \frac{q}{p} \right\rceil\right)^{-1} = [b_1, \dots, b_n]$$

where $b_i \leq -2$ for $1 \leq i \leq n$. With these notations in hand, we define the two integers

$$(1) \quad \begin{aligned} n(p, q) &= |(a_1 + 1) \cdots (a_m + 1)| \cdot |(b_1 + 1) \cdots (b_n + 1)|, \\ m(p, q) &= |(a_1 + 1) \cdots (a_{m-1} + 1)a_m| \cdot |(b_1 + 1) \cdots (b_{n-1} + 1)b_n|. \end{aligned}$$

If $a/b \in (0, 1]$, then we set

$$-b/a = [c_1, \dots, c_k]$$

for $c_i \leq -2$ for $1 \leq i \leq k$ (except when $a/b = 1$, we take $c_1 = -1$ and $k = 1$) and define

$$(2) \quad \Phi_\infty(a/b) = |c_1(c_2 + 1) \cdots (c_k + 1)|$$

and extend it to all of $\mathbb{Q}$ by making it 1-periodic: $\Phi_\infty(r + 1) = \Phi_\infty(r)$. Now for $s \in \mathbb{Q}$, we set

$$(3) \quad \Phi_s(r) = \begin{cases} 0 & \text{if } s = r, \\ \Phi_\infty\left(\frac{s^c \cdot r}{s \cdot r}\right) & \text{otherwise,} \end{cases}$$

where $\frac{a}{b} \bullet \frac{c}{d} = ad - bc$. Here we assume $b, d \geq 0$ (i.e., only numerators can be negative), and s^c is the largest rational number such that $s^c \bullet s = 1$.

For $k \in \mathbb{Z}$ such that $\gcd(k, |pq| - |p| - |q|) = 1$, we define the *exceptional slopes* as

$$(4) \quad e_k = \frac{|pq| - |p| - |q|}{k}.$$

These are the slopes of non-thickenable tori for $T_{p,q}$, see [15, 16]. We also set

$$E = \{k \in \mathbb{Z}_{>1} \mid \gcd(|pq| - |p| - |q|, k) = 1\}.$$

We call $J_k = [e_k^a, e_k)$ the *exceptional interval associated to e_k* , where e_k^a is the smallest rational number satisfying $e_k \bullet e_k^a = 1$. See Section 2.1 for more details. We note that for $(p, q) \neq (2, \pm 3)$ these exceptional intervals are pairwise disjoint, but for $(p, q) = (2, \pm 3)$ they are all nested: $J_k = [0, 1/k)$ for $k > 0$ (c.f. [15, Theorem 1.9]). Now we set

$$J = \bigcup_{k \in E} J_k.$$

We are now ready to classify tight contact structures on $S_r^3(T_{p,q})$ when $r \geq pq - 1$.

Theorem 1.8. *For any torus knot $T_{p,q}$ and $r \in [pq - 1, pq)$, we have*

$$|\text{Tight}(S_r^3(T_{p,q}))| = (m(p, q) - 2n(p, q)) \Phi_{pq}(r) + n(p, q) \Phi_\infty(r).$$

For a positive torus knot $T_{p,q}$ and $r \geq pq$, we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = n(p, q) \Phi_\infty(r).$$

For a negative torus knot $T_{p,q}$, $r \geq pq$ and $n \in \mathbb{N}$, we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = \begin{cases} n(p, q) \Phi_\infty(r), & r \notin J \cup J_1, \\ n(p, q) \Phi_\infty(r) + \Phi_{e_k}(r), & r \in J_k, k \in E \cup \{1\} \text{ and } (p, q) \neq (2, -3), \\ \frac{(n+1)(n+2)}{2} \Phi_{1/n}(r), & r \in [\frac{1}{n+1}, \frac{1}{n}) \text{ and } (p, q) = (2, -3), \\ \infty & r = 0 \text{ and } (p, q) = (2, -3). \end{cases}$$

We will see that these simple formulas already recover most of the known results, and many new results, for small Seifert fibered spaces obtained by surgeries on torus knots.

To deal with sufficiently small surgery coefficients $n \in \mathbb{Z}$, we compute the number of non-loose Legendrian $T_{p,q}$ with $\text{tb} = n$ and no half-convex torsion in their complement; denote this number by $\mathcal{L}_n^X(T_{p,q})$. It turns out that $\mathcal{L}_n^X(T_{p,q})$ is a finite constant for sufficiently small $n \in \mathbb{Z}$. Call this constant $M(p, q)$ and note that it is computed in Formula (9) for negative torus knots and Formula (16) for positive torus knots.

Theorem 1.9. *For a positive torus knot $T_{p,q}$ and $r < pq - p - q$, we have*

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (M(p, q) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (pq - p - q - \lfloor r+1 \rfloor + 1) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2) \Phi_\infty(r) + \sum_{r \in J_k, k \in E} \Phi_{e_k}(r). \end{aligned}$$

For a negative torus knot $T_{p,q}$ and $r < pq - |q|$, we have

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (M(p, q) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (|p + q| + pq - \lfloor r+1 \rfloor + 1) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + n(p, q) \Phi_\infty(r). \end{aligned}$$

Remark 1.10. Theorems 1.8 and 1.9 give explicit formulas for all surgeries on torus knots except for a finite interval $[pq - p - q, pq - 1)$ or $[pq - |q|, pq - 1)$, where we need to use a more involved algorithm to compute $|\text{Tight}(S_r^3(T_{p,q}))|$.

We see the precise overlap with previous results in Section 1.3, but note here that for $r \in [pq - 1, pq + 1]$ these results were previously known by work of Wu [67] and Ghiggini, Lisca, and Stipsicz [27]. For positive torus knots, the above results for $r > pq + 1$ were previously obtained by Ghiggini [26] and partially by the third author [63]. For $r < 0$, the results were previously unknown. For negative torus knots and sufficiently negative r , the results were previously obtained by Matkovič [53], and for $r \in (pq + 1, pq - |p| - |q|)$ the results were known by work of Ghiggini [26] and the third author [63]. For $r > pq - |p| - |q|$, the results are almost all new, except for one family of examples that appeared in the third author's work [63]. Finally, the result for $S_0^3(T_{2,-3})$ is due to Giroux [30] and Honda [36] as it is a torus bundle over S^1 .

1.2.1. Surgeries on negative torus knots. We now state the main result concerning negative torus knots.

Theorem 1.11. *Consider a negative torus knot $T_{p,q}$. Let $\mathcal{L}_n(T_{p,q})$ be the number of Legendrian representatives of $T_{p,q}$ with $\text{tb} = n$ and whose standard neighborhood has tight complement without half-convex torsion. A formula for $\mathcal{L}_n(T_{p,q})$ is given in Equations (6), (7), and (11). For $r < pq - 1$, we have*

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) + n(p, q) \Phi_\infty(r).$$

Remark 1.12. This covers all the cases not handled in Theorems 1.8 and 1.9 above, and overlaps with Theorem 1.9. We will see in Section 2.2 that $M(p, q)$ in Theorem 1.9 is simply the maximum value of $\mathcal{L}_n(T_{p,q})$ minus the number of Legendrian representatives in the tight contact structure on S^3 , and agrees with it for n sufficiently negative. Thus, the theorems give the same count for tight contact structures where they both apply. Unfortunately, for $n \in (pq - |q|, pq - 1)$ there is no closed form for $\mathcal{L}_n(T_{p,q})$, but it is straightforward to determine, as we will see in Section 2.2.

The classification results in this theorem, in a very different form, first appeared in Matkovič's paper [53].

1.2.2. *Surgeries on the positive torus knots.* We now state the main result concerning positive torus knots. We will use the notation from the start of this section.

Theorem 1.13. *Consider a positive torus knot $T_{p,q}$. Let $\mathcal{L}_n(T_{p,q})$ be the number of Legendrian representatives of $T_{p,q}$ with $\text{tb} = n$ and whose standard neighborhood has tight complement without half-convex torsion. A formula for $\mathcal{L}_n(T_{p,q})$ is given in Equations (12), (13), and (19).*

(1) For $r \in [pq - p - q + 2, pq - 1]$ we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) + n(p, q) \Phi_\infty(r)$$

(2) For $r \in [pq - p - q + 1, pq - p - q + 2]$ we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 1) \Phi_{\lfloor r+1 \rfloor}(r) \\ + (n(p, q) - 2) \Phi_\infty(r) + \Phi_{pq-p-q}(r).$$

(3) For $r \in [pq - p - q, pq - p - q + 1]$ we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 2) \Phi_{\lfloor r+1 \rfloor}(r) \\ + (n(p, q) - 2) \Phi_\infty(r) + \Phi_{pq-p-q}(r).$$

(4) For $r < pq - p - q$ and $r \notin J$ we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ + (n(p, q) - 2) \Phi_\infty(r).$$

(5) For $r < pq - p - q$ and $r \in J$,

(a) If $(p, q) \neq (2, 3)$ and $r \in J_k$ for some $k \in E$, then we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ + (n(p, q) - 2) \Phi_\infty(r) + \Phi_{e_k}(r).$$

(b) If $(p, q) = (2, 3)$ and $r \in [1/(n+1), 1/n] \subset J$ then we have

$$|\text{Tight}(S_r^3(T_{2,3}))| = \frac{(n+1)n}{2} \Phi_{1/n}(r).$$

(c) If $r = 0$ we have

$$|\text{Tight}(S_0^3(T_{2,3}))| = \infty$$

This theorem clearly handles all the cases not dealt with in Theorem 1.8 and 1.9. We note that almost all the cases handled in this theorem are new results with the exception of $1/n$ surgeries on the right-handed trefoil, which were originally classified in work of Ghiggini and Van Horn-Morris [29], and some cases are also handled in Ghiggini, Lisca, and Stipsicz's work [28].

In the next two subsections, we illustrate the outcome of the algorithm mentioned above in the case of surgeries on some specific torus knots.

1.2.3. *Surgeries on $(2, \pm(2n+1))$ -torus knots.* In this case, the exceptional slopes from above are

$$e_k = \frac{2n-1}{k},$$

and $J_k = [e_k^a, e_k)$. If $(2n-1)$ and k are relatively prime, then there are non-negative integers a_k and b_k such that $(2n-1)b_k - ka_k = 1$ with $a_k \leq 2n-1$ and $b_k \leq k$. Then, $e_k^a = a_k/b_k$.

Theorem 1.14. *The number of tight contact structures on $S_r^3(T_{2, -(2n+1)})$ for $n \in \mathbb{N}$ is*

$$|\text{Tight}(S_r^3(T_{2, -(2n+1)}))| = \begin{cases} n\Phi_\infty(r) & r \geq -4n-2 \text{ and } r \notin J \cup J_1 \\ n\Phi_\infty(r) - (2n+2 + \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & r < -4n-2 \end{cases}$$

When $r \in J \cup J_1$ we need to consider two cases. If $n = 1$, then either $r = 0$ or there is some unique integer m satisfying $\frac{1}{m+1} \leq r < \frac{1}{m}$. In the former case, we have

$$|\text{Tight}(S_0^3(T_{2, -3}))| = \infty,$$

while in the latter case, we have

$$|\text{Tight}(S_r^3(T_{2, -3}))| = \frac{(m+1)(m+2)}{2}\Phi_{1/m}(r).$$

If $n > 1$ and $r \in J_k$ for some $k \in E \cup \{1\}$, then we have

$$|\text{Tight}(S_r^3(T_{2, -(2n+1)}))| = n\Phi_\infty(r) + \Phi_{e_k}(r).$$

We note that the exact list of contact structures in $\text{Tight}(S_r^3(T_{2, -3}))$ can be given. It is interesting to note that in all cases above $n\Phi_\infty(r)$ of the contact structures come from surgery on non-loose knots in n different overtwisted contact structures, and for $r < -2n-2$ the remainder come from negative contact surgery on Legendrian knots in (S^3, ξ_{std}) . When $r \in J \cup J_1$, the extra contact structures come from Dehn filling complements of non-thickenable tori in the overtwisted contact structure ξ_{2n} .

For positive $(2, 2n+1)$ -torus knots, the exceptional slopes e_k and intervals J_k are the same as for the $(2, -(2n+1))$ -torus knots discussed above.

Theorem 1.15. *The number of tight contact structures on manifolds obtained from surgery on the $(2, 2n+1)$ -torus knot, for $n > 0$, is*

$$|\text{Tight}(S_r^3(T_{2, 2n+1}))| = \begin{cases} 2\Phi_\infty(r) & r \in [2n+1, 2n+2) \cup [4n+2, \infty) \\ 2\Phi_\infty(r) + 2\Phi_{\lfloor r+1 \rfloor}(r) & r \in [2n+3, 4n+2) \\ 2\Phi_\infty(r) + \Phi_{\lfloor r+1 \rfloor}(r) & r \in [2n+2, 2n+3) \\ \Phi_{2n-1}(r) & r \in (2n-1, 2n+1) \\ 0 & r = 2n-1 \\ \Phi_{2n-1}(r) & r < 2n-1 \text{ and } r \notin J. \end{cases}$$

When $r \in J$, we need to consider two cases. For $n = 1$, either $r = 0$ or there is some unique integer m satisfying $\frac{1}{m+1} \leq r < \frac{1}{m}$. In the former case, we have

$$|\text{Tight}(S_0^3(T_{2, 3}))| = \infty,$$

while in the latter case, we have

$$|\text{Tight}(S_r^3(T_{2,3}))| = \frac{m(m+1)}{2} \Phi_{1/m}(r).$$

For $n > 1$ and $r \in J_k$ for some $k \in E$, we have

$$|\text{Tight}(S_r^3(T_{2,2n+1}))| = \Phi_{2n-1}(r) + \Phi_{e_k}(r).$$

Remark 1.16. The $r = 2n - 1$ case of the theorem reproves the non-existence results of Honda and the first author [12] and Lisca and Stipsicz [46]. As mentioned above, for $r \in [4n + 1, 4n + 3]$ the above theorem follows from the classification of contact structures on small Seifert fibered spaces with $e_0 \neq -1, -2$ and on lens spaces. In addition, for $r \in [4n, 4n + 1]$ the classification follows from work of Ghiggini, Lisca, and Stipsicz [28], for $r > 4n + 3$ the result also follows from past work of Ghiggini [26] and the third author [63], and for $r = \frac{1}{k}$ we recover the work of Ghiggini and Van Horn-Morris [29]. For $r \leq 4n$ and not equal to $\frac{1}{k}$ these classification results are new.

1.2.4. *Surgeries on the positive $(5, \pm 8)$ -torus knots.* For the $(5, -8)$ -torus knot, our algorithm produces the following result.

Theorem 1.17. *The number of tight contact structures on manifolds obtained from surgery on the $(5, -8)$ torus knot is*

$$|\text{Tight}(S_r^3(T_{5,-8}))| = \begin{cases} 4\Phi_\infty(r) & \text{if } r \geq -40 \text{ and } r \notin J \cup J_1, \\ 4\Phi_\infty(r) + \Phi_{e_k}(r) & \text{if } r \in J_k \text{ for } k \in E \cup \{1\}, \\ 4\Phi_\infty(r) + 4\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } -41 \leq r < -40, \\ 4\Phi_\infty(r) + 8\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } -42 \leq r < -41, \\ 4\Phi_\infty(r) + (-32 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } r < -42, \end{cases}$$

Here, the intervals J_k are defined as follows. For $e_k = 27/k$ and $k \in E \cup \{1\}$, then there are positive integers a_k and b_k such that $27b_k - ka_k = 1$ with $a_k < 27$ and $b_k < k$. Then $J_k = [a_k/b_k, 27/k)$.

The classification of contact structures on the result of surgery on the $(5, 8)$ -torus knots is as follows.

Theorem 1.18. *The number of tight contact structures on manifolds obtained from surgery on the $(5, 8)$ torus knot is*

$$|\text{Tight}(S_r^3(T_{5,8}))| = \begin{cases} 8\Phi_\infty(r) & \text{if } r \geq 40, \\ 8\Phi_\infty(r) + 8\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 39 \leq r < 40, \\ 8\Phi_\infty(r) + 10\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 37 \leq r < 39, \\ 8\Phi_\infty(r) + (-28 + \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 29 \leq r < 37, \\ 6\Phi_\infty(r) + 2\Phi_{\lfloor r+1 \rfloor}(r) + \phi_{27}(r) & \text{if } 27 \leq r < 29, \\ 6\Phi_\infty(r) + (28 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } r < 27 \text{ and } r \notin J, \text{ and} \\ 6\Phi_\infty(r) + (28 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) + \Phi_{e_k}(r) & \text{if } r \in J_k \text{ for } k \in E. \end{cases}$$

Here, the intervals J_k are defined as follows. For $e_k = 27/k$ and $k \in E$, then there are positive integers a_k and b_k such that $27b_k - ka_k = 1$ with $a_k < 27$ and $b_k < k$. Then $J_k = [a_k/b_k, 27/k)$.

1.3. Small Seifert fibered spaces. Recall that a small Seifert fibered space is a manifold given by the surgery diagram in Figure 1. It is in *normalized form* if $r_i \in (0, 1)$ for all i . In this case we denote the manifold by $Y(e_0; r_1, r_2, r_3)$. The number e_0 is called the *integral Euler number* of the Seifert fibered space. In [27, 67], the contact structures on small Seifert fibered spaces with $e_0 \neq -1, -2$ were classified. There is a simple formula for the number of tight contact structures on such spaces, and they are all Stein fillable. When $e_0 = -1$ or -2 , our understanding is incomplete, and it is for these manifolds that the classification of tight contact structures shows great subtlety. For example, it is with these invariants that we see manifolds not admitting any tight contact structure [12, 46], admitting tight but not symplectically fillable contact structures [13, 28, 43], admitting symplectically fillable contact structures that are not Stein fillable [23, 55], and admitting homotopic but not isotopic contact structures [42, 61]. In addition, the classification of contact structures on these manifolds is only known in a few cases [28, 53, 63], so expanding our understanding of tight contact structures on small Seifert fibered spaces with $e_0 = -1$ or -2 should greatly extend our understanding of the general structure of contact structures on 3-manifolds and better implement certain cut-and-paste operations in symplectic geometry and symplectic embedding questions [4, 22, 48, 49, 50].

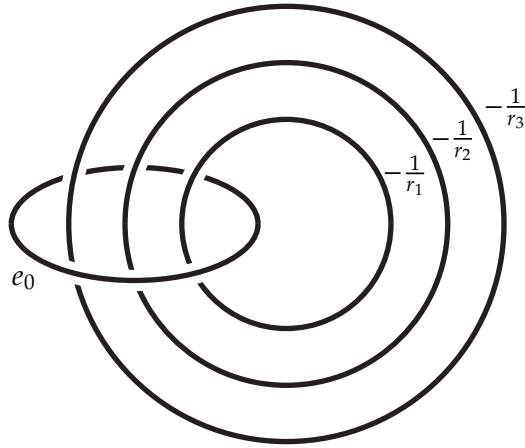


FIGURE 1. A surgery diagram for small Seifert fibered space $Y(e_0; r_1, r_2, r_3)$.

We will see below that surgeries on torus knots almost always produce small Seifert fibered spaces and that we can recover some known classification results, as well as many new ones.

In [59], Moser showed that r surgery on a (p, q) -torus knot is a small Seifert fibered space as long as $r \neq pq$ or $pq \pm 1/n$ for some integer n . On the left-hand side of Figure 2, we see the result of r surgery on the positive torus knot $T_{p,q}$ where $0 < p^* < q$ and $0 < q^* < p$

denote the unique integers satisfying $pp^* \equiv 1 \pmod{q}$ and $qq^* \equiv 1 \pmod{p}$, respectively. Similarly, on the right-hand side of the figure we see the result of r surgery on the negative torus knot $T_{p,q}$, where $q < p^* < 0$ and $0 < q^* < p$ denote the unique integers satisfying $pp^* \equiv -1 \pmod{q}$ and $qq^* \equiv -1 \pmod{p}$, respectively.

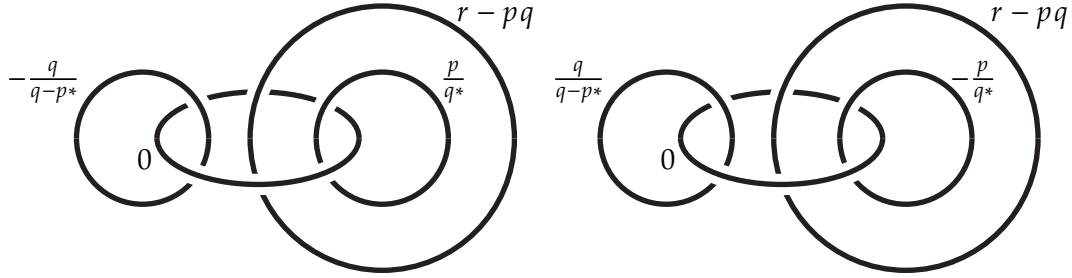


FIGURE 2. Left: surgery diagrams for r surgery on the positive torus knot $T_{p,q}$. Right: surgery diagrams for r surgery on the negative torus knot $T_{p,q}$.

We note that these surgery diagrams are usually not in normalized form. One may perform Rolfsen twists to put the surgery diagram in normalized form. Normalizing the diagram this way we see that for all $n \in \mathbb{N}$ the result of surgery on the (p, q) -torus knot results in

$$(5) \quad S_r^3(T_{p,q}) = \begin{cases} e_0 = -2 & \text{if } r > pq + 1 \\ e_0 \leq -3 & \text{if } r \in (pq, pq + 1) \text{ and } r \neq pq + \frac{1}{n}, \\ \text{lens space} & \text{if } r = pq + \frac{1}{n} \\ L(p, -q) \# L(q, -p) & \text{if } r = pq \\ \text{lens space} & \text{if } r = pq - \frac{1}{n} \\ e_0 \geq 0 & \text{if } r \in (pq - 1, pq) \text{ and } r \neq pq - \frac{1}{n} \\ e_0 = -1 & \text{if } r < pq - 1 \end{cases}$$

The information in these equations can be seen visually in Figure 3. The figures also indicate which surgeries are L -spaces, as this information is important in prior classification results. Recall that a 3-manifold Y is an L -space if its Heegaard Floer homology is as simple as possible: $HF^{\text{red}}(Y, \mathfrak{s}) = 0$ for every Spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y)$.

In the case of a positive (p, q) -torus knot, the classification of tight contact structures on $S_r^3(T_{p,q})$ had previously been completed when $r \geq pq - p - q$. Specifically, when $r > pq + 1$ in work of Ghiggini [26] and the third author [63], for $r \in ((2pq - 1)/2, pq + 1)$ in work of Wu [67] for $r \in (pq - 1, (2pq - 1)/2)$, and in works of Ghiggini, Lisca, and Stipsicz [27, 28] for $r \in (pq - 1, pq)$ and $r \in [pq - 2, pq - 1)$, respectively. For surgery coefficients $r \in (pq - p - q, pq - 1)$, Matkovič [53] reduces the classification problem to a well-understood computation of invariants.

We note that for surgery coefficients $pq + 1/n$, $n \in \mathbb{Z}$, the resulting manifolds are lens spaces and for surgery coefficient pq , the resulting manifold is a connected sum of lens

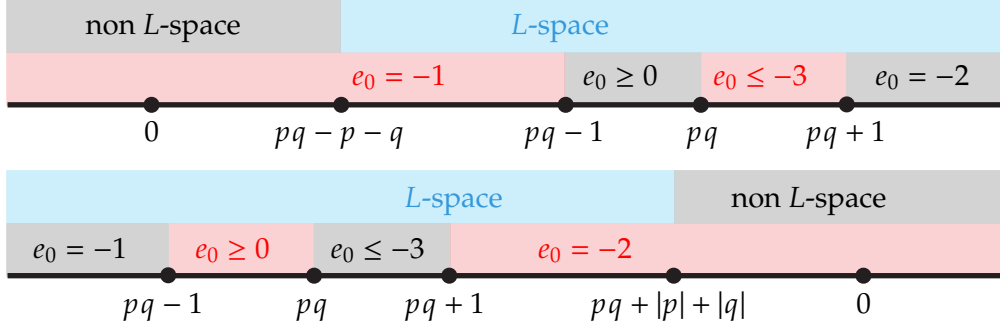


FIGURE 3. The top figure shows the range of values of e_0 for r surgery on the positive (p, q) -torus knot as well as whether it is an L -space or not. The same information is shown in the bottom diagram for r surgery on the negative (p, q) -torus knot. We note that when $r = pq \pm q/n$ we obtain a lens space, when $r = pq$ we obtain a connected sum of lens spaces, and for all other r values we obtain a small Seifert fibered space with indicated e_0 .

spaces. Contact structures were classified on the former by Giroux [30] and Honda [35], and later by Colin [5] (together with the work of Giroux and Honda).

In the case of a negative (p, q) -torus knot, the classification of tight contact structures on $S_r^3(T_{p,q})$ had previously been completed when $r \leq pq - |q| - |p|$; for other slopes the results are new. Specifically, the classification was done in the work of Wu [67] for $r \in ((2pq - 1)/2, pq + 1)$, in the work of Ghiggini, Lisca, and Stipsicz [27] for $r \in (pq - 1, pq)$, by the work of the same authors in [28] for $r \in [pq - 2, pq - 1)$, and by the work of Ghiggini [26] and the third author [63] for $r \in (pq + 1, pq + |p| + |q|)$. For surgery coefficients $r < pq - 1$, Matković [53] reduces the classification problem to a well-understood computation of invariants. We have the same comment as above about lens spaces and their connected sums for surgery coefficients $pq + 1/n$ and pq .

Remark 1.19. For r surgeries on positive (p, q) -torus knots with $r \geq pq - 1$ and r surgeries on negative (p, q) -torus knots with $r \in [pq - 1, pq + |p| + |q|]$, the previous work cited above gives explicit formulas for the number of contact structures on the Seifert fibered space. Of course, our formulas, which look quite different, can be seen to give the same answers. We demonstrate this in Section 4.5. In [28, 53], there were also explicit formulas given for some small Seifert fibered spaces with $e_0 = -1$. While it is not always easy to compare the results there to our work, one can work out many cases (such as for the $(2, 2n + 1)$ -torus knots) and see that once again our computations agree with past work.

Recall that the *maximal twisting number* of a contact structure on a Seifert fibration is the maximal twisting number of a Legendrian representative of a regular fiber with respect to the product framing. If the maximal twisting number is greater than or equal to 0 (which implies that the maximal twisting is 0), then we say the contact structure is *0-twisting*. If the twisting number is less than 0, we say the contact structure is *negatively twisted*. We

distinguish each tight contact structure on $S_r^3(T_{p,q})$ by whether it is 0-twisting or negatively twisted.

Theorem 1.20. *For a negative torus knot $T_{p,q}$, we have*

- (1) *For $r \geq pq$, every tight contact structure on $S_r^3(T_{p,q})$ is negatively twisted.*
- (2) *For $r < pq$, every tight contact structure on $S_r^3(T_{p,q})$ is zero-twisting.*

We now use the notations from Theorem 1.13. For a positive torus knot $T_{p,q}$, we have

- (1) *For $r \geq pq$, every tight contact structure on $S_r^3(T_{p,q})$ is negatively twisted.*
- (2) *For $r \in [pq - p - q, pq)$, every tight contact structure on $S_r^3(T_{p,q})$ is zero-twisting.*
- (3) *For $r < pq - p - q$,*

$$\begin{aligned} & (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ & - (pq - p - q - \lfloor r + 1 \rfloor + 1) \Phi_{\lfloor r+1 \rfloor}(r) \\ & + (n(p, q) - 2) \Phi_{\infty}(r). \end{aligned}$$

tight contact structures, obtained from non-loose Legendrian representatives, are zero-twisting. The other tight contact structures, obtained from (S^3, ξ_{std}) and non-thickenable tori, are negatively twisted.

An algorithm to classify negatively twisting tight contact structures on small Seifert fibrations was achieved in the work of Cavallo and Matkovič [2, 3]. Previously, in the note [17] by the authors of the current paper, a formula was given for the classification on such Seifert fibered spaces obtained by surgery on a positive torus knot. This note was part of a preliminary version of this paper that was released as background for another paper. We note that classifying the 0-twisting contact structures is more challenging and shows more interesting phenomena.

1.4. Surface bundles over circles. Currently, the only complete classification results for tight contact structures on surface bundles over the circle are when the fiber has genus 1. In that case, Honda gave a complete classification of tight contact structures in [36], and Giroux did the same in [30], though he left the tightness of a small number of contact structures undetermined. The only other work trying to understand tight contact structures on surface bundles was done in [38], where Honda, Kazez, and Matić showed that there was a unique tight contact structure on a hyperbolic surface bundle having “extremal Euler class”.

Since torus knots are fibered knots, we see that 0-surgery on a torus knot will result in a surface bundle over the circle. Thus, our results give the first complete classification of tight contact structures on surface bundles over circles with fiber genus larger than 1.

Theorem 1.21. *The result of 0 surgery on the (p, q) -torus knots $S_0^3(T_{p,q})$ is a surface bundle over the circle with fiber genus*

$$\frac{(|p| - 1)(|q| - 1)}{2}.$$

If $pq < 0$ and not $(2, -3)$, then $S_0^3(T_{p,q})$ has exactly

$$n(p, q)$$

tight contact structures, up to isotopy. If $pq > 0$ and not $(2, 3)$, then $S_0^3(T_{p,q})$ has exactly

$$M(p, q)$$

tight contact structures, up to isotopy, where $M(p, q)$ is defined in Equation (16). If $(p, q) = (2, \pm 3)$, then $S_0^3(T_{p,q})$ has infinitely many tight contact structures, up to isotopy.

Remark 1.22. We note that the case of $(2, 3)$ and $(2, -3)$ -torus knots were excluded from the above theorem. In those cases, the surface bundle has genus 1 fibers, were classified in [30, 36], and they gave an explicit list of infinitely many tight contact structures. We recover this in our classification in terms of Dehn fillings of the exceptional slopes $e_k = 1/k$ and their intervals $J_k = [0, 1/k)$ of slopes in the theorems above.

1.5. Symplectic fillability. Now we study fillability of the tight contact structures on $S_r^3(T_{p,q})$. We first consider sufficiently large surgery coefficients.

Theorem 1.23. Suppose $r \geq pq - 1$. Then

- For a positive torus knot $T_{p,q}$, any tight contact structure on $S_r^3(T_{p,q})$ is Stein fillable.
- For a negative torus knots $T_{p,q}$, we need to consider the following cases.
 - (1) If $r \notin J \cup J_1$, then any tight contact structure on $S_r^3(T_{p,q})$ is Stein fillable.
 - (2) Except for the case $(p, q) = (2, -3)$ and $r = 0$, if $r \in J \cup J_1$, then $\sum_{r \in J_k, k \in E \cup \{1\}} \Phi_{e_k}(r)$ contact structures obtained from non-thickenable tori are symplectically fillable. The other tight contact structures on $S_r^3(T_{p,q})$ are Stein fillable.
 - (3) When $(p, q) = (2, -3)$ and $r = 0$, there are infinitely many tight contact structures. Only one is Stein fillable while all others are weakly symplectically fillable, but not strongly fillable.

Next, we determine the fillability when r is sufficiently small. We define the function

$$\eta(p, q) = pq - \max \left\{ \frac{p}{p - p'}, \frac{q}{q'} \right\}.$$

where q'/p' is the largest rational number such that $pq' - p'q = 1$.

Theorem 1.24. Suppose $r < \eta(p, q)$. Then

- For a positive torus knot $T_{p,q}$, we have
 - (1) If $pq - p - q \leq r < \eta(p, q)$, then any contact structure on $S_r^3(T_{p,q})$ is not weakly fillable.
 - (2) If $r < pq - p - q$ and $r \notin J$, then $(pq - p - q - \lfloor r + 1 \rfloor + 1)\Phi_{\lfloor r + 1 \rfloor}(r)$ contact structures on $S_r^3(T_{p,q})$, obtained from negative contact surgery on Legendrian representatives in (S^3, ξ_{std}) , are Stein fillable. The other tight contact structures, obtained from non-loose representatives, are not weakly fillable.

- (3) Except for the case $(p, q) = (2, 3)$ and $r = 0$, if $r < pq - p - q$ and $r \in J$, then $(pq - p - q - \lfloor r + 1 \rfloor + 1) \Phi_{\lfloor r + 1 \rfloor}(r)$ contact structures on $S^3_r(T_{p,q})$, obtained from Legendrian representatives in (S^3, ξ_{std}) , are Stein fillable. Also, $\sum_{r \in J_k, k \in E} \Phi_{e_k}(r)$ tight contact structures, obtained from non-thickenable tori, are symplectically fillable. The other tight contact structures, obtained from non-loose representatives, are not weakly fillable.
- (4) If $(p, q) = (2, 3)$ and $r = 0$, then there are infinitely many tight contact structures. Only one is Stein fillable while all others are weakly symplectically fillable but not strongly fillable.
- For a negative torus knot $T_{p,q}$, $(|p + q| + pq - \lfloor r + 1 \rfloor + 1) \Phi_{\lfloor r + 1 \rfloor}(r)$ contact structures, obtained from Legendrian representatives in (S^3, ξ_{std}) , are Stein fillable. The other tight contact structures, obtained from non-loose representatives, are not weakly fillable.

Remark 1.25. When $\eta(p, q) < r < pq - 1$, we need to apply [54, Theorem 1.1] to each contact structure to determine whether it is Stein fillable or not weakly fillable.

Strong vs Stein fillability. Ghiggini and Van Horn-Morris in [29] completed the classification of tight contact structures on $S^3_{1/k}(T_{2,3})$. This family exhibits a rather curious contact and symplectic geometric properties. In [29], it was proved that there are, up to isotopy, exactly $\frac{k(k-1)}{2}$ tight contact structures on $S^3_{1/k}(T_{2,3})$, and these can conveniently be arranged in a triangle as shown in Figure 4 (with $k = 4$), and labeled $\xi_{i,j}$ where $i = 0, \dots, k - 2$ and $|j| \leq k - i - 2$ with j congruent to $k - i$ modulo 2. It is known that all the $\xi_{i,j}$ are strongly fillable and $\xi_{0,j}$ are Stein fillable. Ghiggini in [23] (see also [29]) showed that $\xi_{i,0}$ is not Stein fillable (or even exactly symplectically fillable). Finally, facing a lack of examples, Ghiggini and Van Horn-Morris in [29] made the following conjecture.

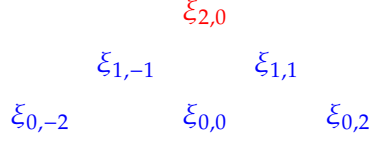
Conjecture 1.26 (Ghiggini and Van Horn-Morris [29]). *The contact manifolds $\xi_{i,j}$ for $i \neq 0$ are not Stein (or exactly symplectically) fillable.*

Recently, the second author [55] proved this conjecture for contact structures on the “interior” of the triangle. More specifically, for $\xi_{i,j}$ with $i = 1, \dots, k - 2$ and $|j| \leq k - i - 2$. But this leaves the Stein fillability of the contact structures on the non-horizontal edges of the triangle unknown. We disprove the conjecture in the following result.

Theorem 1.27. *On the manifold $S^3_{1/4}(T_{2,3})$, the contact structures $\xi_{1,-1}$ and $\xi_{1,1}$ are Stein fillable. That is, all contact structures on the manifold except $\xi_{2,0}$ are Stein fillable.*

The Stein fillability (or perhaps non-fillability) of the contact structures on $S^3_{1/k}(T_{2,3})$ for $k > 4$ remains open. In [10], Sağlam and the first author study the fillability of the contact structures on $S^3_r(T_{2,3})$ for certain $r \in (0, 1)$ and show that there seems to be no apparent structure to the type of fillability for many of the contact structures.

1.6. Heegaard Floer contact invariant. In the process of proving the main theorems, we establish that if $r \notin J$ (resp. $J \cup J_1$ for negative torus knots), then the Heegaard Floer contact

FIGURE 4. The contact structures on $S^3_{1/4}(T_{2,3})$.

invariant is a complete invariant of tight contact structures on $S^3_r(T_{p,q})$. Recall that a contact structure ξ on a 3-manifold Y determines a spin^c structure t_ξ and there is an element $\widehat{c}(\xi) \in \widehat{HF}(-Y, t_\xi)$ associated to ξ .

Theorem 1.28. *A contact structure ξ on $S^3_r(T_{p,q})$ is tight if and only if the Heegaard Floer contact invariant $\widehat{c}(\xi)$ is non-vanishing except when $(p, q) = (2, \pm 3)$ and $r = 0$. Moreover, if $r \notin J$ (resp. $J \cup J_1$ for negative torus knots), then two tight contact structures ξ and ξ' on $S^3_r(T_{p,q})$ are isotopic if and only if their contact invariants agree: $\widehat{c}(\xi) = \widehat{c}(\xi')$.*

For $r \in J$ (resp. $J \cup J_1$ for negative torus knots), we use the technique of partially thickenable tori from Section 5 to establish tightness and distinguish the contact structures on $S^3_r(T_{p,q})$. (Note that some of them are torus bundles, so the results of Honda [36] and Giroux [30] apply.) We believe that $\widehat{c}(\xi)$ is a complete invariant for contact structures on $S^3_r(T_{p,q})$ (except when $(p, q) = (2, \pm 3)$ and $r = 0$, when Giroux torsion prevents it), but it is difficult to verify for the contact structures coming from non-thickenable tori. However, [29] gives evidence that it is true.

Lastly, we show that there are some interesting connections between non-loose torus knots and bordered Floer homology. In particular, we can estimate the number of specific non-loose torus knots in terms of their Alexander polynomial, and hence the number of tight contact structures on $S^3_{n-1}(T_{p,q})$. The mountain ranges for non-loose torus knots contain features such as infinite Xs, an infinite V, and wings; see Section 2.2 for the definitions of these terms. These, and similar information about the mountain ranges for Legendrian trefoil knots in tight S^3 , are all visible in bordered Floer homology of the knot complement. For details, see Section 6.4.

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2. Background results in contact topology

We assume the reader is familiar with standard results in contact geometry as can be found in [20]. We review some classification results of tight contact structures in Section 2.1. We then review the classification of non-loose torus knots in Section 2.2. In Section 2.3, we discuss what is known about non-thickenable tori in the knot type of a torus knot.

2.1. Paths in the Farey graph and contact structures. To discuss the classification of contact structures on solid tori, we will need to use the Farey graph. Details of the Farey graph can be found in [20], but we briefly recall that the Farey graph is in the unit disk D^2 in $\mathbb{R}^2$, with the hyperbolic metric on its interior. We label the point $(0, 1)$ by $0 = 0/1$ and $(0, -1)$ by $\infty = 1/0$ and add a hyperbolic geodesic between these points. Now, for any point on ∂D^2 with positive x -coordinate that is halfway between two labeled points a/b and c/d , we label it by $a/b \oplus c/d$, which is defined to be $(a + c)/(b + d)$. Now add a hyperbolic geodesic between the new point and the two previous points. Iterate this construction. Similarly, label for points on ∂D^2 with negative x -coordinate, except in this case, think of 0 and $0/-1$. We denote the set of vertices in the Farey graph clockwise of a vertex with slope r and anti-clockwise of a vertex of slope s by (r, s) , and similarly for $[r, s)$, $(r, s]$, and $[r, s]$.

Given a torus T^2 and a basis λ, μ for its first homology, we can denote any embedded closed curve γ in T^2 in terms of its homology class $[\gamma] = p\lambda + q\mu$, for relatively prime integers p and q . We will denote this curve by q/p , which is a rational number or ∞ , and we can moreover think of it as a vertex in the Farey graph. So we can think of the vertices in the Farey graph as representing curves on the torus T^2 . It is well-known that two curves on T^2 will give a basis for $H_1(T^2)$ if and only if their corresponding vertices in the Farey graph are connected by an edge. (We note that throughout this paper T^2 will be considered as the boundary of a neighborhood of a knot in S^3 and its basis will be the Seifert longitude followed by its meridian.)

When describing tight contact structures on solid tori, we will need to consider paths in the Farey graph. By a path in the Farey graph from r to s we will always mean a monotone path; that is, each edge in the path moves clockwise from r to s or each edge in the path moves anti-clockwise. Given a minimal path from r to s , we say it is a *continued fraction block* if there is a change of basis taking the vertices of the path to $(1, 2, \dots, n)$ for some positive integer n .

We note that if a number $r < -1$ has continued fraction expansion $r = [a_1, \dots, a_n]$ then the numbers $r^c = [a_1, \dots, a_n + 1]$ and $r^a = [a_1, \dots, a_{n-1}]$ both have an edge in the Farey graph to r and they also have an edge to each other. Moreover, r^c is clockwise of r and r^a is anti-clockwise of r . These are the unique two points with these properties. From this one can see that the minimal path from r clockwise to -1 has vertices starting with $[a_1, \dots, a_n]$ and if $[a_1, \dots, a_k, b]$ is a vertex then the next vertex is $[a_1, \dots, a_k, b + 1]$ where if $b = -2$ we set $[a_1, \dots, a_k, b + 1]$ to be $[a_1, \dots, a_k + 1]$. We note that in terms of continued fractions the vertices $[a_1, \dots, a_k], [a_1, \dots, a_k + 1], \dots, [a_1, \dots, a_{k-1} + 1]$ form a

continued fraction block. Similarly a minimal path from r anti-clockwise to $\lfloor r \rfloor$ is given by $[a_1, \dots, a_n], [a_1, \dots, a_{n-1}], \dots, [a_1]$.

We say a path P in the Farey graph is *decorated* if each edge is assigned either a $+$ or a $-$. We say two decorations on a path differ by *shifting in continued fraction blocks* if their signs agree after possibly interchanging some of the signs in each continued fraction block.

We now define a convenient way to think about solid tori. Consider $T^2 \times [0, 1]$. Foliate $T^2 \times \{0\}$ by curves of slope r and define the space S_r to be the quotient of $T^2 \times [0, 1]$ by these curves. This is easily seen to be a solid torus, and we say S_r is a *solid torus with lower meridian r* . We can similarly quotient $T^2 \times [0, 1]$ by curves of slope r on $T^2 \times \{1\}$, and we call this space S^r a *solid torus with upper meridian r* . Notice that the “standard” solid torus $S^1 \times D^2$ is S_∞ .

We denote the set of isotopy classes of tight contact structures on a solid torus S_r with convex boundary having two dividing curves of slope s by $\text{Tight}(S_r; s)$.

Theorem 2.1 (Giroux [30], Honda [35]). *The set $\text{Tight}(S^r; s)$ is in one-to-one correspondence with the set of minimal paths with all but the first edge decorated from r clockwise to s , up to shuffling in continued fraction blocks.*

From the discussion above it is easy to see that if $-q/p < -1$ and $-q/p = [a_1, \dots, a_n]$ then

$$|\text{Tight}(S^0; s)| = |(a_1 + 1) \cdots (a_{n-1} + 1)a_n|.$$

Lastly, we show that $\Phi_s(r)$ from Equation (3) counts $|\text{Tight}(S_r; s)|$.

Corollary 2.2. *The function $\Phi_s(r)$ counts the number of tight contact structures on the solid torus with lower meridian slope r and convex boundary having two dividing curves of slope s . That is $\Phi_s(r) = |\text{Tight}(S_r; s)|$*

Proof. We first note that $|\text{Tight}(S_\infty; s)| = |\text{Tight}(S^0; -1/s)|$ as there exists a diffeomorphism of T^2 with matrix representation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that induces a diffeomorphism from the solid torus taking S_∞ to the solid torus S^0 , and sending s to $-1/s$. Thus, $\Phi_\infty(r)$ from Equation (2) counts $|\text{Tight}(S_\infty; s)|$ by Theorem 2.1.

Now let $s = a/b$ and $s^c = c/d$. We consider a diffeomorphism on a torus with matrix representation

$$A = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}.$$

Then A sends r to $\frac{s^c \cdot r}{s \cdot r}$ and s to ∞ . Thus we have

$$|\text{Tight}(S_r; s)| = \left| \text{Tight}\left(S_\infty; \frac{s^c \cdot r}{s \cdot r}\right) \right|$$

If $r = s$, then clearly there is no tight contact structure on S_r . This completes the proof. $\square$

Next, we review convex torsion, which is a variant of Giroux torsion with convex boundary.

Definition 2.3. For $n \in \frac{1}{2}\mathbb{N}$ and a small $\epsilon > 0$, consider the following contact manifold:

$$(T^2 \times [-\epsilon, 1 + \epsilon], \ker(\cos(2\pi n z)dx + \sin(2\pi n z)dy)).$$

Perturb $T^2 \times \{0\}$ and $T^2 \times \{1\}$, fixing a single leaf of their characteristic foliations, to make them convex; denote the resulting convex tori by T_0 and T_1 , respectively. The contact manifold bounded by T_0 and T_1 is called a *convex n -torsion layer*. We say a contact manifold (Y, ξ) *contains convex torsion* (resp. *half convex torsion*) if it contains an embedded convex n -torsion layer for some $n \geq 1$ (resp. $n \geq \frac{1}{2}$).

In this paper, we only consider Legendrian knots for which the complement of a standard neighborhood does not contain boundary-parallel half convex torsion, due to the following simple observation.

Proposition 2.4. *Let L be a Legendrian knot with boundary-parallel half convex torsion in its complement. Then Legendrian surgery on L yields an overtwisted contact structure.*

2.2. The classification of Legendrian torus knots. All the statements below that are not proven here can be found in [16, Section 6, 7]. In general, we do not need all the details about the classification of Legendrian torus knots, but we outline the main results we need for our purposes below so as to keep the paper largely self-contained.

2.2.1. Negative torus knots. We begin by considering negative torus knots $T_{p,q}$. Recall, we will always assume that $-q > p > 0$. The classification is based on paths in the Farey graph. Specifically, let P_1 be the minimal path in the Farey graph from q/p anticlockwise to ∞ and let P_2 be the minimal path from q/p clockwise to 0. Notice that decorating all the edges of P_1 except the last one will determine a tight contact structure on the solid torus S_∞ , and similarly, such a decoration on P_2 will determine a contact structure on S^0 . We call (P_1, P_2) a pair of decorated paths for q/p . Gluing these two solid tori together along their common boundary will result in a contact structure ξ_{P_1, P_2} on S^3 . If the concatenated path can be shortened where there is a mismatch of signs on the edges, then the contact structure will be overtwisted. A Legendrian divide L_{P_1, P_2} on the common torus $S_\infty \cap S^0$ is a Legendrian knot in the knot type of $T_{p,q}$ with $\text{tb} = pq$ and tight complement without half-convex torsion. We will use P_1 and P_2 to denote both the decorated and undecorated paths, as the meaning should be clear from context. The contact structures ξ_{P_1, P_2} and $\xi_{-P_1, -P_2}$ are the same, but the corresponding Legendrian divides L_{P_1, P_2} and $L_{-P_1, -P_2}$ are distinct Legendrian knots in this fixed contact structure [16, Theorem 7.16].

In the classification, we are particularly interested in a subset of P_2 . Specifically, let $P_2^\top$ denote the subpath of P_2 from q/p to $\lceil q/p \rceil$. We note ξ_{P_1, P_2} will be tight if and only if the signs of the decorations in P_1 and $P_2^\top$ are the same. In [16] it was shown that the first continued fraction block in either P_1 or P_2 will have length one (this means there is one edge) and if q/p is not $n/2$ for some $n \in \mathbb{Z}$, then only one of the P_i will have a length

one first continued fraction block. Call this continued fraction block B_1 (if $q/p = n/2$ then choose either of the first continued fraction blocks as B_1), then let B_2 be the first continued fraction block in the other path. We then inductively label all the continued fraction blocks of P_1 and $P_2^\top$ as $B_1, \dots, B_k$ by alternating between P_1 and $P_2^\top$, while moving outward from q/p .

We now call a decorated pair of paths (P_1, P_2) *i-inconsistent* if all of the edges in the continued fraction blocks $B_1, \dots, B_{i-1}$ have the same sign, but at least one of the edges in B_i has the opposite sign. In [16, Section 2.5] it was shown that to an *i-inconsistent* pair of paths there is a unique $(i-1)$ -inconsistent pair of paths (P_1^{i-1}, P_2^{i-1}) associated to it and we say these pairs of paths are *compatible*. We can continue and find $(P_1^{i-2}, P_2^{i-2}), \dots, (P_1^2, P_2^2)$ that are all compatible. It was also shown that compatible pairs of paths determine the same contact structure, but they give different Legendrian knots with $\text{tb} = pq$.

Understanding these knots is the key to the classification of non-loose torus knots without half convex torsion. Specifically, the classification is as follows.

- (1) If (P_1, P_2) is 2-inconsistent, then L_{P_1, P_2} destabilizes to infinity [16, Lemma 6.11]. This means that for any $i \in \mathbb{Z}$, there are Legendrian knots L_i such that either $L_{i-1} = S_+(L_i)$ for all i (resp. $L_{i-1} = S_-(L_i)$ for all i), and $L_j = L_{P_1, P_2}$ for some j . In particular, there is a tight contact structure ξ'_{P_1, P_2} on the complement C of a neighborhood of $T_{p, q}$ in which ∂C is convex with dividing slope ∞ and the complement of standard neighborhood of L_i is obtained by attaching a positive (resp. negative) basic slice to (C, ξ'_{P_1, P_2}) . Moreover, if L_{P_1, P_2} is in the family of Legendrian knots related by positive (resp. negative) stabilization, then $L_{-P_1, -P_2}$ is in a family of Legendrian knots related by negative (resp. positive) stabilization. Thus the pair $(\pm P_1, \pm P_2)$ contribute an infinite X to the mountain range for non-loose Legendrian knots in ξ_{P_1, P_2} . See the left-hand diagram in Figure 5. We also note that for any contact structure ξ'_{P_1, P_2} on C as above, any convex torus parallel to the boundary of C will have dividing slope ∞ , meaning that it is non-thickenable.

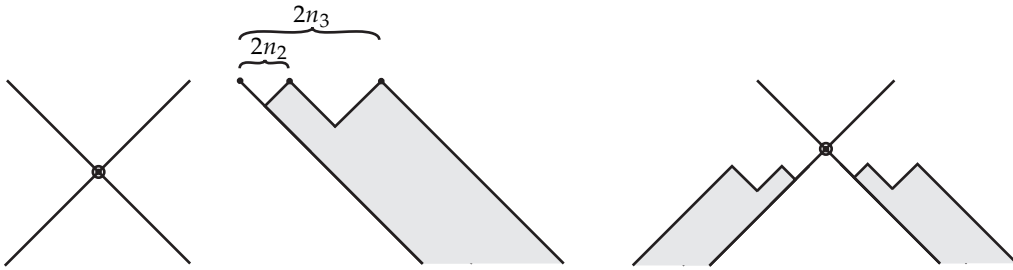


FIGURE 5. On the left we see an infinite X generated by a pair of 2-incompatible pairs of paths. In the middle, we see a wing generated by compatible pairs of decorated paths. The gaps between the peaks in the wings are shown. On the right, we see the complete mountain range for non-loose negative torus knots in some overtwisted contact structure.

- (2) There are $2n(p, q)$ of 2-inconsistent pairs of decorated paths for q/p by [16, Lemma 6.8] (see Formula (1) in Section 1.2 for the definitions of $n(p, q)$ and $m(p, q)$).
- (3) There are $m(p, q)$ decorated paths for q/p . Thus, there are exactly $m(p, q)$ Legendrian realizations of $T_{p,q}$ with $\text{tb} = pq$ and having a standard neighborhood with tight complement without half-convex torsion. More specifically, $2\lceil q/p \rceil$ of the representatives are in (S^3, ξ_{std}) , and the rest are non-loose representatives in overtwisted contact structures. See [16, Lemma 6.9 and Theorem 1.5].
- (4) Let (P_1^j, P_2^j) , for $j = 2, \dots, l$ be a maximal set of compatible paths such that (P_1^j, P_2^j) is j -inconsistent and (P_1^l, P_2^l) is not compatible with any $(l+1)$ -inconsistent pair of decorated paths. Denote by L_j the Legendrian knot $L_{P_1^j, P_2^j}$ with $\text{tb} = pq$. The difference between the rotation number of L_2 and L_j is $2n_{j-1} = 2|s_j \cdot q/p|$, where s_j is the label on the vertex in the continued fraction block B_j that is farthest from q/p . See the center of Figure 5.

From Item (1) above, we know L_2 destabilizes to infinity. We assume that it does so through positive stabilizations (the other case is analogous to this one). Then $S_+^k(L_2)$ is tight for all k and $S_-(L_2)$ is loose. Moreover, L_j for $j > 2$ does not destabilize. Any stabilization of the L_j will remain non-loose as long as their Thurston-Bennequin invariant and rotation number are not below the line in the mountain range formed by the stabilizations of L_2 ; otherwise, it becomes loose. The collection of non-loose knots formed by L_j for $j > 2$ and their stabilizations is Legendrian simple, meaning that they are determined by their Thurston-Bennequin invariants and rotation numbers. The image of this set in the mountain range is shown in the center of Figure 5. This set is called a *wing*. See [16, Section 7.2].

The mountain range of non-loose $T_{p,q}$ torus knots in $\xi_{P_1^j, P_2^j}$ is shown on the right diagram in Figure 5 (recall that this contact structure is independent of j). The only exception is when P_1^2 has all decorations of one sign while P_2 has decorations with the opposite sign, called a *totally inconsistent* pair of paths. In that case, the mountain range is an infinite X together with one extra Legendrian knot L_e with $\text{tb} = |pq| - |p| - |q|$ and $\text{rot} = 0$ that comes from the family of non-thickenable tori discussed in Section 2.3.

We now count the number of non-loose Legendrian realizations of $T_{p,q}$ with a fixed Thurston-Bennequin invariant. From the description above, if $n > pq$, then any Legendrian representative with $\text{tb} = n$ must be on one of the infinite Xs, or the extra Legendrian L_e if $\text{tb} = |pq| - |p| - |q|$. There are $n(p, q)$ of these Xs, so the number of Legendrian knots with $\text{tb} = n$ without half convex torsion in their complement is

$$(6) \quad \mathcal{L}_n(T_{p,q}) = \begin{cases} 2n(p, q) & \text{if } n \neq |pq| - |p| - |q| \\ 2n(p, q) + 1 & \text{if } n = |pq| - |p| - |q|. \end{cases}$$

and their complements are described by adding a basic slice to one of $n(p, q)$ tight contact structures on C with convex boundary having dividing slope ∞ . We also know

$$(7) \quad \mathcal{L}_{pq}(T_{p,q}) = m(p, q).$$

Determining $\mathcal{L}_n(T_{p,q})$ for $n < pq$ is more difficult. From Item (4) above, we know that any non-loose Legendrian representative of $T_{p,q}$ with $\text{tb} < pq$ is obtained from a stabilization of one of the L_{P_1, P_2} for some pair of decorated paths (P_1, P_2) for q/p . Suppose that (P_1, P_2) is i -inconsistent for $i \geq 3$. Assume that all positive (resp. negative) stabilizations of L_{P_1, P_2} remain non-loose. According to Item (4) (see also Figure 5), we know that there is a compatible $(i-1)$ -inconsistent pair of decorated paths (P'_1, P'_2) such that $S_+^{n_{i-1}-n_{i-2}}(L_{P'_1, P'_2}) = S_-^{n_{i-1}-n_{i-2}}(L_{P_1, P_2})$. So any further stabilizations of L_{P_1, P_2} could be counted among the Legendrian knots coming from $L_{P'_1, P'_2}$. Thus, the number of Legendrian knots with $\text{tb} = n$ in the wing spanned by L_{P_1, P_2} is $1 + (pq - n)$ if $pq - n < n_{i-1} - n_{i-2}$, and $n_{i-1} - n_{i-2}$ if $pq - n \geq n_{i-1} - n_{i-2}$. So we define the following function

$$W_i(n) = \begin{cases} 0 & \text{if } n > pq \\ 1 + (pq - n) & \text{if } pq - n < n_{i-1} - n_{i-2} \\ n_{i-1} - n_{i-2} & \text{if } pq - n \geq n_{i-1} - n_{i-2}. \end{cases}$$

that counts the Legendrian knots with $\text{tb} = n$ in the wing spanned by an i -inconsistent pair of paths for $i > 3$. If $i = 3$, we set $n_1 = 0$ and obtain

$$W_3(n) = \begin{cases} 0 & \text{if } n > pq \\ 1 + (pq - n) & \text{if } pq - n < n_2 \\ n_2 & \text{if } pq - n \geq n_2. \end{cases}$$

We also need

$$W_2(n) = 1 \quad \text{for } n \in \mathbb{Z},$$

that counts the Legendrian knots in one part of an X , spanned by 2-inconsistent paths.

We now determine how many i -inconsistent pairs of decorated paths there are. Suppose (P_1, P_2) is a pair of paths for q/p and $B_1, \dots, B_k$ are the continued fraction blocks in $P_1 \cup P_2^\top$ ordered as discussed in the beginning of the section. Let m_i be the number of edges in B_i . The number of ways to decorate the paths in $P_2 - P_2^\top$ is $\lceil q/p \rceil$. Recall that in a k -inconsistent pair of paths, all edges in the continued fraction blocks $B_1, \dots, B_{k-1}$ have the same sign, and B_k contains at least one edge with opposite sign. Thus, we see there are

$$2 \cdot \lceil q/p \rceil \cdot m_k$$

k -inconsistent pairs of decorated paths, as we can choose either of two uniform signs for the edges in $B_1, \dots, B_{k-1}$ and one of $\lceil q/p \rceil$ sign configurations for the edges in $P_2 - P_2^\top$. For B_k , there are $(m_k + 1)$ sign assignments since there are m_k edges (we can shuffle the edges in a continued fraction block), but as we fix at least one sign to be opposite to those in $B_1, \dots, B_{k-1}$, we only have m_k options.

Generally, there are

$$2\lceil q/p \rceil m_i \prod_{j=i+1}^k (m_j + 1)$$

i -inconsistent pairs of decorated paths, as there are $m_j + 1$ sign assignments for the edges in B_j for $j > i$.

We are finally ready to determine the number of Legendrian knots with $\text{tb} < pq$ having tight complements without half convex torsion. Adding all the contributions from all the Legendrian knots L_{P_1, P_2} we see that the total number of non-loose knots in some overtwisted contact structure with $\text{tb} = n$ and no convex half torsion is

$$(8) \quad \mathcal{L}_n^X(T_{p,q}) = 2\lceil q/p \rceil \sum_{i=2}^k \left(W_i(n) \left(m_i \prod_{j=i+1}^k (m_j + 1) \right) \right).$$

We use the notation $\mathcal{L}^X$ as they are all in X , possibly with compatible wings. We can also see that the maximum value of $\mathcal{L}_n^X$ is

$$(9) \quad M(p, q) = 2\lceil q/p \rceil \left(m_2 \prod_{j=3}^k (m_j + 1) + n_2 m_3 \prod_{j=4}^k (m_j + 1) \right. \\ \left. + \sum_{i=4}^k \left((n_{i-1} - n_{i-2}) m_i \prod_{j=i+1}^k (m_j + 1) \right) \right).$$

To see the total number $\mathcal{L}_n(T_{p,q})$ of Legendrian representatives of $T_{p,q}$ with complements having no half convex torsion, we should also consider Legendrian representatives in the tight contact structure on S^3 . These were classified in [11], where it was shown that when $\text{tb} = pq$ there are $2\lceil q/p \rceil$ Legendrian representatives. Moreover, the gaps between the peaks are either $2\lceil q/p \rceil \cdot \lceil q/p \rceil$ or $2\lceil q/p \rceil \cdot \lfloor q/p \rfloor$, and the gaps alternate (recall that $\frac{a}{b} \cdot \frac{c}{d} = ad - bc$). Specifically, the rotation number of the leftmost peak is $-|p + q|$, and the next peak is obtained by adding $2\lceil q/p \rceil \cdot \lceil q/p \rceil$, the following peak is then obtained by adding $2\lceil q/p \rceil \cdot \lfloor q/p \rfloor$, and so on until we reach $|p + q|$. Notice that this means that going from $-|p + q|$ to $|p + q|$ we will be adding $2\lceil q/p \rceil \cdot \lceil q/p \rceil$ exactly $\lceil q/p \rceil$ times and $2\lceil q/p \rceil \cdot \lfloor q/p \rfloor$ exactly $\lceil q/p \rceil - 1$ times. So for $n \leq pq$ the number of Legendrian representatives of $T_{p,q}$ with $\text{tb} = n$ in (S^3, ξ_{std}) is

$$(10) \quad \mathcal{L}_n^T(T_{p,q}) = (|p + q| + pq + 1 - n) - N(\lceil q/p \rceil \cdot (\lceil q/p \rceil \cdot q/p - pq + n - 1)) \\ - N((\lceil q/p \rceil - 1) \cdot (\lfloor q/p \rfloor \cdot q/p - pq + n - 1))$$

where $N(x)$ is x if $x \geq 0$ and 0 otherwise. We use the notation $\mathcal{L}^T$, as they are in tight S^3 . The first term $(|p + q| + pq + 1 - n)$ counts the number of vertices (whose coordinates add to an odd number, so they are possibly in the mountain range) in the mountain range with $\text{tb} = n$ between the extreme rotation numbers $\pm(|p + q| + pq - n)$ realized by Legendrian knots. The other two terms in the formula account for the gaps.

We can finally compute the total number of Legendrian representatives of $T_{p,q}$ whose standard neighborhoods have no half convex torsion in their complement.

$$(11) \quad \mathcal{L}_n(T_{p,q}) = \mathcal{L}_n^X(T_{p,q}) + \mathcal{L}_n^T(T_{p,q}).$$

To compute these formulas, we need to know the numbers n_i and m_i associated with the paths (P_1, P_2) for q/p . These numbers are easily computed from the paths, but can also be computed from the continued fraction

$$q/p = [a_1, \dots, a_n].$$

Now set $i_1 = 1$ and let $i_2, \dots, i_{l-1}$ be the indices of the a_i that are not equal to -2 , and let $i_l = n$. The endpoints of the continued fraction blocks in $P_2^\top$ are

$$v_1 = [a_1, \dots, a_{i_{l-1}} + 1], v_2 = [a_1, \dots, a_{i_{l-2}} + 1], \dots, v_{l-1} = [a_{i_1} + 1]$$

and the number of edges in the continued fraction blocks is

$$|a_{i_l} + 1|, |a_{i_{l-1}} + 2|, \dots, |a_{i_2} + 2|.$$

As for the path P_1 , if $a_{i_l} \neq -2$, the endpoints of the continued fraction blocks in P_1 are

$$w_1 = [a_1, \dots, a_{i_{l-1}}], w_2 = [a_1, \dots, a_{i_{l-1}-1}], \dots, w_{l-1} = [a_1, \dots, a_{i_2-1}], w_l = [a_{i_1}],$$

and if $a_{i_l} = -2$ then we get the same end points except $[a_1, \dots, a_{i_{l-1}}]$ will be omitted.

Similarly, if $a_{i_l} \neq -2$, the number of edges in the continued fraction blocks is

$$1, i_l - i_{l-1} + 1, \dots, i_3 - i_2 + 1$$

and if $a_{i_l} = -2$, then we get the same lengths, except the first 1 is omitted. Now, if $a_{i_l} = -2$, then the labeling of the continued fraction blocks $B_1, \dots, B_k$ above starts with the first continued fraction block in P_2 ; otherwise, it starts with the first continued fraction block in P_1 . So if $a_{i_l} = -2$, then $s_{2j-1} = v_j$ and $s_{2j} = w_j$, giving $n_j = |s_j \cdot q/p|$. Similarly, the lengths m_j of the continued fraction blocks will alternate. If $a_{i_l} \neq -2$, then $s_{2j-1} = w_j$ and $s_{2j} = v_j$, and the n_j are defined in the same way. The m_j will again alternate, but starting with the lengths in P_1 .

2.2.2. Positive torus knots. We now consider positive torus knots $T_{p,q}$. Recall that we always assume that $q > p > 0$. As in the case of negative torus knots, we can construct a Legendrian knot L_{P_1, P_2} with $\text{tb} = pq$ associated with a pair of decorated paths in the Farey graph in the same way. Just as in the case of negative torus knots, for an i -inconsistent pair of paths ($i \geq 3$), there is a unique compatible $(i-1)$ -inconsistent pair of paths (P_1^{i-1}, P_2^{i-1}) . While compatible pairs of paths determine the same contact structure, they all yield different Legendrian knots with $\text{tb} = pq$ in this fixed contact structure.

Unlike the case of negative torus knots, here we consider the full path P_2 instead of the truncated path $P_2^\top$. We begin with the length-1 continued fraction block B_1 starting from q/p , then let B_2 be the first continued fraction block in the other path. We then inductively label all the continued fraction blocks of P_1 and P_2 as $B_1, \dots, B_k$ by alternating between P_1 and P_2 , while moving outward from q/p . Moreover, there is one more type

of “compatibility” between paths. Let $(\widehat{P}_1, \widehat{P}_2)$ be the decorated paths for q/p with all edges labeled with the same sign, say ‘+’ (the same discussion holds for the negative case). We now consider the pair $(\widehat{P}_1^k, \widehat{P}_2^k)$, where all the signs in $\widehat{P}_1^k$ are ‘-’ and all the signs in $\widehat{P}_2^k$ are also ‘-’ except for the edge from $\lceil q/p \rceil$ to ∞ , which is ‘+’. This is a maximally inconsistent pair of paths, and we say it is compatible with the original pair $(\widehat{P}_1, \widehat{P}_2)$. As discussed above, there are pairs $(\widehat{P}_1^i, \widehat{P}_2^i)$ being i -inconsistent for $i = 2, \dots, k-1$, which are compatible with $(\widehat{P}_1^k, \widehat{P}_2^k)$, and hence also compatible with $(\widehat{P}_1, \widehat{P}_2)$.

Now the classification of non-loose positive torus knots without half convex torsion is as follows.

- (1) As in the case of negative torus knots, for a pair of decorated 2-inconsistent paths (P_1, P_2) , there is a contact structure ξ'_{P_1, P_2} on the complement C of a neighborhood of $T_{p,q}$ with convex boundary having two dividing curves of slope ∞ and that any convex torus parallel to ∂C will have dividing slope ∞ . This is true for positive torus knots, except there are two contact structures on C with two dividing curves of slope ∞ that come from (S^3, ξ_{std}) . Specifically, If ξ' is the contact structure on C , which is the complement of the maximal Thurston-Bennequin representative of $T_{p,q}$ in (S^3, ξ_{std}) , then one can add either a positive or a negative basic slice with dividing slopes $pq - p - q$ and ∞ to (C, ξ') to get these contact structures, which we denote ξ_+ and ξ_- . We may further add basic slices to $\xi_{\pm}$ to obtain non-loose Legendrian knots as follows: any positive basic slice added to ξ_+ with dividing slopes ∞ and n will result in a non-loose knot with $tb = n$, similarly, for negative basic slices added to ξ_- ; however, when $n \leq pq - p - q$ the non-loose knot will have half convex torsion in its complement. If we add negative (resp. positive) basic slices b_- (resp. b_+) with slope ∞ and n to ξ_+ (resp. ξ_-), then this will result in a non-loose knot if $n > pq - p - q + 2$. If $n = pq - p - q + 2$, then $b_- \cup \xi_+$ become isotopic to $b_+ \cup \xi_-$ as they are in a continued fraction block and the signs can be shuffled. Thus this will result in a single non-loose Legendrian knot. Finally, if $n < pq - p - q + 2$, then $b_{\mp} \cup \xi_{\pm}$ become overtwisted as they form a path in the Farey graph that can be shortened with opposite signs. Thus, when $n < pq - p - q + 2$, we obtain loose knots. We note that these non-loose knots with complements $(b_{\mp} \cup \xi_{\pm})$ correspond to the pair of paths $(\pm \widehat{P}_1^2, \pm \widehat{P}_2^2)$ discussed above. For more details on this discussion, see [16, Section 7.2.2] and Item (5) below.

As in the case of negative torus knots, any 2-inconsistent pair (P_1, P_2) contributes an infinite X to the mountain range for non-loose Legendrian knots in ξ_{P_1, P_2} as long as $(P_1, P_2) \neq \pm(\widehat{P}_1^2, \widehat{P}_2^2)$. The case of $\pm(\widehat{P}_1^2, \widehat{P}_2^2)$ will be discussed in Item (5) below.

- (2) There are $2n(p, q)$ of 2-inconsistent pairs of decorated paths for q/p by [16, Lemma 6.8] (See Formula (1) in Section 1.2 for the definitions of $n(p, q)$ and $m(p, q)$.)
- (3) There are $m(p, q)$ of decorated paths for q/p . Thus, there are exactly $m(p, q)$ Legendrian realizations of $T_{p,q}$ with $tb = pq$ and a standard neighborhood whose

complement does not contain half convex torsion. See [16, Lemma 6.9 and Theorem 1.5].

- (4) Let (P_1^j, P_2^j) , for $j = 2, \dots, l$ be a maximal set of compatible paths such that (P_1^j, P_2^j) is j -consistent and (P_1^l, P_2^l) is not compatible with any $(l+1)$ -consistent pair of decorated paths. We also assume that $(P_1^2, P_2^2) \neq \pm(\widehat{P}_1^2, \widehat{P}_2^2)$. Denote by L_j the Legendrian knots $L_{P_1^j, P_2^j}$ with $\text{tb} = pq$. The difference between the rotation number of L_2 and L_j is $2n_{j-1} = 2|s_j \cdot q/p|$, where s_j is the label on the vertex in the continued fraction block B_j that is farthest from q/p . (Recall we have listed the continued fraction blocks B_j in P_1 and P_2 .)

From Item (1) above, we know L_2 destabilizes to infinity. We assume that it does so through positive stabilizations (the negative case is analogous to this one). Then $S_+^k(L_2)$ is non-loose for all $k \in \mathbb{N}$. We also know that $S_-(L_2)$ is loose. The L_j for $j > 2$ do not destabilize. Any stabilization of the L_j will remain non-loose as long as their Thurston-Bennequin invariant and rotation number are not below the line in the mountain range formed by the stabilizations of L_2 ; otherwise, it becomes loose. The collection of non-loose knots formed by the L_j and their stabilizations is Legendrian simple (i.e., they are determined by their Thurston-Bennequin invariants and rotation numbers). The image of this set in the mountain range is shown in the middle of Figure 5. This set is called a *wing*. See [16, Section 7.2].

The mountain range of non-loose $T_{p,q}$ torus knots in $\xi_{P_1^j, P_2^j}$ is shown in the left drawing of Figure 6. Notice that the two different wings overlap, but Legendrian knots in one wing are distinct from Legendrian knots in the other wing, even if their classical invariants are the same.

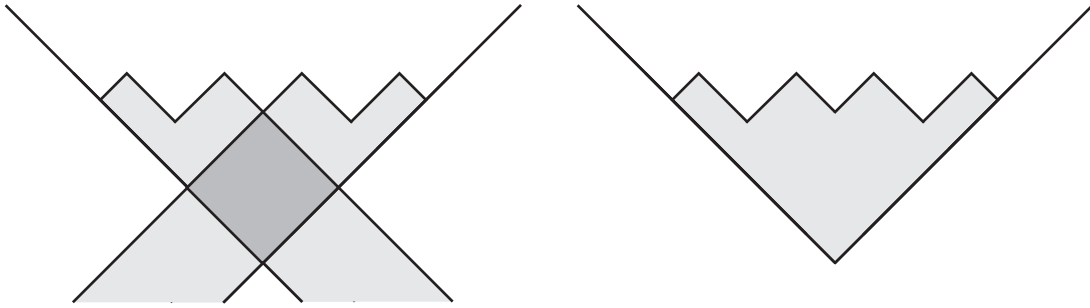


FIGURE 6. On the left, we see the mountain range for non-loose positive torus knots. Note the wings overlap in this case and in the darker shaded region, there are two Legendrian knots for each lattice point in the image (with coordinates that sum to an odd number). On the right, we see the mountain range for non-loose torus knots in ξ_1 that are discussed in Item (5).

- (5) We are left to consider the decorated paths $(\widehat{P}_1^2, \widehat{P}_2^2)$. We note that $\xi_{\widehat{P}_1, \widehat{P}_2} = \xi_1$ (i.e., $d_3(\xi_{\widehat{P}_1, \widehat{P}_2}) = 1$). According to Item (1), the Legendrian knots $L_2^\pm = L_{\pm \widehat{P}_1^2, \pm \widehat{P}_2^2}$ destabilize to infinity, but after L_2^+ is positively stabilized $p + q - 2$ times and L_2^- is negatively stabilized $p + q - 2$ times, they become the equivalent non-loose Legendrian knot, and any further stabilizations result in loose knots. We say $L_2^\pm$ generate an infinite V . See the right of Figure 6. The pairs of paths $\pm(\widehat{P}_1^j, \widehat{P}_2^j)$ for $3 \leq j \leq k$, compatible with $(\widehat{P}_1, \widehat{P}_2)$ also give Legendrian knots $L_j^\pm$ with $\text{tb} = pq$. The difference between the rotation numbers of $L_j^\pm$ and $L_2^\pm$ is $2n_{j-1} = 2|s_j \cdot q/p|$. Stabilizations of the $L_j^\pm$ will remain non-loose as long as their invariants are on or above the infinite V . Moreover, the non-loose Legendrian knots formed from stabilizing the $L_j^\pm$ are Legendrian simple; that is, if two of these Legendrian knots have the same classical invariants, then they are equivalent. See the right-hand side of Figure 6.
- (6) The discussion in Item (1) implies that there is one infinite X that comes from $b_\pm \cup \xi_\pm$ such that Legendrian representatives in it with $\text{tb} \leq pq - p - q$ contain half convex torsion in their complements.

Now, we count the number of Legendrian representatives of $T_{p,q}$ with a fixed Thurston-Bennequin invariant and no half convex torsion in their complements. From the description above, any Legendrian representative with $\text{tb} > pq$ must be on one of the infinite X s or the infinite V . There are $n(p, q) - 1$ of these X s and one V , so the number of non-loose Legendrian knots with $\text{tb} = n > pq$ is

$$(12) \quad \mathcal{L}_n(T_{p,q}) = 2n(p, q).$$

Their complements are described by adding a basic slice to one of $n(p, q)$ tight contact structures on C with convex boundary having dividing slope ∞ . We also know

$$(13) \quad \mathcal{L}_{pq}(T_{p,q}) = m(p, q),$$

as there are $m(p, q)$ of decorated pair of paths for q/p .

Determining $\mathcal{L}_n(T_{p,q})$ for $n < pq$ is more difficult. We will compute $\mathcal{L}_n(T_{p,q})$ in three steps: first, we count the non-loose Legendrian knots in an infinite X (possibly with the attached wings), then count the ones in an infinite V (with wings inside), and finally count the Legendrian knots in tight S^3 .

The computation of the number of Legendrian knots in an infinite X is almost identical to the count for negative torus knots. The exceptions come from one X discussed in Item (6). We note that the mountain range for this infinite X does not contain any wings.

From above, we know that any non-loose Legendrian representative of $T_{p,q}$ with $\text{tb} < pq$ is obtained from a stabilization of one of the L_{P_1, P_2} for some pair of decorated paths (P_1, P_2) for q/p . Suppose that (P_1, P_2) is i -inconsistent for $i \geq 3$. Assume that all positive stabilizations of L_{P_1, P_2} remain non-loose. (Recall either this is true or all negative stabilizations remain non-loose. The argument for the negative case is identical to the

positive case.) According to Item (4), there is a unique compatible $(i-1)$ -inconsistent pair of decorated paths (P'_1, P'_2) such that $S_+^{n_{i-1}-n_{i-2}}(L_{P'_1, P'_2}) = S_-^{n_{i-1}-n_{i-2}}(L_{P_1, P_2})$. Hence any further stabilizations of L_{P_1, P_2} could be counted among the Legendrian knots coming from $L_{P'_1, P'_2}$. Thus, the number of Legendrian knots with $\text{tb} = n$ in the wing spanned by L_{P_1, P_2} is $1 + (pq - n)$ if $pq - n < n_{i-1} - n_{i-2}$ and $n_{i-1} - n_{i-2}$ Legendrian knots if $pq - n \geq n_{i-1} - n_{i-2}$. So we define the following function

$$W_i(n) = \begin{cases} 0 & \text{if } n > pq \\ 1 + (pq - n) & \text{if } pq - n < n_{i-1} - n_{i-2} \\ n_{i-1} - n_{i-2} & \text{if } pq - n \geq n_{i-1} - n_{i-2}. \end{cases}$$

that counts the Legendrian knots with $\text{tb} = n$ in the wing spanned by an i -inconsistent pair of paths for $i > 3$. If $i = 3$, we set $n_1 = 0$ and obtain

$$W_3(n) = \begin{cases} 0 & \text{if } n > pq \\ 1 + (pq - n) & \text{if } pq - n < n_2 \\ n_2 & \text{if } pq - n \geq n_2. \end{cases}$$

We also need

$$W_2(n) = 1 \quad \text{for } n \in \mathbb{Z},$$

that counts the Legendrian knots in one part of X , spanned by 2-inconsistent paths.

We now determine how many i -inconsistent pairs of paths there are. Let (P_1, P_2) be the undecorated pair of paths for q/p , and $B_1, \dots, B_k$ the continued fraction blocks in $P_1 \cup P_2$ ordered as discussed in the beginning of the section. Let m_i be the number of edges in B_i . Recall that in a k -inconsistent pair of paths, all edges in the continued fraction blocks $B_1, \dots, B_{k-1}$ have the same sign, and B_k contains at least one edge with opposite sign. Thus, we see there are $2m_k$ of k -consistent pairs of paths. Similarly, we see that there are $2m_{k-1}(m_k + 1)$ pairs of paths that are $(k-1)$ -consistent; and, in general, there are

$$2m_i \prod_{j=i+1}^k (m_j + 1)$$

pairs of paths that are i -inconsistent.

We are finally ready to determine the number of Legendrian knots with $\text{tb} < pq$ and no half convex torsion in their complements. Adding all the contributions from all the Legendrian knots L_{P_1, P_2} we see that the total number of non-loose knots in X is as follows. If $n > pq - p - q$, then

$$(14) \quad \mathcal{L}_n^X(T_{pq}) = \sum_{i=2}^k W_i(n) \left(2m_i \prod_{j=i+1}^k (m_j + 1) - 2 \right)$$

If $n \leq pq - p - q$, then

$$(15) \quad \mathcal{L}_n^X(T_{pq}) = \left(2m_2 \prod_{j=3}^k (m_j + 1) - 4 \right) + \sum_{i=2}^k W_i(n) \left(2m_i \prod_{j=i+1}^k (m_j + 1) - 2 \right).$$

We note the term -2 in the first equation is because we exclude two of the i -inconsistent pairs of paths that are in the mountain range for the infinite V and will be counted below. The term -4 in the second equation is due to the fact that one of our infinite X consists of Legendrian knots with half convex torsion in their complement if $tb \leq pq - p - q$.

The maximum value of this function among $n \in \mathbb{Z}$ is

$$(16) \quad \begin{aligned} M(p, q) = & \left(2m_2 \prod_{j=3}^k (m_j + 1) - 4 \right) + n_2 \left(2m_3 \prod_{j=4}^k (m_j + 1) - 2 \right) \\ & + \sum_{i=4}^k (n_{i-1} - n_{i-2}) \left(2m_i \prod_{j=i+1}^k (m_j + 1) - 2 \right). \end{aligned}$$

We now turn to the Legendrian knots in the infinite V discussed in Item (5). The extreme rotation numbers when $tb = pq$ are $\pm(p + q - 2)$, and the gaps between the adjacent representatives with $tb = pq$ are $2n_2, 2(n_{i-1} - n_{i-2})$ for $4 \leq i \leq k$. There are two of these gaps as the left part and the right part of V are symmetric. Also, the middle gap is $2(p + q - 2 - 2n_{k-1})$. Thus the number of Legendrian representatives in V with compatible wings for $tb = n < pq$ is

$$(17) \quad \begin{aligned} \mathcal{L}_n^V(T_{p,q}) = & 2N \left((p + q - 1) - (pq - n) - 2N(n_2 - 1 - pq + n) \right. \\ & - 2 \sum_{i=4}^k N((n_{i-1} - n_{i-2}) - 1 - pq + n) \\ & \left. - N((p + q - 2 - 2n_{k-1}) - 1 - pq + n) \right), \end{aligned}$$

where $N(x)$ is x if $x \geq 0$ and equal to 0 if $x \leq 0$. Here, the term $(p + q - 1) - (pq - n)$ is the number of vertices between two extreme representatives on V with $tb = n$ (we only consider potential representatives, so count vertices with $tb + \text{rot} = \text{odd}$). The other terms account for the gaps between the Legendrian representatives in V with $tb = n$.

We note that $\mathcal{L}_n^V(T_{p,q})$ does not contribute to the number of non-loose Legendrian knots when $n < pq - p - q + 2$, so $M(p, q)$ is the total count of non-loose Legendrian with no half convex torsion in their complement for n sufficiently negative.

Finally, in (S^3, ξ_{std}) , there is a unique Legendrian $T_{p,q}$ with $tb = pq - p - q$ and all other Legendrian knots are a stabilization of this one [11]. Thus the number of Legendrian representatives of $T_{p,q}$ in the tight contact structure on S^3 with $tb = n \leq pq - p - q$ is

$$(18) \quad \mathcal{L}_n^T(T_{p,q}) = 1 + (pq - p - q - n).$$

We can finally compute the total number of Legendrian representatives of $T_{p,q}$ whose standard neighborhoods have no half convex torsion in their complement.

$$(19) \quad \mathcal{L}_n(T_{p,q}) = \mathcal{L}_n^X(T_{p,q}) + \mathcal{L}_n^V(T_{p,q}) + \mathcal{L}_n^T(T_{p,q}).$$

To compute these formulas we need to know the numbers n_i and m_i associated with the (undecorated) paths (P_1, P_2) for q/p . These numbers are easily computed from the paths, but can also be computed from the continued fraction

$$q/p = [a_1, \dots, a_n].$$

Now that since $q/p > 1$ all the $a_i \geq 2$. In Section 2.1 we discussed how to create paths in the Farey graph when the $a_i \leq -2$, but we similarly see that for positive a_i , the successive vertices moving anti-clockwise are obtained by subtracting 1 from the last element: $[a_1, \dots, a_{n-1}, a_n + 1]$. And the successive vertices moving clockwise are obtained by dropping the last element: $[a_1, \dots, a_{n-1}]$. We note that in this case we will need to consider the empty continued fraction $[\]$, which we take to be ∞ .

As we did for the negative torus knots, we now set $i_1 = 1$ and let $i_2, \dots, i_{l-1}$ be the indices of the a_i that are not equal to 2, and let $i_l = n$. The endpoints of the continued fraction blocks in P_1 are

$$v_1 = [a_1, \dots, a_{i_{l-1}} - 1], v_2 = [a_1, \dots, a_{i_{l-2}} - 1], \dots, v_{l-1} = [a_{i_1} - 1]$$

and the number of edges in the continued fraction blocks is

$$a_{i_l} - 1, a_{i_{l-1}} - 2, \dots, a_{i_2} - 2.$$

The endpoints of the continued fraction blocks in P_2 are

$$w_1 = [a_1, \dots, a_{i_{l-1}}], w_2 = [a_1, \dots, a_{i_{l-1}-1}], \dots, w_{l-1} = [a_1, \dots, a_{i_2-1}], w_l = [\],$$

if $a_{i_l} \neq 2$. If $a_{i_l} = 2$ then we get the same end points except $[a_1, \dots, a_{i_{l-1}}]$ will be omitted. The number of edges in the continued fraction blocks is

$$1, i_l - i_{l-1} + 1, \dots, i_3 - i_2 + 1, i_2 - i_1$$

if $a_{i_l} \neq 2$. If $a_{i_l} = 2$, then we get the same lengths, except the first 1 is omitted. Now, if $a_{i_l} = -2$, then the labeling of the continued fraction blocks $B_1, \dots, B_k$ above starts with the first continued fraction block in P_2 ; otherwise, it starts with the first continued fraction block in P_1 . So if $a_{i_l} = -2$, then $s_{2j-1} = v_j$ and $s_{2j} = w_j$, giving $n_j = |s_j \cdot q/p|$. Similarly, the lengths m_j of the continued fraction blocks will alternate. If $a_{i_l} \neq -2$, then $s_{2j-1} = w_j$ and $s_{2j} = v_j$, and the n_j are defined in the same way. The m_j will again alternate, but starting with the lengths in P_1 .

2.3. Non-thickenable solid tori. Non-thickenable tori were first found in the work of Honda and the first author [14], and for positive torus knots were classified by LaFountain, the first author, and the third author [15]. The following result will be generalized in Section 5 so that we can understand Dehn fillings of the complements of non-thickenable tori.

Theorem 2.5. *Let C be the complement of a neighborhood of a positive torus knot $T_{p,q}$. For each $k \in \mathbb{N}$, there is a contact structure ξ_k on C that has convex boundary with $2n_k$ dividing curves of slope*

$$s_k = \frac{pq - p - q}{k}, \quad n_k = \gcd(pq - p - q, k),$$

such that any convex torus T in C parallel to ∂C has dividing slope s_k . Moreover, for $k > 1$, (C, ξ_k) embeds in (S^3, ξ_{std}) in exactly two different ways, denote the complement of these embeddings by $N_k^\pm$; and, for $k = 1$, there is a unique embedding of (C, ξ_k) in (S^3, ξ_{std}) , denote its complement N_1 .

If S is a solid torus in the knot type of $T_{p,q}$ that is contained in $N_k^\pm$ and has convex boundary with dividing slope in $(s_k^a, s_k]$, then any torus in the complement of S and isotopic to ∂S has dividing slope in $[s, s_k]$ where s is the dividing slope of ∂S . The same is true for solid tori in N_1 .

The tori $N_k^\pm$ and N_1 are called *non-thickenable solid tori*, and the solid tori S , as in the theorem, are called *partially thickenable tori*. We note N_1 is the neighborhood of the maximal Thurston-Bennequin invariant Legendrian realization of $T_{p,q}$ and the other $N_k^\pm$ are not neighborhoods of Legendrian knots.

For negative torus knots, the non-thickenable tori were classified by Mukherjee, the first author, and the second author [16]. Again, in Section 5 we will consider a generalized “partial thickenable result” necessary for our main results.

Theorem 2.6. *Let C be the complement of a neighborhood of a negative torus knot $T_{p,q}$. For each $k \in \mathbb{N}$, there is a contact structure ξ_k on C that has convex boundary with $2n_k$ dividing curves of slope*

$$s_k = \frac{|pq| - |p| - |q|}{k}, \quad n_k = \gcd(|pq| - |p| - |q|, k),$$

such that any convex torus T in C parallel to ∂C has dividing slope s_k . Moreover, for $k > 1$, $(C(p, q, \xi_k))$ embeds in $(S^3, \xi_{|pq|-|p|-|q|+1})$ in exactly two different ways, denote the complement of these embeddings by $N_k^\pm$; and, for $k = 1$, there is a unique embedding of $C(p, q, \xi_k)$ in $(S^3, \xi_{|pq|-|p|-|q|+1})$, denote its complement N_1 .

Here ξ_n denotes the overtwisted contact structure with d_3 -invariant n .

3. The classification strategy

In this section, we will give an algorithm that counts the number of tight contact structures on $S_r^3(T_{p,q})$. We then justify this algorithm and show how to construct the contact structures. Finally, we prove the main results from the introduction based on this algorithm. We apply the algorithm to several examples in Section 4. Finally, we remind the reader that, we have implemented this algorithm in Rust [18]. A [web version](#) [19] of the algorithm is also available for easy accessibility.

3.1. The algorithm. Here, we will give an algorithm that counts the number of tight contact structures on $S_r^3(T_{p,q})$ up to isotopy. In the next subsection, we will show how to construct these contact structures and justify the algorithm.

The algorithm takes as input a pair of relatively prime integers (p, q) satisfying $|q| > p > 1$, and a rational number r . We compute the following numbers:

- (a) the number $\lfloor r + 1 \rfloor$, the least integer greater than r ,
- (b) the number $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q})$. See Section 2.2 for an algorithm to compute $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q})$ based on [16],
- (c) the number $n(p, q)$ from Equation (1),
- (d) the slopes

$$e_k = \frac{|pq| - |p| - |q|}{k},$$

the sets

$$E = \{k \in \mathbb{Z}_{>1} \mid \gcd(k, |pq| - |p| - |q|) = 1\},$$

and the intervals $J_k = [e_k^a, e_k)$ where e_k^a is defined in Section 2.1. Also, the set

$$J = \bigcup_{k \in E} J_k$$

- (e) the numbers $\Phi_s(r)$ from Equation (3), for $s = \infty, \lfloor r + 1 \rfloor, e_k$.

Remark 3.1. In Section 1.2, we use $E \cup \{1\}$ and $J \cup J_1$ for negative torus knots. Since e_1 is an integer and thus corresponds to a slope of the standard neighborhood of a Legendrian representative, we do not include it for positive torus knots. However, the main theorems for negative torus knots in Section 1.2 can be stated more concisely if we include e_1 as a non-thickenable slope. Since the algorithm in this section uses the number of Legendrian representatives, we use E and J for both positive and negative torus knots.

Then the number of tight contact structures on $S_r^3(T_{p,q})$ is the following.

I. A negative torus knot $T_{p,q}$:

- (i) If $r \notin J$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + n(p, q) \Phi_\infty(r). \end{aligned}$$

In particular, if $r < 0$ or $r \geq |pq| - |p| - |q|$, then the above formula applies.

- (ii) If $r \in J$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + n(p, q) \Phi_\infty(r) + \sum_{r \in J_k, k \in E} \Phi_{e_k}(r). \end{aligned}$$

- (iii) If $r = 0$ and $(p, q) = (2, -3)$, then

$$|\text{Tight}(S_r^3(T_{p,q}))| = \infty.$$

Remark 3.2. Case (iii) is a special case of Case (ii), as $0 \in J_k$ for every $k \in E$ when $(p, q) = (2, -3)$. If $(p, q) \neq (2, -3)$, then r can only be in one of J_k for $k \in E$, so the sum in case (ii) has at most one term.

II. A positive torus knot $T_{p,q}$:(i) If $r \geq pq - p - q + 2$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2)\Phi_\infty(r) + \Phi_{pq-p-q}(r). \end{aligned}$$

(ii) If $r \in [pq - p - q + 1, pq - p - q + 2]$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 1) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2)\Phi_\infty(r) + \Phi_{pq-p-q}(r). \end{aligned}$$

(iii) For $r \in [pq - p - q, pq - p - q + 1]$ we have

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 2) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2)\Phi_\infty(r) + \Phi_{pq-p-q}(r). \end{aligned}$$

(iv) For $r < pq - p - q$ and $r \notin J$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2)\Phi_\infty(r). \end{aligned}$$

In particular, if $r < 0$, then the above formula applies.(v) For $r < pq - p - q$ and $r \in J$, then

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 4) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + (n(p, q) - 2)\Phi_\infty(r) + \sum_{r \in J_k, k \in E} \Phi_{e_k}(r). \end{aligned}$$

(vi) and if $r = 0$ and $(p, q) = (2, 3)$, then

$$|\text{Tight}(S_0^3(T_{2,3}))| = \infty$$

Remark 3.3. Again, Case (vi) is a special case of Case (v), as $0 \in J_k$ for every $k \in E$ when $(p, q) = (2, 3)$. If $(p, q) \neq (2, 3)$, then r can only be in one of J_k for $k \in E$, so the sum in case (ii) has at most one term.

We note that the only term in the above algorithm that does not have a simple, closed-form expression is $\mathcal{L}_n(T_{p,q})$ for $n \in (pq - p - q, pq - 1]$ for positive torus knots and $(pq - |q|, pq - 1]$ for negative torus knots. This algorithm gives an easily computable formula except in the indicated range, and one must implement the algorithm in Section 2.2 to compute $\mathcal{L}_n(T_{p,q})$ in that range, which is already implemented in [18].

3.2. Justification of the algorithm. In this section, we state our four main results (Proposition 3.4 through Proposition 3.7) and establish the algorithm described in the previous subsection. We prove the first two of these results in this section, and the remaining two will be established in subsequent sections.

Recall, in Section 1.2 and 2.3 we discussed non-thickenable solid tori in the knot type of a torus knot $T_{p,q}$. In particular, these tori have convex boundary with dividing slopes e_k , from Equation (4), and associated intervals $J_k = [e_k^a, e_k)$.

Proposition 3.4. *Every tight contact structure on $S_r^3(T_{p,q})$ comes from*

- (1) *negative contact $(r - \lfloor r + 1 \rfloor)$ -surgery on a Legendrian realization of $T_{p,q}$ with $\text{tb} = \lfloor r + 1 \rfloor$ in some contact structure such that the complement of its standard neighborhood does not contain half-convex torsion, or*
- (2) *Dehn filling the complement (C, ξ_k) , discussed in Section 2.3, of a non-thickenable torus in the knot type of $T_{p,q}$ where $r \in J_k$ and $k \in E$.*

We note that the Legendrian knots discussed in Item (1) of the proposition are either Legendrian representatives of $T_{p,q}$ in the tight contact structure on S^3 or non-loose Legendrian representatives of $T_{p,q}$ in some overtwisted contact structure on S^3 .

This result gives a list of all possible tight contact structures, but we must now verify that they are indeed tight and determine when some of the constructed contact structures are isotopic. We begin by identifying the overcount resulting from Proposition 3.4.

Proposition 3.5. *The contact structures constructed in Item (1) of Proposition 3.4 contain the following overcount of possible tight contact structures.*

- (1) *For positive torus knot $T_{p,q}$ and $r \in [pq - p - q + 1, pq - p - q + 2)$, there is an overcount of*

$$(2n(p, q) - 1) \Phi_{\lfloor r+1 \rfloor}(r) - (n(p, q) - 2) \Phi_{\infty}(r) - \Phi_{pq-p-q}(r).$$

- (2) *For positive torus knot $T_{p,q}$ and $r \in [pq - p - q, pq - p - q + 1)$, there is an overcount of*

$$(2n(p, q) - 2) \Phi_{\lfloor r+1 \rfloor}(r) - (n(p, q) - 2) \Phi_{\infty}(r) - \Phi_{pq-p-q}(r).$$

- (3) *For positive torus knot $T_{p,q}$ and $r < pq - p - q$, there is an overcount of*

$$(n(p, q) - 2) (2\Phi_{\lfloor r+1 \rfloor}(r) - \Phi_{\infty}(r)).$$

- (4) *For any torus knot and r not considered above, there is an overcount of*

$$n(p, q)(2\Phi_{\lfloor r+1 \rfloor}(r) - \Phi_{\infty}(r)).$$

In the next two results we see that the contact structures in Proposition 3.4 are all tight and distinct, except as indicated in Proposition 3.5.

Proposition 3.6. *The contact structures constructed in Item (2) of Proposition 3.4 are all tight, distinct from those constructed in Item (1), and Dehn fillings of the complements of distinct non-thickenable tori are distinct. Moreover, distinct choices of contact structures on the solid torus for Dehn filling result in distinct tight contact structures.*

We note that when $(p, q) \neq (2, \pm 3)$ the J_k are disjoint, so the last statement in Proposition 3.6 is automatic, but when $(p, q) = (2, \pm 3)$ the intervals are not distinct, and so the statement is necessary there.

Proposition 3.7. *The contact structures constructed in Item (1) of Proposition 3.4 are all tight, and they are distinct except for the overcount from Proposition 3.5. Moreover, these contact structures are distinguished by their non-vanishing Heegaard Floer contact invariants.*

At the end of this section, we prove Propositions 3.4 and 3.5, while Proposition 3.6 and Proposition 3.7 will be proven in Section 5 and Section 6, respectively. Before we do this, however, we first establish the algorithm from the previous subsection.

Proof of the algorithm in Section 3.1. According to Proposition 3.4, an upper bound on the number of tight contact structures on $S_r^3(T_{p,q})$ is given by

$$\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) \cdot \Phi_{\lfloor r+1 \rfloor}(r) + \sum_{r \in J_k, k \in E} \Phi_{e_k}(r),$$

as the first term corresponds to contact structures coming from Item (1) in Proposition 3.4, and the second term bounds those coming from Item (2).

Now taking into account the overcounts discussed in Proposition 3.5, we can see that the upper bounds on the number of tight contact structures on $S_r^3(T_{p,q})$ agree with the actual number of tight contact structures given in the algorithm in Section 3.1. For example, if $pq < 0$ and $r \notin J$, then the overcount from Proposition 3.5 is $n(p, q)(2\Phi_{\lfloor r+1 \rfloor}(r) - \Phi_\infty(r))$ and subtracting this from the formula above gives Item (i) in the algorithm. Similarly, Items (ii) and (iii) follow by taking account of the contribution from $r \in J$. The results for $pq > 0$ follow similarly.

We also note that in Item (i) for positive torus knots, $2\Phi_\infty(r) = \Phi_{pq-p-q}(r)$. So we are left to see that all these contact structures are indeed tight and distinct. Tightness is immediate from Propositions 3.6 and 3.7. To show they are all different, we notice that Proposition 3.6 tells us that the contact structures that come from Dehn filling complements of non-thickenable solid tori in the knot type $T_{p,q}$ are all distinct and cannot come from contact surgery on a Legendrian knot. Moreover, Proposition 3.7 implies that any two negative contact surgeries not identified as being the same in Proposition 3.5 are actually distinct. In fact, they even have different Heegaard Floer contact invariants, except when $(p, q) = (2, \pm 3)$ and $r = 0$. In this case, the surgeries yield torus bundles, which admit infinitely many tight contact structures and were classified by Giroux [30] and Honda [36]. Thus, the upper bounds are sharp and the algorithm computes the number of tight contact structures. $\square$

We now establish the upper bound on the number of tight contact structures given in Proposition 3.4.

Proof of Proposition 3.4. Let K be the surgery dual of $T_{p,q}$ in $S_r^3(T_{p,q})$, that is, K is the core of the solid torus glued to the complement of $T_{p,q}$ to perform the surgery. We will use coordinates on a neighborhood of K that come from the longitude-meridian coordinates on $T_{p,q}$ (notice that the boundary of a neighborhood of K in $S_r^3(T_{p,q})$ can also be thought

of as the boundary of a neighborhood of $T_{p,q}$ in S^3). That is, a neighborhood of K is S_r , a solid torus with lower meridian r .

Given any tight contact structure ξ on $S_r^3(T_{p,q})$, we take a neighborhood N of K . Note that we can always choose N to be $S_r(s)$, which has convex boundary with two dividing curves of some slope s . We now consider the case when we can choose such a neighborhood so that $s \in \mathbb{Z}$. In this case, we can remove N from $S_r^3(T_{p,q})$ and glue in a solid torus N' , which is $S_\infty(s)$ with lower meridian ∞ . Then, we extend the contact structure on $S_r^3(T_{p,q}) \setminus N$ over the solid torus N' by the unique tight contact structure (note the dividing curves are longitudinal). This will result in a contact structure ξ' on S^3 , and N' will be the neighborhood of a Legendrian knot L' . The contact structure ξ' on S^3 might be either tight or overtwisted. In the former case, $(S_r^3(T_{p,q}), \xi)$ is obtained from (S^3, ξ_{std}) by positive or negative contact surgery on L' . However, for negative torus knots, we know that any positive contact surgery is overtwisted [6, 44] and for positive torus knots, according to Item (1) in Section 2.2.2, we know that any tight positive contact surgery is equivalent to some negative contact surgery on a non-loose Legendrian representative. Now, if ξ' is overtwisted, it is clear that L' is non-loose since the complement of N' is tight. Thus $(S_r^3(T_{p,q}), \xi)$ is obtained from contact surgery on a non-loose knot. However, positive contact surgery on a non-loose Legendrian torus knot always produces an overtwisted contact structure. To see this, recall that in [16, Section 6.2.3] it was shown that all non-loose Legendrian (p, q) -torus knots destabilize to one with $tb = pq$ or to ∞ , except for one when $pq < 0$. In the former case, [16, Lemma 6.14] implies that positive contact surgery will be overtwisted, while in the latter case, positive contact surgery is obviously overtwisted since there will be half-convex Giroux torsion in the surgery solid torus. For the unique Legendrian realization mentioned for the case when $pq < 0$, any stabilization of it will result in a Legendrian that does destabilize to infinity; thus positive contact surgery on it will also be overtwisted. As we may assume that $s = \lfloor r + 1 \rfloor$ by shrinking the torus N if necessary, we see that any such ξ will come from negative contact surgery on a Legendrian knot in ξ_{std} or a non-loose Legendrian knot with no half-convex torsion in its complement, as the existence of half-convex torsion in the complement of a standard neighborhood of a Legendrian knot implies that any contact surgery on it will produce an overtwisted contact structure.

We now consider the case that the slope s cannot be chosen to be an integer; then we assume that s is chosen to be as large as possible (that is, any solid torus containing N will also have dividing slope s). Now replace N with a solid torus N' with lower meridian ∞ and extend the contact structure on the complement of N by a universally tight contact structure on N' . This gives a contact structure on S^3 , and N' is a non-thickenable solid torus, that is, ξ is obtained by Dehn filling the complement of a non-thickenable torus in some contact structure on S^3 and extending the contact structure over the surgery solid torus. $\square$

Remark 3.8. In the proof of Proposition 3.4, one might try to use the result of Wand [65] that Legendrian surgery preserves tightness, which implies that (+1)-contact surgery preserves overtwistedness. However, this does not imply that any positive surgery preserves overtwistedness, unlike negative contact surgery. In fact, Baldwin and the first author [1] found a counterexample.

We will now discuss the overcount of tight contact structures constructed in Proposition 3.4.

Proof of Proposition 3.5. For the discussion below, we recall that Item (1) in Section 2.2.2 states that given any contact structure on the complement C of a neighborhood of $T_{p,q}$, that has no convex Grioux torsion and convex boundary with two dividing curves of slope ∞ , any convex torus parallel to ∂C will have dividing slope ∞ , with two exceptions when $pq > 0$. To understand this case, let L be the unique maximal Thurston-Bennequin invariant Legendrian representative of $T_{p,q}$ in (S^3, ξ_{std}) and let (C', ξ') be the contact structure on the complement of its standard neighborhood. The two exceptional contact structures will be obtained from (C', ξ') by adding a basic slice with dividing slopes $pq - p - q$ and ∞ .

We start with Item (4) in the proposition. If r is not as in Items (1) or (2), then we note that among the non-loose Legendrian knots counted by $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q})$, there will be $2n(p, q)$ that are part of $n(p, q)$ infinite X's. Thus the complement of 2 of these Legendrian knots, say L_1 and L_2 , (that are on the same X) are obtained from a unique tight contact structure ξ' on the complement C of $T_{p,q}$ in S^3 with dividing set consisting of two curves of slope ∞ by attaching a thickened annulus with dividing slopes $\lfloor r+1 \rfloor$ and ∞ . But the path from r to $\lfloor r+1 \rfloor$ might be part of a continued fraction block with the edge from $\lfloor r+1 \rfloor$ to ∞ . Thus, the right count for the number of contact structures coming from contact surgery on L_1 and L_2 is $\Phi_\infty(r)$ and not $2\Phi_{\lfloor r+1 \rfloor}(r)$. Thus, we have an overcount of $2\Phi_{\lfloor r+1 \rfloor}(r) - \Phi_\infty(r)$ and given that there are $n(p, q)$ such pairs of Legendrians, we obtain the claimed overcount.

Now suppose that $pq > 0$ and $r \in (pq - p - q, pq - p - q + 1)$. There are 2 non-loose Legendrian knots with $tb = pq - p - q + 2$ on an infinite X whose complements come from adding a contact structure on a thickened torus to (C', ξ') discussed above and 1 non-loose Legendrian knot with $tb = pq - p - q + 2$ at the vertex in the infinite V. Thus, any r -surgery effecting a contact surgery on these 3 Legendrian knots will be equivalent to some Dehn filling of (C', ξ') (that is, a positive contact surgery on L). So all the contact structures obtained by negative contact surgery on these 3 Legendrian knots are also achieved by positive contact surgery on L , and these are counted by

$$\Phi_{pq-p-q}(r).$$

Thus, taking into account the discussion for Item (4) in the theorem, we see that contact surgery on Legendrian knots with $tb = \lfloor r+1 \rfloor$ yields at most

$$(\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 1) \Phi_{\lfloor r+1 \rfloor}(r) + (n(p, q) - 2) \Phi_\infty(r) + \Phi_{pq-p-q}(r)$$

distinct tight contact structures.

For positive torus knot $T_{p,q}$ and $r \in [pq - p - q, pq - p - q + 1)$, there are 2 non-loose Legendrian knots whose complements are built from (C', ξ') by adding a contact structure on a thickened torus (see Item (1) in Section 2.2.2 to see why the third Legendrian becomes loose when stabilized) so the correct formula to use is

$$(\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q) + 2) \Phi_{\lfloor r+1 \rfloor}(r) + (n(p, q) - 2) \Phi_{\infty}(r) + \Phi_{pq-p-q}(r).$$

Finally, for positive torus knot $T_{p,q}$ and $r < pq - p - q$, the Legendrian knots coming from (C', ξ') are either loose or have half convex torsion in their complement, so they do not contribute any further. Thus, the overcount will be

$$(2n(p, q) - 4) \Phi_{\lfloor r+1 \rfloor}(r) - (n(p, q) - 2) \Phi_{\infty}(r).$$

This completes our discussion of the obvious overcounting of tight contact structures in Proposition 3.4. $\square$

3.3. Proof of the main theorems. We begin by establishing Theorem 1.8 that computes the number of tight contact structures on $S_r^3(T_{p,q})$ when $r \geq pq - 1$.

Proof of Theorem 1.8. For positive torus knots, we notice that $r \geq pq$ cannot be contained in any of the intervals J_k . Thus, Item (i) in the algorithm for the number of tight contact structures on $S_r^3(T_{p,q})$ gives

$$(\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) + n(p, q) \Phi_{\infty}(r),$$

tight contact structures (here, $2\Phi_{\infty}(r) = \Phi_{pq-p-q}(r)$). However, from Section 2.2 we know that in this range $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) = 2n(p, q)$. This gives us the claimed formula of

$$|\text{Tight}(S_r^3(T_{p,q}))| = n(p, q) \Phi_{\infty}(r).$$

This concludes the case for positive torus knots and $r \geq pq$.

For negative torus knots and $r \geq pq$, the same computation works, except here there can be contributions from Dehn filling the complements of the non-thickenable tori, which contribute $\Phi_{e_k}(r)$ for each J_k containing r .

Finally, we consider the case when $r \in [pq - 1, pq)$. Notice $r \notin J_k$ for any k . So the desired formula

$$|\text{Tight}(S_r^3(T_{p,q}))| = (m(p, q) - 2n(p, q)) \Phi_{pq}(r) + n(p, q) \Phi_{\infty}(r)$$

follows from the algorithm in Section 3.1 and the fact that $\mathcal{L}_{pq}(T_{p,q}) = m(p, q)$ as discussed in Section 2.2. $\square$

We now consider the case when the surgery coefficient r is sufficiently negative.

Proof of Theorem 1.9. For positive torus knots, this follows directly from Items (iv) and (v) of the algorithm by noting that when $r < pq - p - q$, we have

$$\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) = M(p, q) + (1 + pq - p - q - n),$$

since $\mathcal{L}_{\lfloor r+1 \rfloor}^V(T_{p,q}) = 0$, $\mathcal{L}_{\lfloor r+1 \rfloor}^X(T_{p,q}) = M(p, q)$, and $\mathcal{L}_{\lfloor r+1 \rfloor}^T(T_{p,q}) = (1 + pq - p - q - \lfloor r + 1 \rfloor)$, as was shown in Section 2.2.

For negative torus knots, it directly follows from Items (i) and (ii) by noting that when $r < pq - |q|$, we have

$$\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) = M(p, q) + (|p + q| + pq + 1 - \lfloor r + 1 \rfloor),$$

since $\mathcal{L}_{\lfloor r+1 \rfloor}^{\times}(T_{p,q}) = M(p, q)$, and $\mathcal{L}_{\lfloor r+1 \rfloor}^{\top}(T_{p,q}) = (|p + q| + pq + 1 - \lfloor r + 1 \rfloor)$, as was shown in Section 2.2. $\square$

We now turn to Theorem 1.11 and Theorem 1.13 that concern surgeries $r < pq - 1$ on a negative, respectively positive, torus knots.

Proof of Theorem 1.11 and 1.13. The first theorem is exactly the statement of Item (i) in the algorithm, and the second is exactly Items (ii) through (vi). $\square$

4. Examples

In this section, we illustrate our main theorems by giving the classification of tight contact structures on $S_r^3(T_{p,q})$ when (p, q) is $(2, \pm(2n+1))$ and $(5, \pm 8)$. In Section 4.5, we will see that the formulas given in this paper for small Seifert fibered spaces with $e_0 \neq -1, -2$ obtained from surgery on torus knots agree with the formulas that are in the literature.

4.1. Surgeries on $(2, -(2n+1))$ -torus knots. Here, we prove Theorem 1.14. Following the algorithm in Section 3.1, to compute the number of contact structures on $S_r^3(T_{2, -(2n+1)})$, we need to compute $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{2, -(2n+1)})$, $n(2, -(2n+1))$, e_k and e_k^a .

To do so, we first compute the continued fractions of q/p and $(q/p - \lceil q/p \rceil)^{-1}$:

$$\begin{aligned} -\frac{2n+1}{2} &= [-n-1, -2] \\ \left(-\frac{2n+1}{2} - (-n)\right)^{-1} &= [-2] \end{aligned}$$

From Equation (1), we see that

$$\begin{aligned} n(2, -(2n+1)) &= n \\ m(2, -(2n+1)) &= 4n \end{aligned}$$

According to the Formulas (6), (7), and (11) in Section 2.2.1, we see that

$$\mathcal{L}_k(T_{2, -(2n+1)}) = \begin{cases} 2n & \text{if } k > -4n-2 \text{ and } k \neq 2n-1, \\ 2n+1 & \text{if } k = 2n-1, \\ -2-k & \text{if } k \leq -4n-2. \end{cases}$$

We also have

$$e_k = \frac{2n-1}{k}.$$

If $(2n-1)$ and k are relatively prime, then there are non-negative integers a_k and b_k such that $(2n-1)b_k - ka_k = 1$ with $a_k \leq 2n-1$ and $b_k \leq k$. In this case, $e_k^a = a_k/b_k$. Thus we see that $J_k = \lfloor a_k/b_k, (2n-1)/k \rfloor$.

If $r \notin J \cup J_1$, then Item (i) of the algorithm in Section 3.1 gives (note that $J \subset J \cup J_1$)

$$\begin{aligned} |\text{Tight}(S_r^3(T_{2,-(2n+1)})))| &= (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{2,-(2n+1)}) - 2n(2, -(2n+1))) \Phi_{\lfloor r+1 \rfloor}(r) \\ &\quad + n(2, -(2n+1))\Phi_\infty(r). \end{aligned}$$

If $r < -4n - 2$, then $r \notin J \cup J_1$. Thus this simplifies to

$$|\text{Tight}(S_r^3(T_{2,-(2n+1)})))| = \begin{cases} n\Phi_\infty(r) & r \geq -4n - 2 \text{ and } r \notin J \cup J_1, \\ n\Phi_\infty(r) - (2n+2 + \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & r < -4n - 2. \end{cases}$$

Now we consider the case $r \in J \cup J_1$. If $n > 1$ and $r \in J_1$, then $r \notin J$ so we apply Item (i) in the algorithm in Section 3.1 to obtain

$$\begin{aligned} |\text{Tight}(S_r^3(T_{2,-(2n+1)})))| &= n\Phi_\infty(r) + \Phi_{\lfloor r+1 \rfloor}(r) \\ &= n\Phi_\infty(r) + \Phi_{e_1}(r). \end{aligned}$$

The second equality holds as $J_1 = [2n-2, 2n-1)$ and hence $\lfloor r+1 \rfloor = 2n-1 = e_1$.

If $n > 1$ and $r \in J_k$ for some $k \in E$, then we apply Item (ii) in the algorithm to obtain

$$|\text{Tight}(S_r^3(T_{2,-(2n+1)})))| = n\Phi_\infty(r) + \Phi_{e_k}(r), \quad r \in J_k, \quad k \in E,$$

as all J_k are pairwise disjoint.

If $n = 1$ and $r \in J \cup J_1$, then $r \in J_k$ for $1 \leq k \leq m$ where $m = \lceil 1/r - 1 \rceil$. Also, $e_k = 1/k$ and $r \in [0, 1)$. Now we apply Item (ii) to obtain

$$\begin{aligned} |\text{Tight}(S_r^3(T_{2,-3}))| &= \Phi_{\lfloor r+1 \rfloor}(r) + \Phi_\infty(r) + \sum_{k=2}^m \Phi_{e_k}(r) \\ &= \sum_{k=0}^m \Phi_{1/k}(r) \\ &= \frac{(m+1)(m+2)}{2} \Phi_{1/m}(r). \end{aligned}$$

The second equality holds as $\Phi_{\lfloor r+1 \rfloor}(r) = \Phi_1(r)$. The third equality holds since $\Phi_{1/(m-i)}(r) = (i+1)\Phi_{1/m}(r)$ for $0 \leq i \leq m$. $\square$

4.2. Surgeries on $(2, 2n+1)$ -torus knots. Now we establish Theorem 1.15. Following the algorithm in Section 3.1, to compute the number of contact structures on $S_r^3(T_{2,2n+1})$, we need to compute $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{2,2n+1})$, $n(2, 2n+1)$, the e_k and e_k^a .

To do so, we first compute the continued fractions of $(p/q - 1)^{-1}$ and $(q/p - \lceil q/p \rceil)^{-1}$:

$$\begin{aligned} \left(\frac{2}{2n+1} - 1 \right)^{-1} &= \overbrace{[-2, \dots, -2, -3]}^{n-1} \\ \left(\frac{2n+1}{2} - (n+1) \right)^{-1} &= [-2] \end{aligned}$$

From Equation (1), we see that

$$\begin{aligned} n(2, 2n + 1) &= 2, \\ m(2, 2n + 1) &= 6. \end{aligned}$$

According to the Formulas (12), and (13) in Section 2.2.2 we see that

$$\mathcal{L}_k(T_{2,2n+1}) = \begin{cases} 4 & \text{if } k > 4n + 2, \text{ and} \\ 6 & \text{if } k = 4n + 2. \end{cases}$$

To compute $\mathcal{L}_k(T_{2,2n+1})$ for smaller value of k , we use Formula (19) in Section 2.2.2 and need to compute $\mathcal{L}_k^X(T_{2,2n+1})$, $\mathcal{L}_k^V(T_{2,2n+1})$, and $\mathcal{L}_k^T(T_{2,2n+1})$. Following the Formulas (14), (15), (17), and (18) in Section 2.2.2, we obtain

$$\mathcal{L}_k^X(T_{2,2n+1}) = \begin{cases} 2 & \text{if } k > 2n - 1, \text{ and} \\ 0 & \text{if } k \leq 2n - 1, \end{cases}$$

and

$$\mathcal{L}_k^V(T_{2,2n+1}) = \begin{cases} 2 & \text{if } k > 4n + 2 \\ 4 & \text{if } 2n + 3 < k \leq 4n + 2, \\ 3 & \text{if } k = 2n + 3, \\ 2 & \text{if } k = 2n + 2, \\ 1 & \text{if } k = 2n + 1, \text{ and} \\ 0 & \text{if } k < 2n + 1, \end{cases}$$

and finally

$$\mathcal{L}_k^T(T_{2,2n+1}) = \begin{cases} 0 & \text{if } k > 2n - 1, \text{ and} \\ 2n - k & \text{if } k \leq 2n - 1. \end{cases}$$

Thus we have

$$\mathcal{L}_k(T_{2,2n+1}) = \begin{cases} 4 & \text{if } k > 4n + 2, \\ 6 & \text{if } 2n + 3 < k \leq 4n + 2, \\ 5 & \text{if } k = 2n + 3, \\ 4 & \text{if } k = 2n + 2, \\ 3 & \text{if } k = 2n + 1, \\ 2 & \text{if } k = 2n, \text{ and} \\ 2n - k & \text{if } k \leq 2n - 1. \end{cases}$$

Now we are ready to count the tight contact structures. By applying Item (i) of the algorithm in Section 3.1 for $r \geq 2n + 1$, we obtain

$$|\text{Tight}(S_r^3(T_{2,2n+1}))| = \begin{cases} 2\Phi_\infty(r) & \text{if } r \geq 4n + 2, \\ 2\Phi_\infty(r) + 2\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 2n + 3 \leq r < 4n + 2, \\ 2\Phi_\infty(r) + \Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 2n + 2 \leq r < 2n + 3, \text{ and} \\ 2\Phi_\infty(r) & \text{if } 2n + 1 \leq r < 2n + 2. \end{cases}$$

The $2\Phi_\infty(r)$ term comes from the fact that $\Phi_{pq-p-q}(r) = 2\Phi_\infty(r)$ if $r \geq pq - p - q + 2$. By applying Items (ii) and (iii) in the algorithm for $r \in [2n - 1, 2n + 1)$, we obtain

$$|\text{Tight}(S_r^3(T_{2,2n+1}))| = \begin{cases} \Phi_{2n-1}(r) & \text{if } 2n - 1 < r < 2n + 1, \\ 0 & \text{if } r = 2n - 1, \end{cases}$$

Finally, if $r < 2n - 1$ then we have two cases. If $r \notin J$, then we apply Item (iv) of the algorithm and obtain

$$|\text{Tight}(S_r^3(T_{2,2n+1}))| = \Phi_{2n-1}(r),$$

which holds as $(2n - \lfloor r + 1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) = \Phi_{2n-1}(r)$ when $r < 2n - 1$.

If $n \neq 1$ and $r \in J_k$ for some $k \in E$, we apply Item (v) of the algorithm to obtain

$$|\text{Tight}(S_r^3(T_{2,2n+1}))| = \Phi_{2n-1}(r) + \Phi_{e_k}(r),$$

as all J_k are pairwise disjoint.

If $n = 1$ and $r \in J$, then $r \in J_k$ for $2 \leq k \leq m$ and $m = \lceil 1/r - 1 \rceil$. Now we apply Item (v) to obtain

$$\begin{aligned} |\text{Tight}(S_r^3(T_{2,2n+1}))| &= \Phi_{2n-1}(r) + \sum_{k=2}^m \Phi_{e_k}(r) \\ &= \sum_{k=1}^m \Phi_{1/k}(r) \\ &= \frac{m(m+1)}{2} \Phi_{1/m}(r). \end{aligned}$$

The second equality holds since $e_k = 1/k$ and $2n - 1 = e_1 = 1$ when $n = 1$. The last equality holds since $(i + 1) \cdot \Phi_{1/m}(r) = \Phi_{1/(m-i)}(r)$ for $0 \leq i \leq m - 1$. $\square$

4.3. Surgeries on $(5, -8)$ -torus knots. In this subsection, we prove Theorem 1.17. According to the algorithm in Section 3.1, to compute the number of contact structures on $S_r^3(T_{5,-8})$, we need to compute $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{5,-8})$, $n(5, -8)$, $m(5, -8)$ the e_k and e_k^a .

To do so, we first compute the continued fractions for q/p and $(q/p - \lceil q/p \rceil)^{-1}$:

$$\begin{aligned} -\frac{8}{5} &= [-2, -3, -2], \\ \left(-\frac{8}{5} - (-1)\right)^{-1} &= [-2, -3] \end{aligned}$$

Thus, we obtain

$$n(5, -8) = 4 \text{ and } m(5, -8) = 16.$$

So, according to the formulas (6), (7) in Section 2.2.1 we see that

$$\mathcal{L}_n(T_{5,-8}) = \begin{cases} 8 & \text{if } n > -40 \text{ and } n \neq 27, \\ 9 & \text{if } n = 27, \\ 16 & \text{if } n = -40. \end{cases}$$

To determine the number of Legendrian realizations for $\text{tb} < -40$ is more involved. We use the formulas (8), (10) discussed in Section 2.2.1. One first easily computes that $m_1 = 1, m_2 = 2$, and $m_3 = 1$, as well as $s_1 = -3/2, s_2 = -2$, and $s_3 = -1$. So we see that $n_1 = 1, n_2 = 2$, and $n_3 = 3$. Plugging this information into the formula for the number of non-loose knots with no half convex torsion in their complement and $\text{tb} = n \leq -40$ gives

$$\mathcal{L}_n^X(T_{5,-8}) = 2(4 + W_3(n)) = \begin{cases} 10 & \text{if } n = -40 \text{ and } n \neq 29, \\ 12 & \text{if } n < -40. \end{cases}$$

In addition, the number of Legendrian representatives in (S^3, ξ_{std}) is

$$\mathcal{L}_n^T(T_{5,-8}) = \begin{cases} 2 & \text{if } n = -40, \\ 4 & \text{if } n = -41, \\ -36 - n & \text{if } n \leq -42. \end{cases}$$

Thus,

$$\mathcal{L}_n(T_{5,-8}) = \begin{cases} 12 & \text{if } n = -40, \\ 16 & \text{if } n = -41, \\ -24 - n & \text{if } n \leq -42. \end{cases}$$

Finally, we note that $e_k = 27/k$. If $\gcd(27, k) = 1$, then there are positive integers a_k and b_k such that $27b_k - ka_k = 1$ with $a_k < 27$ and $b_k < k$. In this case, $e_k^a = a_k/b_k$. Thus for $k \in E$ we see that $J_k = [a_k/b_k, 27/k)$.

When $r \notin J \cup J_1$, Item (i) in the algorithm in Section 3.1 gives (note that $J \subset J \cup J_1$)

$$(\mathcal{L}_{\lfloor r+1 \rfloor}(T_{5,-8}) - 2n(5, -8)) \Phi_{\lfloor r+1 \rfloor}(r) + n(5, -8) \Phi_\infty(r).$$

Moreover, if $r \leq -40$, then $r \notin J \cup J_1$. Thus this simplifies to

$$|\text{Tight}(S_r^3(T_{5,-8}))| = \begin{cases} 4\Phi_\infty(r) & \text{if } r \geq -40 \text{ and } r \notin J \cup J_1, \\ 4\Phi_\infty(r) + 4\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } \lfloor r+1 \rfloor = -40, \\ 4\Phi_\infty(r) + 8\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } \lfloor r+1 \rfloor = -41, \\ 4\Phi_\infty(r) + (-32 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } r < -42. \end{cases}$$

If $r \in J_k$ for some $k \in E \cup \{1\}$ then we have

$$|\text{Tight}(S_r^3(T_{5,-8}))| = 4\Phi_\infty(r) + \Phi_{e_k}(r),$$

since such r must be larger than -40 . We also note that when $\lfloor r+1 \rfloor = 27$, the term $(\mathcal{L}_{27}(T_{5,-8}) - 2n(5, -8))\Phi_{\lfloor r+1 \rfloor}(r) = \Phi_{\lfloor r+1 \rfloor}(r)$ reduces to $\Phi_{e_1}(r)$ as $e_1 = 27$. Hence we, obtain a consistent formula for $r \in J \cup J_1$.

4.4. Surgeries on $(5, 8)$ -torus knots. This section contains the proof of Theorem 1.18. According to the algorithm in Section 3.1, to compute the number of contact structures on $S_r^3(T_{5,8})$ we must compute $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{5,8})$, $n(5, 8)$, $m(5, 8)$, and the e_k and e_k^a .

The continued fraction for $(5/8 - 1)^{-1} = -8/3$ is $[-3, -3]$ and to compute $n(5, 8)$ from Equation (1) we need to compute $(8/5 - 2)^{-1} = -5/2$ which has continued fraction $[-3, -2]$. Thus, we see that

$$n(5, 8) = 8 \text{ and } m(5, 8) = 24.$$

According to the formulas (12), (13) in Section 2.2.2, we see that

$$\mathcal{L}_{\lfloor r+1 \rfloor}(T_{5,8}) = \begin{cases} 16 & \text{if } \lfloor r+1 \rfloor > 40 \text{ and} \\ 24 & \text{if } \lfloor r+1 \rfloor = 40. \end{cases}$$

To determine the number of Legendrian realizations for $\lfloor r+1 \rfloor < 40$ is more involved. We follow the process discussed in Section 2.2.1. One first easily computes that $m_1 = 1, m_2 = 2, m_3 = 1$, and $m_4 = 1$, as well as $s_1 = 3/2, s_2 = 2, s_3 = 1$, and $s_4 = \infty$. So we see that $n_1 = 1, n_2 = 3, n_3 = 3$, and $n_4 = 5$. Plugging this information into the formulas (14), (15) gives

$$\mathcal{L}_n^X(T_{5,8}) = \begin{cases} 16 & \text{if } 27 < n \leq 39, \\ 14 & \text{if } n \leq 27. \end{cases}$$

Using the formula (17), we also compute

$$\mathcal{L}_n^V(T_{5,8}) = 2(4 + W_3(n)1) = \begin{cases} 8 & \text{if } n = 40, \\ 10 & \text{if } n = 39, 38, \\ 9 - (37 - n) & \text{if } n = 37, \dots, 29, \text{ and} \\ 0 & \text{if } n < 29. \end{cases}$$

Using the formula (18), we can compute

$$\mathcal{L}_n^T(T_{5,8}) = 1 + 27 - n$$

for $n \leq 27$ and 0 otherwise. Thus for $n < 40$ we have

$$\mathcal{L}_n(T_{5,8}) = \begin{cases} 26 & \text{if } n = 39, 38, \\ 25 - (37 - n) & \text{if } 28 \leq n \leq 37 \\ 15 + (27 - n) & \text{if } n < 28. \end{cases}$$

Finally, we note that $e_k = 27/k$. If 27 and k are relatively prime, then there are positive integers a_k and b_k such that $27b_k - ka_k = 1$ with $a_k < 27$ and $b_j < k$. In this case, $e_k^a = a_k/b_k$. Thus for $k \in E$ we see that $J_k = [a_k/b_k, 27/k)$.

Now, according to our algorithm, we have for $r \geq 29$

$$|\text{Tight}(S_r^3(T_{5,8}))| = \begin{cases} 8\Phi_\infty(r) & \text{if } r \geq 40, \\ 8\Phi_\infty(r) + 8\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } \lfloor r+1 \rfloor = 40, \\ 8\Phi_\infty(r) + 10\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } \lfloor r+1 \rfloor = 38, 39, \text{ and} \\ 8\Phi_\infty(r) + (-28 + \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) & \text{if } 29 \leq r < 37. \end{cases}$$

For $r \in [27, 29)$ we have

$$|\text{Tight}(S_r^3(T_{5,8}))| = 6\Phi_\infty(r) + 2\Phi_{\lfloor r+1 \rfloor}(r) + \Phi_{27}(r).$$

We note that this formula encompasses Items (ii) and (iii) in the algorithm. Finally for $r < 27$ and $r \notin J$, we have

$$|\text{Tight}(S_r^3(T_{5,8}))| = 6\Phi_\infty(r) + (28 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r),$$

and if $r \in J_k$ for some $k \in E$, then we have

$$|\text{Tight}(S_r^3(T_{5,8}))| = 6\Phi_\infty(r) + (28 - \lfloor r+1 \rfloor)\Phi_{\lfloor r+1 \rfloor}(r) + \Phi_{e_k}(r).$$

4.5. Tight contact structures on small Seifert fibered spaces with $e_0 \neq -1, -2$. When a small Seifert fibered space with $e_0 \neq -1$ or -2 is also obtained by surgery on a torus knot (see Section 1.3), we can compute the number of tight contact structures by using our formulas or the formulas given in [27, 67]. In this section, we will show that despite these formulas looking quite different, they produce the same number. Of course, we know theoretically that they must be the same, but it is interesting to see the relationship between the formulas. We will explain the relation between the formulas for negative torus knots; the case of positive torus knots follows similarly.

Case 1: $e_0 \leq -3$. Tight contact structures on small Seifert fibered spaces with $e_0 \leq -3$ were first classified by Wu in [67]. A surgery diagram for $Y(e_0; r_1, r_2, r_3)$ is given in Figure 1. Consider the continued fraction expansion

$$-1/r_i = [a_1^i, \dots, a_{n_i}^i].$$

For $e_0 \leq -3$, Wu showed that

$$|\text{Tight}(Y(e_0; r_1, r_2, r_3))| = |e_0 + 1| \prod_{i=1}^3 \left(\prod_{j=1}^{n_i} |a_j^i + 1| \right).$$

We note that if we realize the link in Figure 1 by Thurston-Bennequin invariant -1 Legendrian unknots, then the number above just computes all the possible negative contact surgeries realizing $Y(e_0; r_1, r_2, r_3)$. For example, $\prod_{j=1}^{n_i} |a_j^i + 1|$ is simply the number of tight contact structures on a solid torus with convex boundary having dividing slope -1 and lower meridian $-1/r_i$.

As discussed in Section 1.3, r surgery on a negative (p, q) -torus knot will have $e_0 \leq -3$ if $r \in (pq, pq+1)$. More specifically, $e_0 = -k-2$ if $r \in \left(\frac{(k+1)pq+1}{k+1}, \frac{kpq+1}{k} \right)$. Referring to Figure 2, we see that the surgery coefficients on the vertical unknots are $r' = r - pq, q/(q-p^*)$, and

$-p/q^*$. If we negatively Rolfsen twist the unknot with framing $q/(q - p^*)$, the framing changes to $-q/p^*$ and the horizontal unknot has framing -1 . Notice that $r' \in (1/(k+1), 1/k)$, so if we perform $k+1$ negative Rolfsen twists on that unknot, its framing will change from r' to $r'' < -2$, and it will convert the framing on the horizontal unknot to $-k-2$. So to apply Wu's formula, we need to set $r_1 = q^*/p$, $r_2 = p^*/q$, and $r_3 = -1/r''$.

If we let $q/p = [a_1, \dots, a_m]$ then it is well-known that $-q/p^* = [a_m, \dots, a_1]$, [34]. (We note that the minus sign here arises from our definition of p^* in Section 1.3, which makes it negative.) We also have the lesser-known continued fraction relation.

Lemma 4.1. *Using the notation from Section 1.2, $-p/q^* = [b_n, \dots, b_1]$.*

We prove the lemma below, but first note that, with this established, Wu's formula gives

$$|\text{Tight}(S_r^3(T_{p,q}))| = (k+1)n(p, q) \prod_{j=1}^{n_3} |a_j^3 + 1|.$$

Since the surgery slope r is not in J , our algorithm in Section 3.1 says

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{[r+1]}(T_{p,q}) - 2n(p, q)) \Phi_{[r+1]}(r) + n(p, q) \Phi_\infty(r).$$

Since all non-loose Legendrian knots with $\text{tb} = pq + 1$ destabilize to infinity, the first term in the formula is zero. Thus we have

$$|\text{Tight}(S_r^3(T_{p,q}))| = n(p, q) \Phi_\infty(r).$$

To compute $\Phi_\infty(r)$ we must consider a path from ∞ anti-clockwise to r we recall that we took $r \in \left(\frac{(k+1)pq+1}{k+1}, \frac{kpq+1}{k}\right)$ and note that $\frac{kpq+1}{k} = (pq+1) \oplus (k-1)(pq)$. So the path from ∞ to r contains the path from ∞ to $\frac{kpq+1}{k}$, which is a continued fraction block of length k . The path from $\frac{kpq+1}{k}$ to r is equivalent, after changing basis, to the path $1/k$ to r' ; and that path is equivalent via the change of basis $\begin{pmatrix} 1 & -(k+1) \\ 0 & 1 \end{pmatrix}$ to the path from -1 to r'' . Thus

$$\Phi_\infty(r) = (k+1) \prod_{j=1}^{n_3} |a_j^3 + 1|,$$

verifying that our formula agrees with Wu's in this case.

Proof of Lemma 4.1. Recall that the a_j 's and b_i 's are defined by

$$\left(\frac{q}{p} - \left\lceil \frac{q}{p} \right\rceil\right)^{-1} = [b_1, \dots, b_n] \text{ and } \frac{q}{p} = [a_1, \dots, a_m].$$

So we have

$$[b_1, \dots, b_n] = \left(\frac{q}{p} - \left\lceil \frac{q}{p} \right\rceil\right)^{-1} = \left(\frac{q}{p} - (a_1 + 1)\right)^{-1} = -\frac{p}{q - (a_1 + 1)p}.$$

Thus $[b_n, \dots, b_1] = -p/(q - p(a_0 + 1)p)^*$. Now $q - p(a_0 + 1)p \equiv q \pmod{p}$ and $0 < q - (a_0 + 1)p < p$, so we see that $[b_n, \dots, b_1] = -p/q^*$. $\square$

Case 2: $e_0 \geq 0$. Tight contact structures on small Seifert fibered spaces with $e_0 \geq 1$ were first classified by Wu in [67], and this was generalized to $e_0 \geq 0$ by Ghiggini, Lisca and Stipsicz in [27]. We will focus on the later formulas as they handle all the cases in a similar fashion.

As discussed in Section 1.3, r surgery on a negative (p, q) -torus knot will result in a small Seifert fibered space when $r \in (pq - 1, pq)$. More specifically, if $r \in \left(\frac{kpq-1}{k}, \frac{(k+1)pq-1}{k+1}\right)$ for some k , then $e_0 = k - 1$. Referring to Figure 2, we see that the surgery coefficients on the vertical unknots are $r' = r - pq$, $q/(q - p^*)$, and $-p/q^*$. If we negatively Rolfsen twist the unknot with framing $q/(q - p^*)$, the framing changes to $-q/p^*$, and the horizontal unknot has framing -1 . Notice that $r' \in (-1/k, -1/(k + 1))$, so if we perform k positive Rolfsen twists on the unknot with framing r' , its framing will change to $r'' < -2$, and it will convert the framing on the horizontal unknot to $k - 1$. This is how we see that $e_0 = k - 1$. However, to apply the Ghiggini-Lisca-Stipsicz formula, we do not perform this last set of Rolfsen twists and instead only apply one positive Rolfsen twist to the $r' = (r - pq)$ -framed curve to obtain a curve with framing $r''' \in (-1/(k - 1), -1/k)$, and the horizontal curve now has framing 0. Figure 7 shows a Kirby diagram for a 4-manifold whose boundary is this Seifert fibered space.

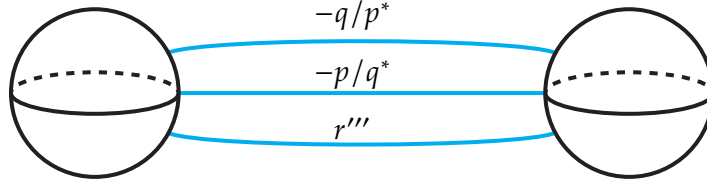


FIGURE 7. A Kirby diagram with boundary $S_r^3(T_{p,q})$.

We now consider the case when $k > 1$ (this corresponds to $e_0 > 0$). In this case we see that $r''' = [-1, a_1^1, \dots, a_{n_1}^1]$. Now if $q/p = [a_1, \dots, a_m]$ and $\left(\frac{q}{p} - \left\lfloor \frac{q}{p} \right\rfloor\right)^{-1} = [b_1, \dots, b_n]$, then from above we see that $-p/q^* = [a_m, \dots, a_1]$ and $-q/p^* = [b_n, \dots, b_1]$. By [27, Theorem 1.1] we have:

$$|\text{Tight}(S_r^3(K_{p,q}))| = \left| \left(a_m \prod_{i=1}^{m-1} (a_i - 1) \right) \left(b_n \prod_{i=1}^{n-1} (b_i - 1) \right) \left(\prod_{i=1}^{n_1} (a_i^1 - 1) \right) \right|.$$

We note that the curves in Figure 7 can be taken to be Legendrian knots with $\text{tb} = 0$, and then this formula simply computes the number of different negative contact surgeries on this link that realize the appropriate smooth surgery. We see that two of the terms give

$$m(p, q) = \left| \left(a_m \prod_{i=1}^{m-1} (a_i - 1) \right) \left(b_n \prod_{i=1}^{n-1} (b_i - 1) \right) \right|.$$

So we can write the formula from [27] as $|\text{Tight}(S_r^3(K_{p,q}))| = m(p, q) \left| \prod_{i=2}^{n_1} (a_i^1 - 1) \right|$. The term $\left| \prod_{i=2}^{n_1} (a_i^1 - 1) \right|$ counts the number of contact structures on a solid torus with convex

boundary having dividing slope 0 and lower-meridian r''' . But this is the same as the number of contact structures on the solid torus with 0-sloped convex boundary and lower-meridian r' . (One may check that their continued fractions differ by adding a -2 after the -1 in the expansion for r''' , so the formula gives the same number). Moreover, by a change of coordinates the path in the Farey graph from 0 to r' can be thought of as a path from pq to r .

Since the surgery slope r is not in J our algorithm in Section 3.1 says

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) + n(p, q) \Phi_\infty(r).$$

$\lfloor r+1 \rfloor = pq$ so $\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) = m(p, q)$ by 7 in Section 2.2. We also note that $\Phi_\infty(r) = 2\Phi_{pq}(r)$ since there is an edge in the Farey graph from ∞ to pq and that edge is not part of a continued fraction block in the path from pq to r . Thus

$$\begin{aligned} |\text{Tight}(S_r^3(T_{p,q}))| &= (m(p, q) - 2n(p, q)) \Phi_{pq}(r) + n(p, q) 2\Phi_{pq}(r) \\ &= m(p, q) \left| \prod_{i=2}^{n_1} (a_i^1 - 1) \right|, \end{aligned}$$

agreeing with the Ghiggini-Lisca-Stipsicz formula.

We are left to check the case when $k = 1$ (when $e_0 = 0$). In this case we see that $r''' = [a_0^3, a_1^3, \dots, a_{n_3}^3]$, with $a_0^3 \leq -2$. To fit with [27], we set $[a_m, \dots, a_1]$ equal to $[a_0^1, \dots, a_{m-1}^1]$ (that is $a_0^1 = a_m$, and so on), $[b_n, \dots, b_1]$ equal to $[a_0^2, \dots, a_{n-1}^2]$ and $n_1 = m-1$ and $n_2 = n-1$. With that [27, Theorem 1.1] shows that

$$\begin{aligned} |\text{Tight}(S_r^2(T_{p,q}))| &= \left| \left(\prod_{i=1}^3 (a_0^i + 1) - \prod_{i=1}^3 a_0^i \right) \prod_{i=1}^3 \left(\prod_{j=1}^{n_i} (a_j^i + 1) \right) \right|. \\ &= |a_0^1 a_0^2 + a_0^1 a_0^3 + a_0^2 a_0^3 + a_0^1 + a_0^2 + a_0^3 + 1| \left| \prod_{i=1}^3 \left(\prod_{j=1}^{n_i} (a_j^i + 1) \right) \right|. \end{aligned}$$

Our algorithm yields

$$|\text{Tight}(S_r^3(T_{p,q}))| = (\mathcal{L}_{\lfloor r+1 \rfloor}(T_{p,q}) - 2n(p, q)) \Phi_{\lfloor r+1 \rfloor}(r) + n(p, q) \Phi_\infty(r).$$

We need to understand r and r''' better. We know $r \in (pq-1, (2pq-1)/2)$ so there is some l such that $r \in [(l+1)(pq-1) \oplus pq, l(pq-1) \oplus pq)$. Thus $r' \in [-\frac{l+1}{l+2}, -\frac{l}{l+1})$. Since r''' is obtained from r' by a single positive Rolfsen twist, we see that $r''' \in [-l-1, -l)$, and so $a_0^3 = -l-1$. Now, after changing coordinates, as above, the path from r''' to $-l$ goes to the path from r' to $-1/(l+1)$, which again, after changing coordinates, is the same as the path from $l(pq-1) \oplus pq$ to r . Thus the number of tight contact structures on the solid torus with convex boundary having dividing curves of slope $l(pq-1) \oplus pq$ and lower-meridian r , is given by $|\prod_{j=1}^{n_3} (a_j^3 + 1)|$. Notice that the path from ∞ anti-clockwise to $l(pq-1) \oplus pq$

is a continued fraction block of length $l + 1$. Thus we see that $\Phi_\infty(r) = (l + 2)|\prod_{j=1}^{n_3}(a_j^3 + 1)|$ and similarly $\Phi_{pq}(r) = (l + 1)|\prod_{j=1}^{n_3}(a_j^3 + 1)|$. So our formula above becomes

$$((m(p, q) - 2n(p, q))|a_0^3| + n(p, q)|a_0^3 - 1|) \left| \prod_{j=1}^{n_3} (a_j^3 + 1) \right|.$$

We also note that

$$m(p, q) = |a_0^1(a_1^1 + 1) \cdots (a_{n_1}^1)a_0^2(a_1^2 + 1) \cdots (a_{n_2}^2 + 1)| \text{ and} \\ n(p, q) = |(a_0^1 + 1) \cdots (a_{n_1}^1)(a_0^2 + 1) \cdots (a_{n_2}^2 + 1)|.$$

Thus

$$\begin{aligned} & ((m(p, q) - 2n(p, q))|a_0^3| + n(p, q)|a_0^3 - 1|) \\ &= (|a_0^3||a_0^1a_0^2 - 2(a_0^1 + 1)(a_0^2 + 1)| - |(a_0^3 - 1)(a_0^1 + 1)(a_0^2 + 1)|) \left| \prod_{i=1}^2 \left(\prod_{j=1}^{n_i} (a_j^i + 1) \right) \right| \\ &= |a_0^1a_0^2 + a_0^1a_0^3 + a_0^2a_0^3 + a_0^1 + a_0^2 + a_0^3 + 1| \left| \prod_{i=1}^2 \left(\prod_{j=1}^{n_i} (a_j^i + 1) \right) \right|, \end{aligned}$$

and so our formula agrees with the one from [27].

It is interesting to note that the difference between the cases $k > 1$ and $k = 1$. In the former case, the path in the Farey graph from ∞ anti-clockwise to r started with the jump from ∞ to pq and then started a new continued fraction block so we saw that $\Phi_\infty(r) = 2\Phi_{pq}(r)$, and so there was a cancellation in our formula to give the simple answer of $m(p, q)\Phi_{pq}(r)$. However, when $k = 1$, the first continued fraction block in the path from ∞ to r had length longer than 1, and this prevented an easy relation between the terms in our formula.

If one did not notice that the edge from ∞ to pq when $k = 1$ was part of a continued fraction block, then one would have obtained the formula $\Phi_\infty(r) = 2(l + 1)|\prod_{j=1}^{n_3}(a_j^3 + 1)|$ instead of the correct $(l + 2)|\prod_{j=1}^{n_3}(a_j^3 + 1)|$. This overcount is because the first edge is part of a continued fraction block. In the paper [27], when $e_0 = 0$, they see that all the contact structures come from negative contact surgery on the link in Figure 7, but there is an overcount there due to the ability to “switch some basic slices between the legs in the path” (see the paper for details). These two overcounts are exactly the same, even though they are explained in different ways.

5. Partially thickenable tori

In this section, we prove Theorem 3.6. To prove this theorem, we first strengthen, in the next section, the result about partially thickenable tori discussed in Section 2.3.

5.1. Generalizing non-thickenability. We now strengthen Theorem 2.5 and 2.6 in the following two theorems, one for the trefoils and one for all other torus knots.

Theorem 5.1. *Let C be the complement of the $(2, \pm 3)$ -torus knot in S^3 . For each positive integer k there is a tight contact structure ξ_k on C for which ∂C is convex with two dividing curves of slope $1/k$ such that any convex torus in (C, ξ_k) parallel to the boundary has dividing slope $1/k$. Moreover, suppose a minimally twisting contact structure on $T^2 \times [0, 1]$, with dividing slopes s_0 and $s_1 = 1/k$, is glued to the boundary of (C, ξ_k) . If $s_0 > 0$, then any convex torus in the resulting manifold, parallel to the boundary, has dividing slope in $[s_0, 1/k]$ and would be contained in a thickened torus whose complement is (C, ξ_k) .*

Theorem 5.2. *For $(p, q) \neq (2, \pm 3)$, we define*

$$\mathcal{I} := E \cup \{1\} = \{n \in \mathbb{Z}_{\geq 1} \mid \gcd(|pq| - |p| - |q|, n) = 1\}.$$

Let C be the complement of the (p, q) -torus knot in S^3 . For every $k \in \mathbb{Z}_{\geq 1}$ and $n_k = \gcd(k, |pq| - |p| - |q|)$, there is a contact structure ξ_k on C with convex boundary with $2n_k$ dividing curves of slope $e_k = \frac{|pq| - |p| - |q|}{k}$ such that any convex torus in (C, ξ_k) that is parallel to ∂C has dividing slope e_k . Moreover, for $k \in \mathcal{I}$ and $s \in (e_k^a, e_k]$, suppose a minimally twisting contact structure on $T^2 \times [0, 1]$, with dividing slopes $s_0 = s$ and $s_1 = e_k$, is glued to the boundary of (C, ξ_k) . Then, any convex torus in the resulting manifold, parallel to the boundary, has dividing slope in $[s, e_k]$ and would be contained in a thickened torus whose complement is (C, ξ_k) .

Remark 5.3. We note that these two theorems imply that the contact structures formed by gluing the thickened tori as in the theorems to (C, ξ_k) is tight, since in an overtwisted contact structure any slope on a torus parallel to the boundary could be realized.

We begin by giving a specific decomposition of C from [15]. (The construction in [15] was only for positive torus knots, but as observed in [16], the same construction works for negative torus knots too.) Put $T_{p,q}$ on a Heegaard torus for S^3 . We can think of S^3 as a neighborhood $\mathcal{T} = T^2 \times [-1, 1]$ of the Heegaard torus, with the Heegaard torus being $T^2 \times \{0\}$, union two solid tori V_1 and V_2 . The union of these solid tori are a neighborhood of a Hopf link $F_1 \cup F_2$. We can form a neighborhood N of $T_{p,q}$ by taking an annulus neighborhood of $T_{p,q}$ on $T^2 \times \{0\}$ times $[-1, 1]$. Let A' be the complementary annulus on $T^2 \times \{0\}$ to the one used to form the neighborhood.

Then C is $V_1 \cup V_2 \cup (A' \times [-1, 1])$. Notice that one may cut C along A' to get two solid tori (isotopic to $V_1 \cup V_2$). It will also be useful to see how to build C from $V_1 \cup V_2$. To this end notice that in $A' \times [-1, 1]$ there is an annulus $A = T_{p,q} \times [-1, 1]$ and $A' \times [-1, 1]$ may be thought of as a thickening of the annulus A . Using standard longitude-meridian coordinate on ∂V_i we see that annulus A intersects ∂V_1 in a (p, q) curve and intersects ∂V_2 in a (q, p) curve.

For the convenience of visualizing C we notice that $S^3 - (F_1 \cup F_2)$ is $T^2 \times \mathbb{R}$ and we can choose coordinate on T^2 so that $T_{p,q}$ is a slope ∞ curve, call this coordinate system on T^2 , $\mathcal{C}'$. See Figure 8. In this figure, V_1 is simply $(-\infty, -1] \times T^2$ with F_1 added back in, and

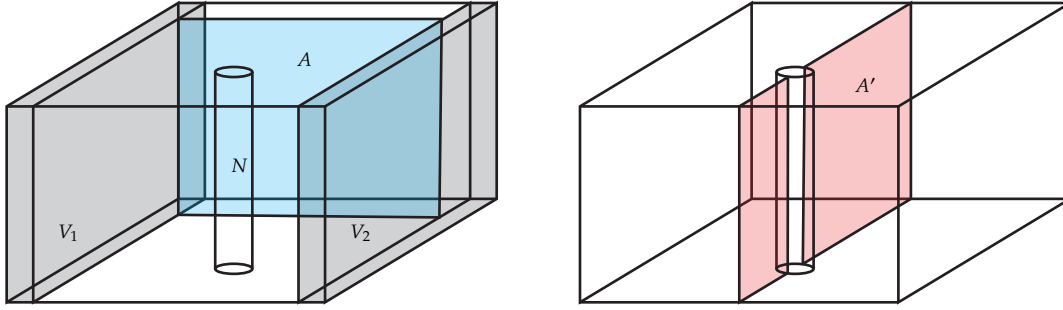


FIGURE 8. The complement C of a neighborhood N of the torus knot $T_{p,q}$ in S^3 .

similarly for V_2 . Here we see A is simply a vertical annulus. We are now ready to construct the contact structure in Theorems 5.1 and 5.2. Again, the construction below is only for positive torus knots, but a similar construction also works for negative torus knots. See [16, Section 6.3].

Construction 5.4. We start with the unique tight contact structure on V_1 with convex boundary having two dividing curves of slope $-m_1 = -(qk + 1)$ and similarly on V_2 but with slope $-m_2 = -(pk + 1)$, where k is a non-negative integer. In the coordinate system $\mathcal{C}'$, these slopes are $\frac{pm_1+q}{p'm_1+q'}$ and $\frac{qm_2+p}{q'm_2+p'}$, so A intersects the dividing curves on ∂V_1 and ∂V_2 the same number of times. Thus, we can take an I invariant contact structure on $A \times I$ where the dividing curve on A runs from one boundary component to another and intersects the boundary components of A the same number of times the dividing curves on ∂V_i do. The contact structure on $A \times I$ can be glued to the ones on the V_i to give a contact structure ξ_k on C .

In Section 3 of [15] it was shown that ξ_k is universally tight and ∂C is convex with dividing slope $s_k = \frac{pq-p-q}{k}$ and $2n_k$ dividing curves where $n_k = \gcd(pq - p - q, k)$. Lemma 3.3 in [15] showed that any convex torus in ∂C with two dividing curves that was parallel to the boundary was contact isotopic to the boundary, or, to say another way, cobounded with the boundary an I -invariant contact structure on $T^2 \times I$.

We note the important fact that (C, ξ_1) is the complement of the unique maximal Thurston-Bennequin invariant (p, q) torus knot in (S^3, ξ_{std}) . $\square$

We begin with a simple observation about this construction.

Lemma 5.5. *Using the notation from the construction above, consider (C, ξ_k) . Let L_i be the Legendrian core of V_i (recall the V_i are standard neighborhoods of Legendrian knots) and let $\hat{A}$ be an extension of A to an (immersed) annulus with boundary on the L_i (so one boundary component of $\hat{A}$ will wrap p times around L_1 and the other will wrap q times around L_2). Let $\hat{V}_i$ be any small standard neighborhood of L_i , we can isotope $\hat{A}$ so that it intersects ∂V_i in ruling curves and is convex. Let C' be the union of $\hat{V}_1, \hat{V}_2$, and an I -invariant neighborhood of the part of $\hat{A}$ outside of*

$\widehat{V}_1 \cup \widehat{V}_2$. Then there is a contact isotopy from (C, ξ_k) to $(C', \xi_k|_{C'})$. Said another way, any small neighborhood of $L_1 \cup L_2 \cup \widehat{A}$ will contain a neighborhood that is contact isotopic to (C, ξ_k) .

Proof. From Construction 5.4, we know $(C', \xi_k|_{C'})$ is contactomorphic to (C, ξ_k) , so the contact structure on $C \setminus C'$ is an I -invariant contact structure on $T^2 \times I$. Thus the result follows. $\square$

We need a few lemmas before turning to the proof of Theorem 5.1. These are largely adaptations of lemmas in Section 3 of [15], though given our more general situation, they require more care.

Lemma 5.6. *Let ξ' be a minimally twisting contact structure on $T^2 \times [0, 1]$, with dividing slopes s_0 and $s_1 = e_k$. Glue this to (C, ξ_k) along $T^2 \times \{1\}$ and obtain (C_0, ξ) . If $s_0 \in (e_k^a, e_k]$, then there can be no bypass for ∂C in $T^2 \times [0, 1]$ attached along a ruling curve of slope $0'$. (The slope $0'$ refers to the 0 slope in the coordinates $\mathfrak{C}'$ described above.)*

Proof. If a bypass was attached along this slope, then there would be a convex torus in $T^2 \times [0, 1]$ with dividing slope e_k^a , contradicting the fact that ξ' is minimally twisting. $\square$

Lemma 5.7. *Using the notation from Construction 5.4, consider (C, ξ_k) and assume that $\gcd(k, pq - p - q) = 1$. Let V_1 and V_2 be the solid tori used in the construction of C and A the annulus used. Let ξ' be a minimally twisting contact structure on $T^2 \times [0, 1]$, with dividing slopes s_0 and $s_1 = e_k$. Glue this to (C, ξ_k) along $T^2 \times \{1\}$ to get (C_0, ξ) . If $s_0 \in (e_k^a, e_k]$ and $\widehat{A}$ is any convex annulus properly embedded in $C_0 \setminus (V_1 \cup V_2)$ with boundary ruling curves on $\partial V_1 \cup \partial V_2$ parallel to ∂A such that $\widehat{A}$ is isotopic to A , then the dividing curves on $\widehat{A}$ run from one boundary component to the other and there is a contact isotopy of (C_0, ξ) taking $V_1 \cup V_2 \cup A$ to $V_1 \cup V_2 \cup \widehat{A}$.*

Proof. We first consider the case that $\widehat{A}$ is disjoint from A . Clearly, the dividing curves of $\widehat{A}$ run from one boundary component to the other, because if not, then we would have a bypass for ∂C in $T^2 \times [0, 1]$ contradicting Lemma 5.6. (To see this notice that by Lemma 5.5 the complement $V_1 \cup V_2 \cup N(A)$, where $N(A)$ is a small neighborhood of A disjoint from $\widehat{A}$, is isotopic to (C, ξ_k) and hence its complement in (C, ξ_k) is an I invariant contact structure on $T^2 \times I$ which can be added to $(T^2 \times [0, 1], \xi')$ to get another thickened torus contactomorphic to $(T^2 \times [0, 1], \xi')$ and containing the forbidden bypass.)

Now A and $\widehat{A}$ cobound $A \times [0, 1]$ with each $A \times \{t\}$ intersecting ∂V_i in a ruling curve. So there is a sequence of bypass attachments that will take A to $\widehat{A}$. But after each bypass attachment, the dividing curves still run from one boundary component to the other by the argument above. Thus it was a trivial bypass attachment and so after choosing a different product structure on $A \times [0, 1]$, we can assume that each $A \times \{t\}$ is convex by Lemma 2.10 in [37]. So we have a contact isotopy preserving the V_i and taking A to $\widehat{A}$.

If $\widehat{A}$ is not disjoint from A then we can use isotopy discretization [5] (or [11] for its use in studying Legendrian knots). This gives a sequence of convex annuli A_i starting with A and ending with $\widehat{A}$ such that each A_i has boundary made of ruling curves on ∂V_i and each

A_{i+1} obtained from attaching a bypass to A_i . Thus, each A_i is disjoint from the next one, and so using the above argument, we inductively build an isotopy from the annulus A to the annulus $\widehat{A}$. $\square$

We now move on to the proof of Theorems 5.1 and 5.2 concerning adding contact structures on thickened tori to (C, ξ_k) .

Proof of Theorems 5.1 and 5.2. With the above lemmas in place, the proof is very similar to the proof of [15, Proposition 3.7], but we sketch the argument here for the convenience of the reader.

Let ξ' be a minimally twisting contact structure on $T^2 \times [0, 1]$, with dividing slopes $s_0 \in (e_k^n, e_k]$ and $s_1 = e_k$. Glue this to (C, ξ_k) along $T^2 \times \{1\}$ to get (C_0, ξ) .

Assume for contradiction that T is a convex torus in (C_0, ξ) with dividing slope outside of $[s_0, e_k]$. We can take an annulus $\widetilde{A}$ with one boundary component a ruling curve on ∂C_0 and the other a dividing curve on T (so $\widetilde{A}$ is an annulus whose slope agrees with the dividing slope of T). There are enough bypasses $D_1, \dots, D_n$ along $\widetilde{A}$ for ∂C_0 to take ∂C_0 to T . (That is T is obtained from ∂C_0 by these n bypass attachments.) If these bypasses were contained in $(T^2 \times [0, 1], \xi')$ then this would be a contradiction as we would see a torus T' with the same dividing slope as T in $T^2 \times [0, 1]$, contradicting the fact that $(T^2 \times [0, 1], \xi')$ is minimally twisting. We now argue that $(T^2 \times [0, 1], \xi')$ can be isotoped in (C_0, ξ) so that it contains the D_i . This contradiction will establish the theorem.

To this end let L_i be the Legendrian core of V_i (as in Lemma 5.5, see also Construction 5.4) and let $\widehat{A}$ be the annulus from Lemma 5.5. Set $X = L_1 \cup L_2 \cup \widehat{A}$. As observed in Lemma 5.5, (C, ξ_k) can be isotoped to be contained in any arbitrarily small neighborhood of X . We now show that we can isotopy X (so that (C, ξ_k) can still be isotoped into any small neighborhood of X) so that it is disjoint from the bypasses D_i . This will complete the proof. We will argue this for a single D_i , which we denote D , and observe that the same argument can be applied to the other D_j .

We may isotope the L_i (and hence all of X) so that they are disjoint from ∂D . A contact isotopy of D will now make D transverse to X . So $X \cap D$ consists of closed circles, vertices (corresponding to the intersection of D with the L_i), and arcs. Now let $N(L_i)$ be a small standard neighborhood of L_i , we can isotope X so that $X \setminus [(N(L_1) \cup N(L_2)) \cap X]$ is a convex annulus $\widehat{A}'$ with boundary ruling curves on $\partial N(L_i)$, and still intersecting D transversely. By choosing the $N(L_i)$ sufficiently small, we can assume that $N(L_i) \cap D$ is a collection of disks.

We now show how to remove the closed curves of intersection between D and X . Take an innermost such simple closed curve (that is, one that bounds a disk that does not contain any further intersections with X). From above, this curve is part of the intersection of D with $\widehat{A}'$. So we can isotope $\widehat{A}'$ to remove this circle of intersection points (notice that this may remove other intersection points of $\widehat{A}'$ with D , or other D_i , but it will not increase the number of intersections). We can make the isotoped $\widehat{A}'$ convex and by Lemma 5.7 we

see that this can be made a contact isotopy. So the new X will have the same properties as the old X , but have (at least) one fewer circle of intersection with D .

Now let γ be an arc of intersections between D and X that connects two vertices. Take a sub arc γ' that is disjoint from $N(L_1) \cup N(L_2)$. We may isotop $\hat{A}'$ as above to remove γ' from the intersection of D and X . This will turn the arc between two vertices into two arcs, each going from a vertex to the boundary of D . Thus $D \cap X$ consists only of properly embedded arcs and “stars”, that is a vertex with edges going from it to the boundary. Now if γ is an outermost properly embedded arc (that is it separates off a disk from D that has no other intersections with X), then it comes from an intersection with $\hat{A}'$ and so can be removed as above.

We are left to consider outermost stars. The vertex of such a star comes from the intersection with L_1 or L_2 ; we assume L_1 , with the other case being analogous. In this case, the star will have p edges. We can consider the subset N consisting of the union of $N(L_1)$, $N(L_2)$, and a small I invariant neighborhood of $\hat{A}'$. Recall from Lemma 5.5 we know that N is contact isotopic to (C, ξ_k) . We may choose N so that its intersection with D is a neighborhood of the star. See Figure 9 for the case of $p = 3$. The boundary of $N \cap D$

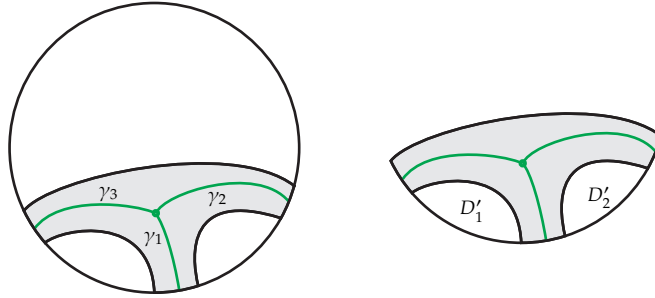


FIGURE 9. On the left is the intersection of a neighborhood of X with D near an outermost star. On the right is this disk D' together with the subdisks D'_1 and D'_2 .

consists of p arcs, $\gamma_1, \dots, \gamma_p$, and we assume that γ_p separates a D into two disks, one of which contains the star and no other intersections with X (this is the outermost condition on the star). Call this disk D' . Now, each of the γ_i for $i < p$ will separate off a disk D'_i from D' that does not contain the star. Let T' be the result of pushing ∂N across one of the D'_i , $i < p$, and making it convex. This torus is contained in the complement of C and hence by Lemma 5.5 in $(T^2 \times [0, 1], \xi')$. Thus, the slope of its dividing curves must be in $[s_0, e_k] \subset (e_k^a, e_k]$. But notice that we can assume that T' contains a ruling curve on $\partial N(L_2)$ since we do not need to move that part of ∂C to push across D'_i . This ruling curve has twisting (relative to the framing coming from $\partial N(L_2)$) equal to $-(pq(k-1) + p + q)$ (this is because the ruling curves have slope (p, q) and the dividing curves on $\partial N(L_2)$ have slope $-(q(k-1) + 1)$, see Construction 5.4). But the ruling curve of this slope on a convex torus with dividing slope in $(e_k^a, e_k]$ will intersect a dividing curve greater than or

equal to $-(pq(k-1) + p + q)$ with equality if and only if the slope is e_k (and there are only two dividing curves). To see this, recall that the slope of this ruling curve in the Seifert coordinates on ∂C is pq , which is outside the interval $(e_k^a, e_k]$, and thus its intersection with a curve of slope in the interior of the interval will be some positive linear combination of its intersection with e_k and e_k^a , and the intersection with e_k is $-(pq(k-1) + p + q)$. Thus, we can think of T as obtained from ∂C by a sequence of bypass attachments from one side and at every step the slope of the dividing curves does not change. That is, this is a contact isotopy. Thus, by a contact isotopy, we can push ∂C over all the D'_i , $i < p$. Now the disk D' is entirely contained in C . Pushing ∂C over D' will not change the dividing slope since we know that all convex tori inside of C parallel to the boundary have dividing slope equal to ∂C and hence are contact isotopic to ∂C . Thus, we have removed the outermost star from D .

By successively removing outermost disk and stars from D , we eventually make D disjoint from X and thus contained in $(T^2 \times [0, 1], \xi')$, obtaining our contradiction and thus establishing the theorem.

The final statement in the theorems clearly follows from the above arguments. $\square$

5.2. Dehn fillings of non-thickenable tori. In this section, we will establish Proposition 3.6. Recall this proposition claims that the result of Dehn filling the complement (C, ξ_k) , discussed in Section 2.3, of a non-thickenable torus in the knot type of $T_{p,q}$ where $r \in J_k$ and $k \in E \cup \{1\}$ is a tight contact structure, they are all distinct from negative contact surgery on Legendrian knots, and Dehn filling (C, ξ_k) and (C, ξ_l) for $l \neq k$ gives distinct contact structures. When $(p, q) \neq (2, \pm 3)$ the last statement is obvious since the J_k are disjoint, so it is only meaningful for $(p, q) = (2, \pm 3)$. Lastly, we show distinct choices of contact structures on the solid torus for Dehn filling result in distinct contact structures.

Proof of Proposition 3.6. We first prove tightness. Given (C, ξ_k) and some $r \in J_k = [e_k^a, e_k]$, let ξ' be any tight contact structure on the solid torus S_r with convex boundary having dividing curves of slope s_k . (Recall the notation S_r of a solid torus with lower meridian r from Section 2.1.) We need to show that (M, ξ) , obtained by gluing (C, ξ_k) and (S_r, ξ') together, is tight. To this end, suppose (M, ξ) contains an overtwisted disk D . Let L be a Legendrian representative of the dual surgery curve in (S_r, ξ') , and L' the result of some number of stabilizations of L . Then L' can be isotoped to be disjoint from D . Thus, D is in the complement of a neighborhood, a standard neighborhood of L' . But since $r \in J_k$ we know that the slope of the dividing curves on the boundary of a neighborhood N' of L' are in J_k too (we are measuring the slopes with respect to the original longitude-meridian coordinates on $T_{p,q}$). Since N' is isotopic to a subset of (S_r, ξ') , we know that its complement is (C, ξ_k) with a thickened torus $(T^2 \times [0, 1], \xi'')$ attached, where the boundary slopes on this thickened torus are $s_0 \in J_k$ and s_k . But from either Theorem 5.1 or 5.2, see Remark 5.3, we know that this manifold is tight, contradicting the existence of D in it.

To prove the Dehn fillings of (C, ξ_k) are distinct from negative contact surgery on Legendrian knots, we establish the following property. Let K' be the surgery dual of $T_{p,q}$

in the (M, ξ) . Any solid torus in the knot type of K' with convex boundary has dividing slope bounded above by e_k . Notice that when this is established, we will see that (M, ξ) could not have come from negative contact surgery on a Legendrian knot, as we know that in such a manifold we can find solid tori in the knot type of K' with integer slopes larger than e_k . (Notice that we are not considering the non-destabilizable Legendrian knots with $\text{tb} = pq$ when $pq < 0$, since negative contact surgery on them will never smoothly agree with (M, ξ) . For example, the resulting small Seifert fibered spaces will have distinct values for e_0 .) We also notice that when the above property is established, we can also distinguish the Dehn fillings of (C, ξ_k) and (C, ξ_l) for $l \neq k$, and hence complete the proof.

Suppose there is a solid torus S' in (M, ξ) with convex boundary having dividing slope larger than e_k . Let T be the boundary of (C, ξ_k) in (M, ξ) . We know that $T' = \partial S'$ and T are smoothly isotopic. So we can use a standard discretization of isotopy argument [5, 37] that says there is a sequence of convex tori $T_1, \dots, T_k$ such that $T_1 = T$, $T_k = T'$, and T_i is obtained from T_{i-1} by a bypass attachment. We will inductively show that T_i is contained in a solid torus S_i with convex boundary having dividing slope e_k and whose complement is (C, ξ_k) . This will lead to a contradiction, as T' cannot be in such a solid torus (as it would imply the solid torus is overtwisted); thus, S' could not have existed. Clearly, T_1 satisfies the inductive hypothesis with $S_1 = S_r$. Now assume that T_i satisfies the inductive hypothesis. If T_{i+1} is obtained from T_i by attaching a bypass inside the solid torus it bounds, then clearly it also satisfies the hypothesis with $S_{i+1} = S_i$. Now, assume that the bypass is attached from the outside of the solid torus. Then note that (M, ξ) with S_i removed is (C, ξ_k) with a thickened torus added, where the outer boundary of the thickened torus is larger than e_k^a . Thus, by Theorems 5.1 and 5.2 the torus T_{i+1} has (C, ξ_k) in its complement and so there is a torus S_{i+1} as claimed.

Lastly, suppose we choose two different tight contact structures on S_r . Note that Dehn filling (C, ξ_k) using S_r is equivalent to Dehn filling (C, ξ_k) using $S_{e_k^a}$, followed by a sequence of Legendrian surgeries on the core of $S_{e_k^a}$ and its (stabilized) push-offs. Distinct contact structures on S_r correspond to different choices of stabilizations. Recall that $|e_k \cdot e_k^a| = 1$, so there is a unique tight contact structure on $S_{e_k^a}$, so there is a unique tight contact structure on $S_{e_k^a}(T_{p,q})$ from the Dehn filling. By Theorem 1.23 for negative torus knots and Theorem 1.24 for positive torus knots, we know this tight contact structure has non-vanishing contact invariant. According to the results of Plamenevskaya [61] and Simone [62], different Spin^c structures on Weinstein cobordisms from a fixed contact structure induce distinct contact invariants on the other end. This implies that distinct choices of stabilizations yield non-isotopic contact structures. $\square$

Remark 5.8. We note that the last argument in the proof could be proven by a state transition argument similar to the one earlier in the proof.

6. Bordered contact invariants and non-loose knots

We construct contact structures on $S^3_r(T_{p,q})$ via negative contact surgery on Legendrian knots in contact structures on S^3 , which may be overtwisted. To show that the resulting contact structure is tight, we use the bordered contact invariants to compute its Heegaard Floer contact invariant.

In Section 6.1, we review some relevant facts about bordered contact invariants from [58]. The reader may consult [68, 69] for general background on bordered sutured Floer homology and [33] for the immersed curves. In Section 6.2, we characterize the bordered contact invariants of the complement of a non-loose Legendrian representative for $T_{p,q}$. In Section 6.3, we prove a general Legendrian surgery result on non-loose L-space knots and prove Theorem 1.28 and Proposition 3.7.

6.1. Bordered contact invariants and immersed curves. Let $(Y, \mathcal{F}, \Gamma)$ be a bordered sutured 3-manifold; this is a manifold with boundary such that $\mathcal{F} \subset \partial Y$ is parametrized, and Γ is a set of dividing curves on $\partial Y \setminus \mathcal{F}$. Also let $\mathcal{A} = \mathcal{A}(\mathcal{F})$ be the strand algebra associated to the surface $\mathcal{F}$. To $(Y, \mathcal{F}, \Gamma)$ we may associate a type-A module $\widehat{BSA}(Y)$, which is an $\mathcal{A}_\infty$ module over $\mathcal{A}$, as well as a type-D structure $\widehat{BSD}(Y)$. There is an algebraic pairing of bordered modules that recovers the sutured Floer homology of a pair of bordered sutured manifolds glued along their common parametrized boundaries: if $(Y_1, \mathcal{F}, \Gamma_1)$ and $(Y_2, -\mathcal{F}, \Gamma_2)$ are bordered sutured 3-manifolds, then

$$SFC(Y_1 \cup_{\mathcal{F}} Y_2, \Gamma_1 \cup \Gamma_2) \cong \widehat{BSA}(Y_1) \boxtimes \widehat{BSD}(Y_2) \cong \widehat{BSD}(Y_2) \boxtimes \widehat{BSA}(Y_1).$$

For a bordered sutured 3-manifold $(Y, \mathcal{F}, \Gamma)$ and any idempotent $\iota \in \mathcal{A}$, there exists a bordered sutured manifold $\mathcal{W}_\iota = (\mathcal{F} \times [0, 1], -\mathcal{F}, \Gamma_\iota)$ called the *elementary cap* associated to ι . We have the following natural identification:

$$(20) \quad \widehat{BSA}(Y) \cdot \iota = \widehat{BSA}(Y) \boxtimes \widehat{BSD}(\mathcal{W}_\iota) \cong SFC(Y, \Gamma \cup \Gamma_\iota)$$

Now suppose ξ is a contact structure on Y such that for some idempotent $\iota \in \mathcal{A}$, the contact structure induces dividing set $\Gamma \cup \Gamma_\iota$. Then there is a well-defined contact invariant $c_A(\xi) \in \widehat{BSA}(Y)$, characterized by

$$(21) \quad c_A(\xi) = c_A(\xi) \cdot \iota, \quad [c_A(\xi)] = EH(\xi) \in SFH(Y, \Gamma \cup \Gamma_\iota)$$

under the identification (20).

Given $c_A(\xi)$ we may also define $c_D(\xi) = c_A(\xi) \boxtimes \iota^\vee \in \widehat{BSA}(Y) \boxtimes \widehat{BSDD}(AZ)$. Here, AZ is the Auroux-Zarev Heegaard Diagram depending only on $\mathcal{F}$, and $\iota^\vee$ is a specific generator corresponding to the idempotent ι . We have the following pairing theorem, which allows us to recover the sutured contact invariant by pairing type-A and type-D invariants.

Theorem 6.1 (Theorem 1.2 of [58]). *If $(Y_1, \mathcal{F}, \Gamma_1)$ and $(Y_2, -\mathcal{F}, \Gamma_2)$ are bordered sutured 3-manifolds with contact structures ξ_1 and ξ_2 , respectively (compatible with the parametrization of $\mathcal{F}$), then*

$$[c_A(\xi_1) \boxtimes c_D(\xi_2)] = EH(\xi_1 \cup \xi_2) \in SFH(Y_1 \cup_{\mathcal{F}} Y_2, \Gamma_1 \cup \Gamma_{\mathcal{T}\mathcal{W}} \cup \Gamma_2)$$

Here, $\xi_1 \cup \xi_2$ is the *gluing* of two contact structures, so some twisting on the dividing set appears. Hence, our resulting dividing set is $\Gamma_1 \cup \Gamma_{\mathcal{T}\mathcal{W}} \cup \Gamma_2$ with a twisting slice $\mathcal{T}\mathcal{W}$ from AZ diagram inserted. See [58, Section 2] for more details.

For a given parametrized surface $\mathcal{F}$, Mathews [52] has demonstrated a correspondence between homology classes in $\mathcal{A}(\mathcal{F})$ and tight contact structures on $\mathcal{F} \times [0, 1]$ with elementary dividing sets. For a cycle $a \in \mathcal{A}(\mathcal{F})$, let us denote by ξ_a the corresponding contact structure on $\mathcal{F} \times [0, 1]$.

Theorem 6.2 (Corollary 1.9 of [58]). *Let $(Y, \mathcal{F}, \Gamma)$ be a bordered sutured manifold with a compatible contact structure ξ . Then for every cycle $a \in \mathcal{A}(\mathcal{F})$, we have*

$$c_A(\xi \cup \xi_a) = m_2(c_A(\xi), a).$$

For a bordered sutured 3-manifold with torus boundary $(Y, \mathcal{F}, \Gamma)$, if $\mathcal{F} = T^2 \setminus D^2$ and Γ is a trivial arc in D^2 , unlinked with respect to the parametrizing arcs (see Figure 10a), then we can identify $\widehat{BSA}(Y, \mathcal{F}, \Gamma)$ with $\widehat{CFA}(Y, \mathcal{F})$ and work of Hanselman–Rasmussen–Watson [33] interprets the bordered invariants and the pairing theorem in terms of immersed curves in the punctured torus $T^2 \setminus \{*\}$. Indeed, the punctured torus is canonically identified with $\mathcal{F}$ so that the generators of the bordered module correspond to intersection points between the immersed curves and the parametrizing arcs. In the case of a type-A module, the m_i operations are encoded with certain arcs as in Figure 10b. There is a similar correspondence in the type-D case.

The pairing of type-A and type-D modules is performed by homotoping the associated curves to lie in the upper right and lower left of the torus (in the usual square fundamental domain, thought of as the complement of the parametrizing arcs which are the edges of the square) except for purely vertical and horizontal arcs. The intersection points between the type-A and type-D arcs correspond to generators of the paired complex. The differential involves counting immersed bigons between generators; see Figure 10c.

A convenient model for the type-A or type-D structure can be obtained from such an immersed curve diagram by “pulling tight” all the curve components, treating the puncture as a disk with small radius. In such a basis, there are no nontrivial differentials, so $\widehat{CFA} = H_*(\widehat{CFA})$ and $\widehat{CFD} = H_*(\widehat{CFD})$ as $\mathcal{A}_\infty$ -modules [39]. For such a “pulled tight” basis, it is possible to also give a contact geometric interpretation of higher $\mathcal{A}_\infty$ -operations.

Proposition 6.3 (Min–Park–Varvarezos [57]). *Let (Y, ξ) be a contact 3-manifold with convex torus boundary. Suppose the induced dividing curve Γ_s is a pair of parallel curves of slope s . Let $\Gamma_{s'}$ be a pair of parallel curves on ∂Y of slope s' such that $|s \cdot s'| = 1$, and $\mathcal{F}$ a parametrization of ∂Y with basis s' and s . Suppose $EH(\xi) \neq 0$ and that in the basis corresponding to the “pulled-tight” immersed curves,*

$$c_A(\xi) = m_4(y, \rho_3, \rho_2, \rho_1) + \text{possibly other basic generators}$$

for some $y \in \widehat{BSA}(-Y)$. Then for the positive basic slice $(B(\Gamma_s, \Gamma_{s'}), \xi_0)$, $EH(\xi \cup \xi_0) \neq 0$.

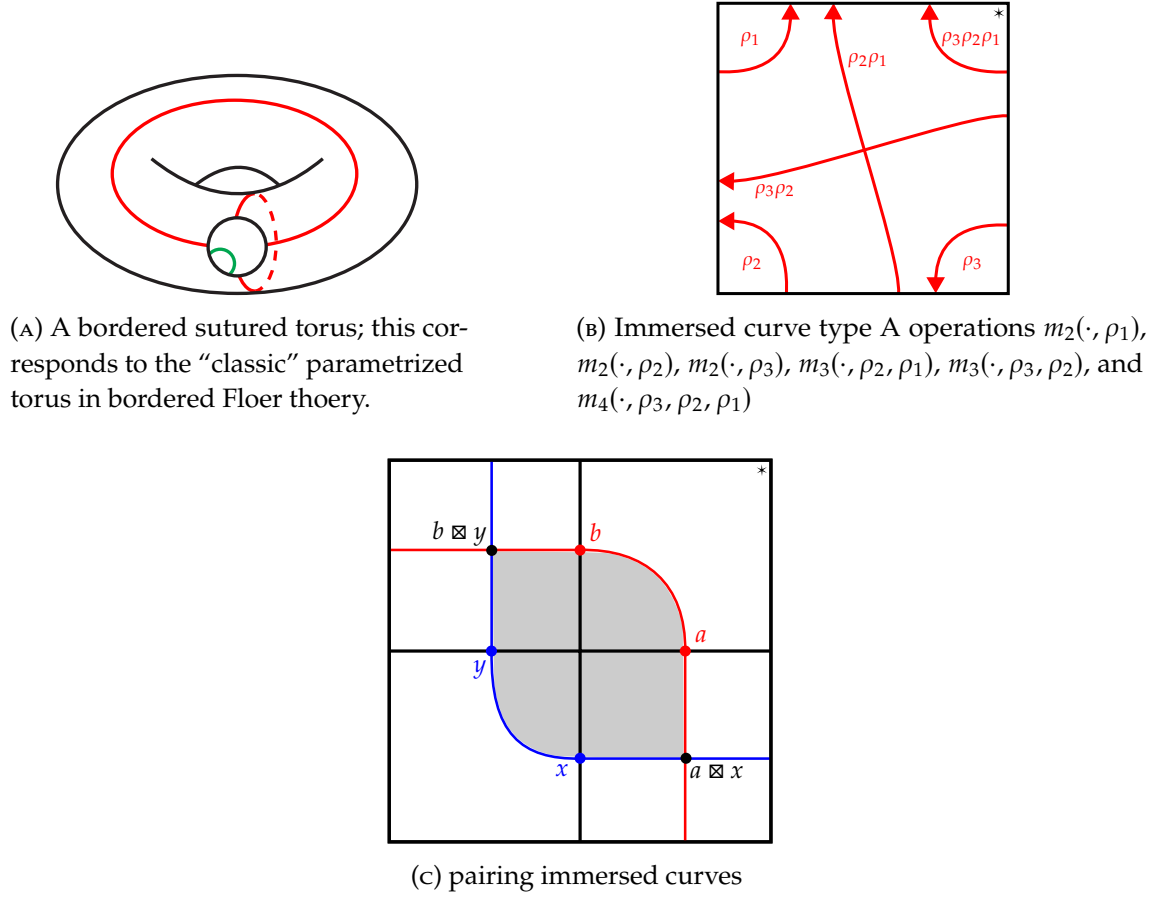


FIGURE 10. Immersed curves for torus boundary

In the case of the complement of a knot $K \subset S^3$, the immersed curve invariant encodes several invariants from knot Floer homology. There is a standard lift of the immersed curve invariant of $S^3 \setminus K$ to the infinitely punctured cylinder, such that the meridian edge lifts to the vertical unbounded line, whereas the Seifert longitude is the horizontal closed curve. When the curves are “pulled tight”, all but one component (the so-called “distinguished component” γ_0 of the immersed (multi)curve can be homotoped to lie within an arbitrarily small neighborhood of the vertical line passing through the punctures. The component γ_0 wraps once around the longitudinal direction. See Figure 11 for an example. The intersection points of the immersed curve with the vertical line correspond to generators of $HFK(S^3, K)$, and the height of each intersection corresponds to the Alexander grading of its corresponding generator. In the “pulled tight” configuration, the maximal and minimal height of such intersections are $g(K)$ and $-g(K)$. The heights where the non-vertical segment of γ_0 intersect the vertical line correspond to Alexander gradings $\tau(K)$ and $-\tau(K)$.

We will be interested in the case of L-space knots. For us, an *L-space knot* is a knot which admits an L-space surgery; in particular, all torus knots in S^3 are L-space knots. If $K \subset S^3$

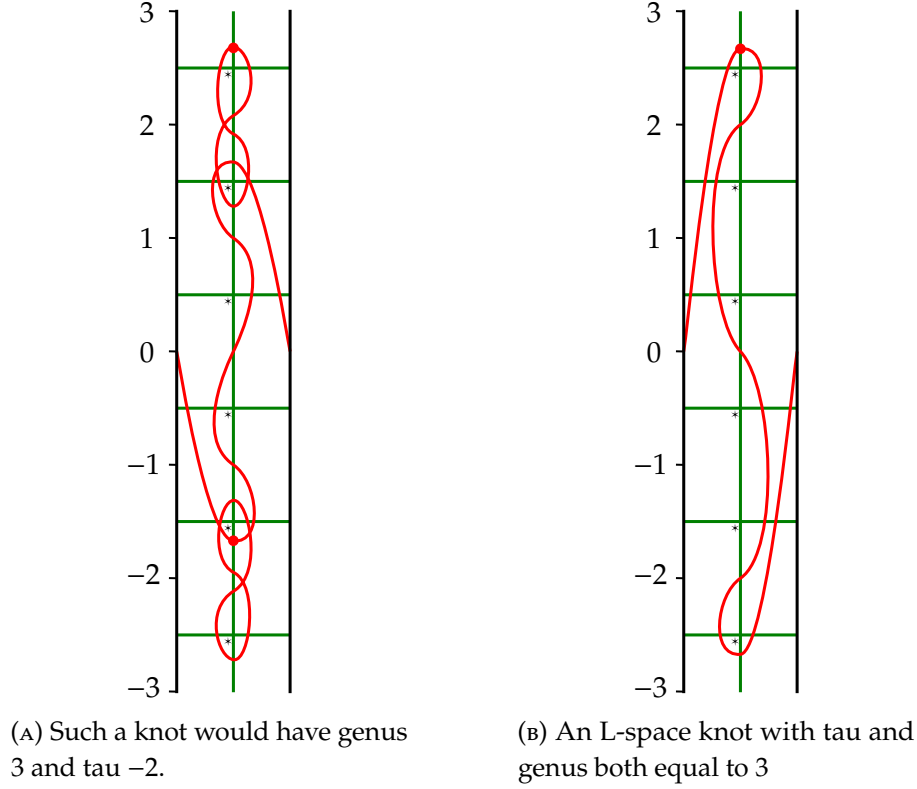
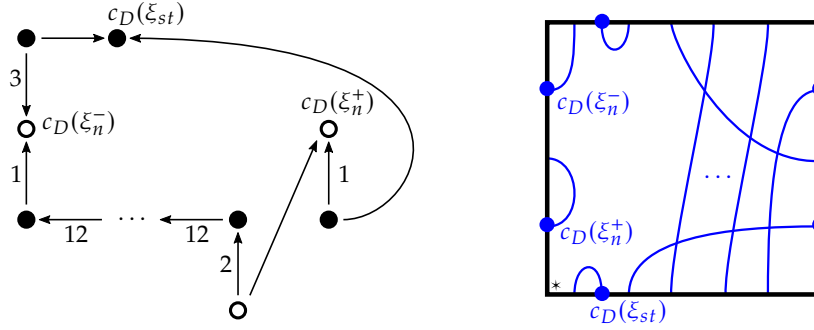
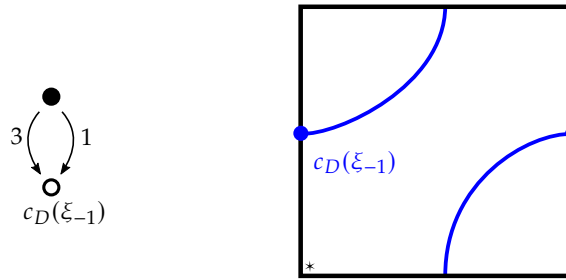


FIGURE 11. Example of lifted immersed curve invariants for knot complements. The marked intersection points are at heights corresponding to the genera and tau values of the knots.

is an L-space knot, then for each Alexander grading $s \in \mathbb{Z}$, $\text{rk HFK}(K, s) \leq 1$ [60]. The immersed curve invariant of K is known to have a special form: in the cylindrical lift, it consists of a single embedded curve that “weaves” through the punctures without any backtracking. This follows from the correspondence between CFK and immersed curves described in [32]; see also [64, Section 4.1]. In particular, $\tau(K) = \pm g(K)$ for such knots; see Figure 11b for an example.

In [58, Section 4.3] the second and fourth authors computed the contact invariants for an n -framed solid torus $S^1 \times D^2$. Consider $n < 0$. Recall that the contact invariants live in $\widehat{CFD}(-S^1 \times D^2)$, so that we must consider bordered sutured invariants of the $-n$ -framed solid torus. That is, the parametrization consists of a longitude λ and a curve of slope $-1/n$. There is a unique tight contact structure ξ_{st} on $(S^1 \times D^2, \Gamma_\lambda)$, where Γ_λ is a pair of longitudes. There are $|n|$ tight contact structures on $(S^1 \times D^2, \Gamma_{1/n})$; we distinguish two of them: ξ_n^+ and ξ_n^- , characterized by the fact that $\xi_n^\pm \cup B_\pm(\Gamma_{1/n}, \Gamma_\lambda) = \xi_{st}$, where $B_\pm(\Gamma, \Gamma')$ denotes a (positive or negative) basic slice from dividing curve Γ to Γ' . For Legendrian surgery, we specialize to the case $n = -1$. In this case, both ξ_{-1}^+ and ξ_{-1}^- are the same

(A) A type D structure for an $-n$ -framed solid torus along with its corresponding immersed curve

(B) A type D structure for a 1-framed solid torus with its corresponding immersed curve.

FIGURE 12. Type D invariants for an n -framed solid torus and the special case of $n = -1$

contact structure as Γ_{-1} is longitudinal, hence we denote it by ξ_{-1} . Figure 12 shows the type-D invariant corresponding to the 1-framed solid torus as well as the type-D contact invariant for ξ_{-1} .

Let K be a knot in S^3 and $Y = S^3 \setminus N(K)$. We parametrize ∂Y with the meridian μ and a longitude λ . Under this parametrization, $\widehat{CFA}(Y)$ has generators in two idempotents, which we denote by ι_0 and ι_1 corresponding to the meridian and longitude, respectively.

According to [33], the immersed curve is naturally thought of as a subset of $\partial Y \setminus \{*\} = T_\bullet^2$. Therefore, reparametrization of the boundary corresponds to simply applying the appropriate mapping class to the punctured torus. In particular, if Y is the complement of a knot, changing the framing corresponds to a shear transformation. This can be seen in the cylindrical lift as well; see Figure 13 for an example. In that example, vertical, horizontal, and slope -2 lines are shown. The fundamental domains for the torus invariant with parametrization ϕ_0 are the horizontal rectangles, while the fundamental domains for the parametrization ϕ_{-2} are the parallelograms with vertical and diagonal sides.

We note that diagonal lines of different slopes intersect the immersed curve invariant at potentially different number of points, corresponding to the different ranks of $SFH(C, \Gamma_s)$ for different slopes s . Moreover, this difference arises only from the intersections with the unique non-vertical segment of γ_0 . In the “pulled tight” configuration, these variable

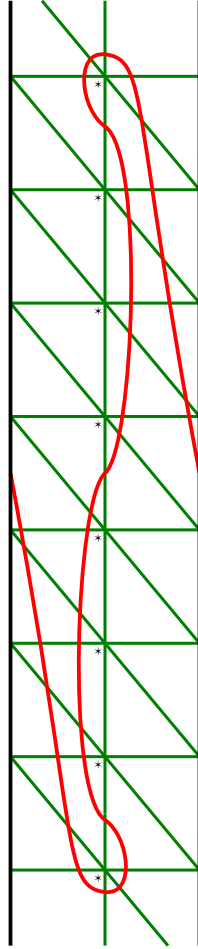


FIGURE 13. Immersed curve in the cylinder: the vertical line corresponds to the meridian, horizontal lines are Seifert longitudes, and the diagonal lines are (-2) -framed longitudes.

generators are related to one another via $m_3(\cdot, \rho_2, \rho_1)$. This is exactly the “unstable chain” described in [41, Chapter 11].

6.2. Bordered contact invariants for torus knot complements. Let (C, Γ_μ) be a sutured 3-manifold with torus boundary which is the complement of an L-space knot $K \subset S^3$, where Γ_μ is a pair of meridional dividing curves. Here, we use the notation $C(\infty)$ for (C, Γ_μ) . Similarly, we use the notation $C(n)$ for (C, Γ_n) where Γ_n is a pair of sutures with slope n . Let ξ_∞ be a contact structure on $C(\infty)$. Note that $SFH(-C(\infty))$ may be canonically identified with $\widehat{HFK}(S^3, -K)$, so that $EH(\xi_\infty)$ can be identified with a homology class in $\widehat{HFK}(S^3, -K, s(\xi_\infty))$, where $s(\xi_\infty)$ is the Alexander grading corresponding to the Spin^c structure associated to ξ_∞ .

In general, we parametrize the boundary of C so that one parametrizing arc corresponds to a curve of slope m and the other corresponds to a curve of slope n . Let us denote this bordered manifold by $C(m, n)$.

We would like to categorize certain kinds of generators in $\widehat{CFA}(C(\infty, n))$ according to their $\mathcal{A}_\infty$ operations. This will allow us to locate contact invariants from the gluing behavior. From the natural identifications (20), we have: $\widehat{CFA}(C(\infty, n)) \cdot \iota_0 = SFC(C(\infty))$ and $\widehat{CFA}(C(\infty, n)) \cdot \iota_1 = SFC(C(n))$.

First, we locate generators which are potential contact invariants for $C(\infty)$.

Definition 6.4. Let $x \in \widehat{CFA}(C(\infty, n))$ be a homogeneous (with respect to Spin^c grading) cycle in the idempotent corresponding to the meridian. We say x is (algebraically) inner-type if $[m_2(x, \rho_1)] \neq 0$ and $[m_2(x, \rho_3)] \neq 0$. If, moreover, the Alexander grading of $x \in SFH(C(\infty)) = \widehat{HFK}(S^3, K)$ is extremal; i.e., $|A(x)| = |\tau(K)| = g(K)$, then we say x is algebraically V-type. Otherwise, we say x is algebraically X-type.

Remark 6.5. If K is a positive L-space knot (that is, $\tau(K) > 0$) or $p \geq -2g(K)$ then $\widehat{CFA}(C(\infty, n))$ has no V-type generators. This is because in these cases, for an extremal-grading generator x , at least one of $m_2(x, \rho_1)$ and $m_2(x, \rho_3)$ is nullhomologous; see Figure 14.

We give conditions for a contact structure to have an inner-type contact invariant:

Lemma 6.6. Let C be the complement of an L-space knot $K \subset S^3$ admitting a contact structure ξ_∞ on $C(\infty)$. In particular, suppose $(B_\pm(\infty, n), \xi_n^\pm)$ is a positive/negative basic slice from Γ_μ to Γ_n . If $EH(\xi_\infty \cup \xi_n^\pm) \neq 0$ then $c_A(\xi_\infty) \in \widehat{CFA}(-C(\infty, n))$ is algebraically inner-type.

Proof. By Theorem 6.2, we know that $m_2(c_A(\xi_\infty), \rho_1) = c_A(\xi_\infty \cup \xi_n^+)$ and $m_2(c_A(\xi_\infty), \rho_3) = c_A(\xi_\infty \cup \xi_n^-)$. Moreover, by (21), $[c_A(\xi_\infty \cup \xi_n^\pm)] = EH((\xi_\infty \cup \xi_n^\pm))$ which is nonzero by hypothesis. Hence $c_A(\xi_\infty)$ satisfies the definition of an inner-type generator. $\square$

Since K is an L-space knot, the $\text{rk } SFH(C(\infty), \mathfrak{s}) = 1$ where $\mathfrak{s}$ is the Spin^c structure of ξ_∞ . and hence is generated by a single generator x . If $c_A(\xi_\infty)$ is nontrivial in homology, it follows that $c_A(\xi_\infty) = x \in \widehat{CFA}(-C(\infty, n))$ for every meridian-longitude parametrization (μ, λ_n) .

Let us consider the case where x is X-type. From the grading restriction, we see that in the immersed curve invariant, x is only adjacent to “nearly vertical” segments of the pulled-tight immersed curves; see Figures 14b and 14a. Those segments remain nearly vertical for any meridian-longitude parametrization (μ, λ_n) . Hence the local picture will be as in Figure 15a. In particular, $m_2(x, \rho_1)$ and $m_2(x, \rho_3)$ are nontrivial in homology for every n . As a consequence, we conclude the following:

Corollary 6.7. Let C be the complement of an L-space knot $K \subset S^3$ admitting a contact structure ξ_∞ on $C(\infty)$. If $c_A(\xi_\infty)$ is X-type, then for every $p \in \mathbb{Z}$, attaching a basic slice $(B_\pm(\infty, n), \xi_n^\pm)$ to $C(\infty)$ results in a tight contact structure on $C(n)$; in particular, $EH(\xi_\infty \cup \xi_n^\pm) \neq 0$.

A similar result holds for V-type invariants. In particular, suppose x is a top-grading V-type generator. From the structure of the immersed curve, we see that there is a “nearly vertical” segment immediately down and to the right of x . Hence, in any meridian-longitude parametrization, there is a nontrivial ρ_3 action. On the other side, we see there is a nontrivial ρ_1 action exactly when the longitudinal framing is less than $2g$. In that case, the local model is as in Figure 15c. The situation is reversed in the case of a bottom-grading V-type invariant; see Figure 15d. By Theorem 6.2, we conclude:

Corollary 6.8. *Let C be the complement of an L-space knot $K \subset S^3$ admitting a contact structure ξ_∞ on $C(\infty)$. Suppose $c_A(\xi_\infty)$ is V-type. If $c_A(\xi_\infty)$ is maximal grading, then for every n , $EH(\xi_\infty \cup \xi_n^-) \neq 0$, whereas $EH(\xi_\infty \cup \xi_n^+) \neq 0$ exactly when $n \geq 2g + 1$. On the other hand, if $c_A(\xi_\infty)$ is minimal grading, then for every n , $EH(\xi_\infty \cup \xi_n^+) \neq 0$, whereas $EH(\xi_\infty \cup \xi_n^-) \neq 0$ exactly when $n \geq 2g + 1$.*

According to this corollary, an infinite X of a torus knot (except for one) corresponds to X-type generators. That is, the complement of a Legendrian representative in an infinite X thickens to ξ_∞ such that $c_A(\xi_\infty)$ is an algebraically X-type generator. Also, the infinite V and X obtained by half convex Lutz twist from (S^3, ξ_{std}) (see [16, Section 2]) of a torus knot corresponds to V type generators.

Next, we describe generators which are potential contact invariants for a wing-type Legendrian knot complement. It is most natural to describe them in a basis corresponding to the “pulled-tight” configuration of immersed curves. Recall that this “pulled-tight” basis can be thought of as one without nontrivial differentials, or in other words, a basis for $H_*(\widehat{CFA}(C))$, with the (equivalent) $\mathcal{A}_\infty$ module structure.

Definition 6.9. Let $x \in \widehat{CFA}(C(\infty, n))$ be a *basic* generator in the idempotent corresponding to the longitude. We say x is (algebraically) *W-type* if:

- (i) $m_2(x, \rho_2) = 0$ and x is not in the image of $m_4(\cdot, \rho_3, \rho_2, \rho_1)$;
- (ii) x is not in the image of $m_2(\cdot, \rho_1)$ or $m_2(\cdot, \rho_3)$

More generally, let $x \in SFC(-C(n); \mathfrak{s}) \subset \widehat{CFA}(-C(\infty, n))$ be homogeneous (with respect to Spin^c grading). We say x is (algebraically) *W-type* if, in the “pulled tight” basis, x is a nontrivial linear combination of basis elements at least one of which is a W-type generator and x satisfies condition (i).

In terms of the immersed curve invariant, the curve near a W-type basic generator does not immediately turn towards the vertical parametrizing arc; rather, the curve has at least one vertical pass both above and below the generator; see Figure 15b. The geometric motivation for this is that, as explained in Section 2.2.1, wing-type Legendrians neither thicken to infinity nor do their complements admit a tight bypass attachment to $C(\infty)$. By Theorem 6.2 and Proposition 6.3, the contact invariant of a wing-type Legendrian complement must satisfy condition (i).

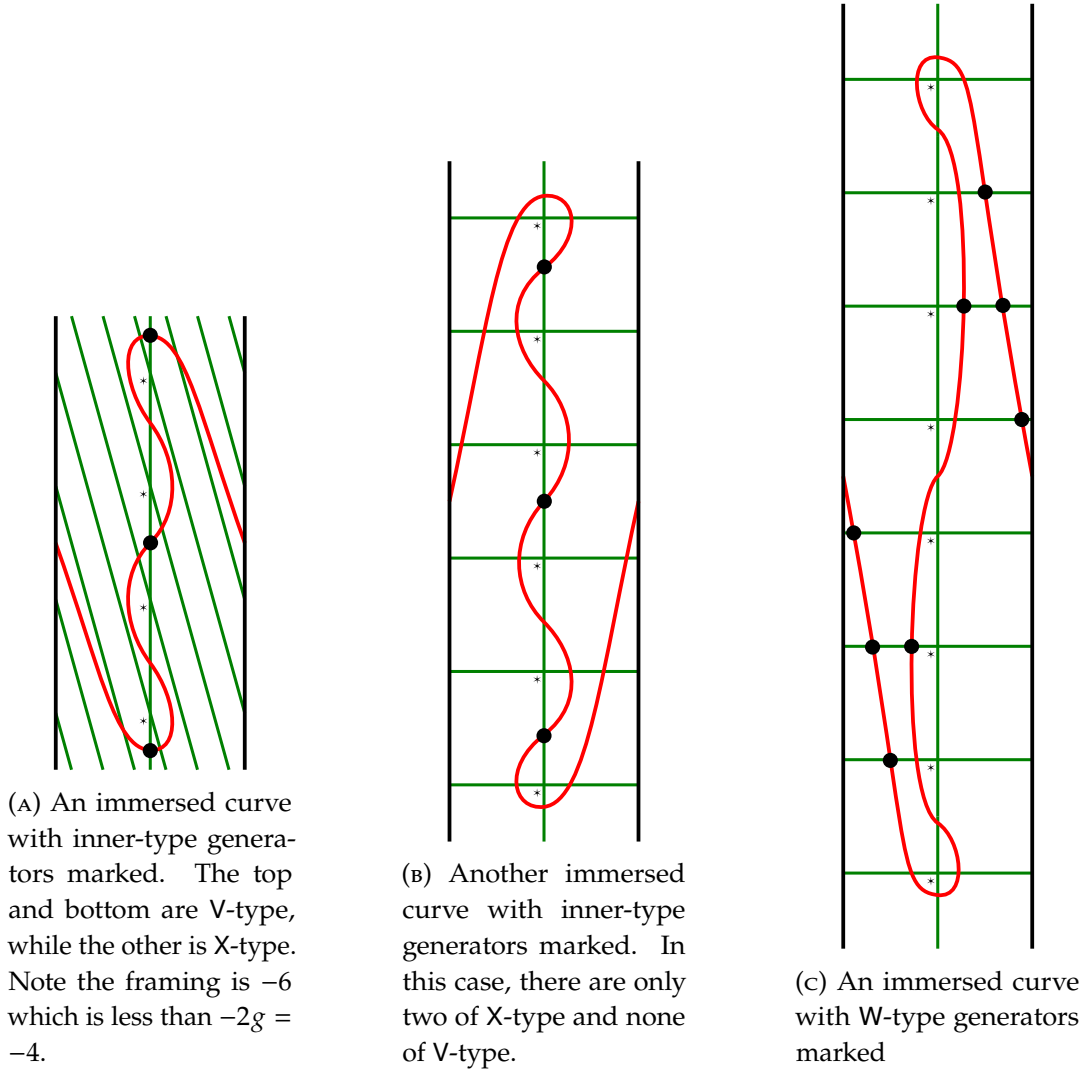


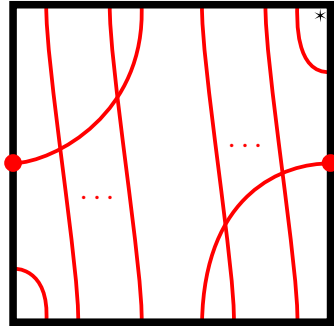
FIGURE 14. Types of generators on the cylinder

Remark 6.10. If x is a W-type basic generator, the vertical behavior of the immersed curve near x described earlier implies:

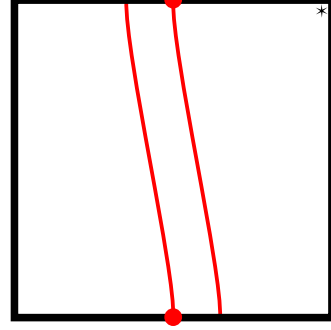
- (i) $m_3(x, \rho_2, \rho_1) \neq 0$ and
- (ii) $x = m_3(y, \rho_2, \rho_1)$ for some y .

In [57], it is also shown that for two Legendrian knot complements $\xi_1, \xi_2 \in C(n)$, if the Legendrians have a common stabilization, in particular, $\xi_1 \cup B_+(n, n-1) = \xi_{stab} = \xi_2 \cup B_-(n, n-1)$, with $EH(\xi_{stab}) \neq 0$, then in the basis corresponding to the “pulled-tight” immersed curves, we have:

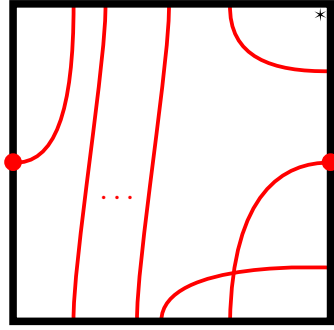
$$c_A(\xi_2) = m_3(c_A(\xi_1), \rho_2, \rho_1) + \text{possibly other basic generators}$$



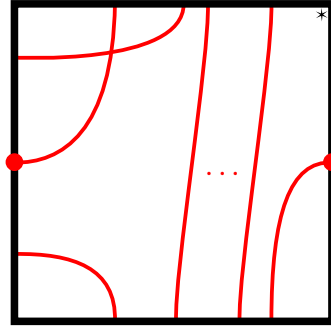
(A) An X-type generator



(B) A W-type generator



(C) A V-type generator in the maximal grading



(D) A V-type in the minimal grading

FIGURE 15. Types of generators on the torus.

More generally, for a pair contact structures that become the same after attaching k positive and k negative basic slices, respectively (as in the case of the complements of Legendrians which become the same after k stabilizations), their invariants are related by a composition of k such m_3 operations if their shared stabilization has non-vanishing EH invariant. This is the other geometric motivation for the definition of W-type: since wing-type Legendrians are related to X and V-type Legendrians via stabilizations, we should expect their contact invariants to be in the image of $m_3(\cdot, \rho_2, \rho_1)$.

We claim the complement of a wing-type Legendrian torus knot has contact invariant that, in the “pulled-tight” basis, is a W-type basic generator.

We first consider the wing-type Legendrian complements with the maximal Thurston-Bennequin invariant, i.e., $tb = pq$:

Lemma 6.11. *Let (C, ξ) be the complement of a wing-type Legendrian representative of $T_{p,q}$ with $tb = pq$. Then $c_A(\xi)$ is a W-type generator in the “pulled-tight” basis.*

Proof. Recall that a wing type Legendrian knot L is associated to an infinite X or an infinite V . That means, after some number of all positive (or all negative) stabilizations, L^{stab} lies on an infinite X or a V so that its complement contains a $(C(\infty), \xi_\infty)$ without half convex torsion. By Lemma 6.6, we know that $c_A(\xi_\infty)$ is algebraically inner type. Moreover, by Corollaries 6.7 and 6.8, the complement of the (standard neighborhood of the) stabilization L^{stab} has nonvanishing EH invariant. Thus, by Theorem 6.2 and Remark 6.10, $c_A(\xi)$ satisfies

$$c_A(\xi) + \text{other generators} = \underbrace{m_3(\cdot, \rho_2, \rho_1) \circ \cdots \circ m_3(\cdot, \rho_2, \rho_1) \circ m_2(\cdot, \rho_1)}_j (c_A(\xi_\infty))$$

for some $j \geq 1$. We note that the right-hand side of the above equation is homologous to $m_{j+2}(c_A(\xi_\infty), \rho_{12}, \dots, \rho_{12}, \rho_1)$ by the $\mathcal{A}_\infty$ relations. Because $|pq| > 2g$ and L is an L-space knot, there is only one generator in each Spin^c grading as pq is in the range of L-space surgeries (this can also be seen from the immersed curve via reparametrization: a lift to the cylinder of a line of slope $-pq$ will intersect the immersed curve at most once). Thus, we may conclude that $c_A(\xi)$ must be a basic generator so that there are no other generators in the above sum.

Let us consider the case when L is associated to an infinite X via negative stabilizations (the case of positive stabilizations is symmetric). By the structure of the immersed curve of L-space knot complements, this means x must be one of the generators labelled w_j in Figure 16a or the generator labelled y . Notice that each of the generators labelled w is a W-type basic generator. However, we can rule out y from contributing to $c_A(\xi)$ because $m_2(y, \rho_2) \neq 0$, while $m_2(c_A(\xi), \rho_2) = 0$ by Theorem 6.2. Hence $c_A(\xi)$ is W-type.

Now we consider the case where the wing is related to a V-type Legendrian, which we may assume to be via only negative stabilizations (both sides of the V are reachable by either all positive or all negative stabilizations). We see in Figure 16b that $c_A(\xi)$ must be one of the generators labelled w_j or the one labelled y . Once again, each of the former is W-type, so we only need to rule out the possibility that $c_A(\xi) = y$. To see this, we observe that the conjugate contact structure $\bar{\xi}_\infty$ has non-vanishing invariant, as shown in Figure 16b. Moreover, $m_2(c_A(\bar{\xi}_\infty), \rho_3) = y$. Therefore, y is the contact invariant of the (conjugate) V-type Legendrian complement with $\text{tb} = pq$. Since $pq > 2g - 1$, the Spin^c structure of each wing-type Legendrian complement with $\text{tb} = pq$ is distinct from that of the V-type complements with $\text{tb} = pq$. Therefore, $c_A(\xi)$ and y have different gradings so that $c_A(\xi) \neq y$, as desired. $\square$

We next investigate the behavior of W-type generators under stabilization.

Lemma 6.12. *Let $c_A(\xi) \in \widehat{\text{CFA}}(-C(\infty, n))$ be a W-type basic generator in the “pulled-tight” basis. If the immersed curve directly above $c_A(\xi)$ continues vertically twice or turns left after the first vertical pass, then $c_A(\xi \cup B_+(n, n-1)) \in \widehat{\text{CFA}}(-C(\infty, n-1))$ is a W-type basic generator as well.*

Similarly, if the immersed curve directly below $c_A(\xi)$ continues vertically twice or turns right after the first vertical pass, then $c_A(\xi \cup B_-(n, n-1)) \in \widehat{\text{CFA}}(-C(\infty, n-1))$ is a W-type basic generator as well.

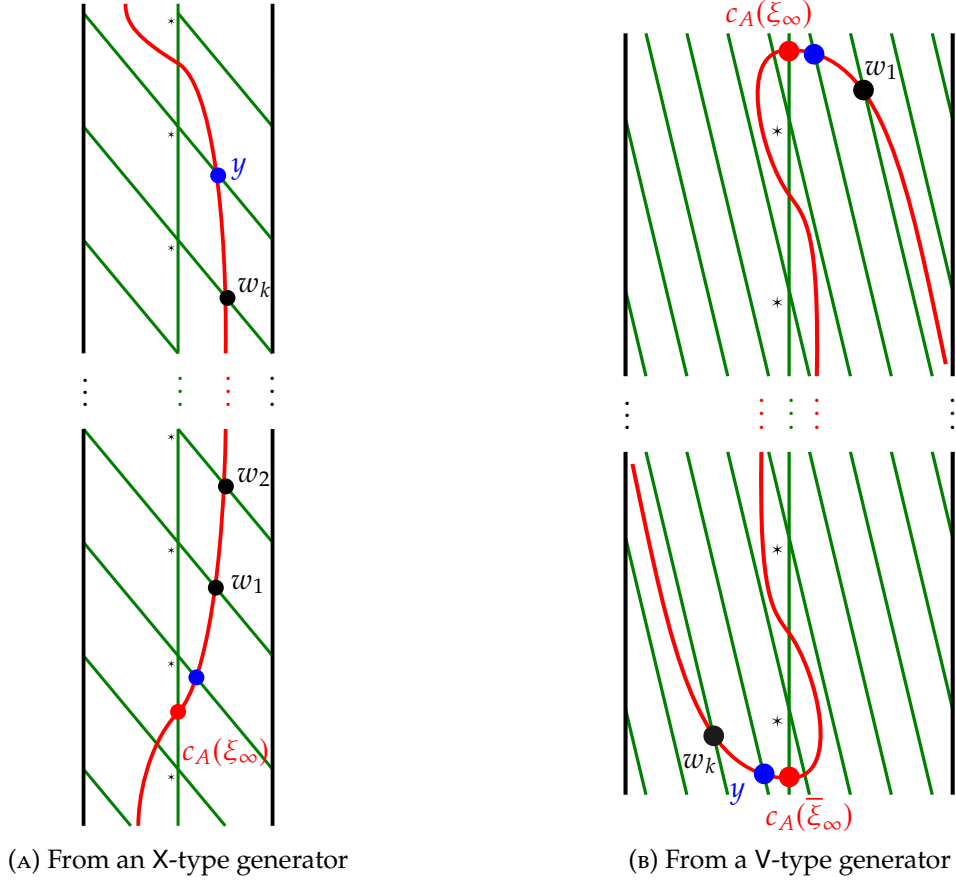


FIGURE 16. W-type generators for two kinds of wing-type Legendrian complements.

Proof. Figure 17 shows the local pictures of the immersed curves near $c_A(\xi)$ in the two cases. Three parametrizations are shown: (μ, λ_{-n}) , $(\lambda_{-n}, \lambda_{-n+1})$, and (μ, λ_{-n+1}) indicated by the square and parallelogram fundamental toroidal domains. By Theorem 6.2, $c_A(\xi \cup B_+(n, n-1)) = m_2(c_A(\xi), \rho_1)$, and hence we have that $c_A(\xi \cup B_+(n, n-1)) = y$ in $\widehat{CFA}(-C(n, n-1))$.

By the naturality of contact invariant under reparametrization [57], we have that $c_A(\xi \cup B_+(n, n-1)) = y$ in $\widehat{CFA}(-C(\infty, n-1))$. From the figure, we see that y is a W-type basic generator, as desired.

The argument is analogous for the case of $B_-(n, n-1)$. □

We are now in a position to prove that wing-type Legendrian torus knot complements have contact classes that are W-type generators and in fact are distinguished by their contact classes.

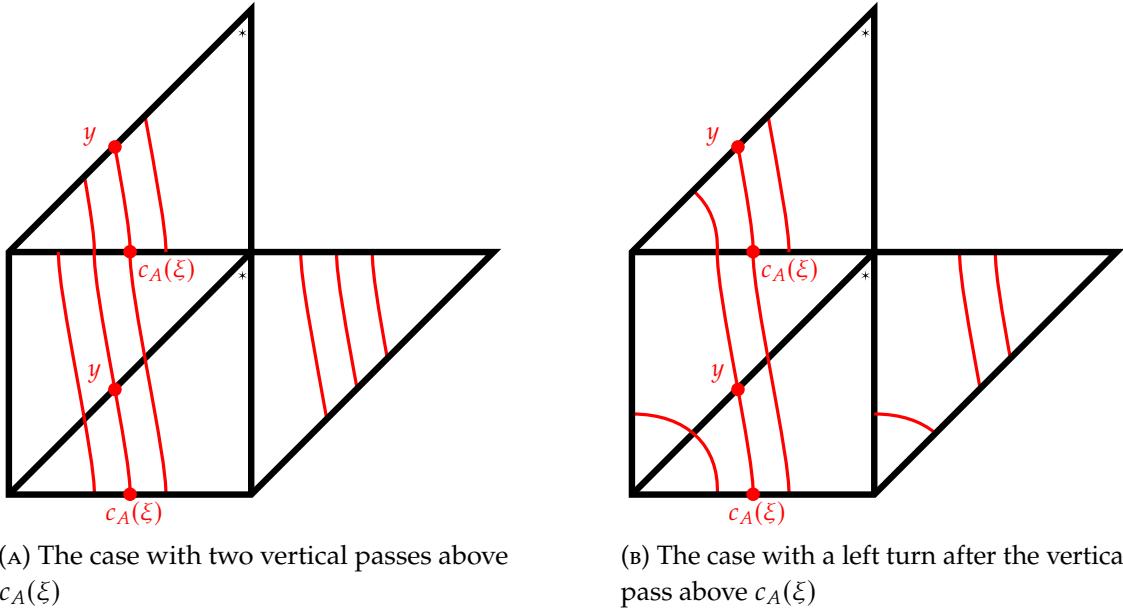


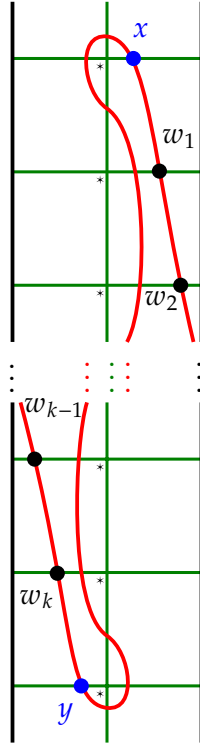
FIGURE 17. The effect of stabilization on certain kinds of W-type generators.

Proposition 6.13. *Let (C, ξ) be the complement of a wing-type Legendrian torus knot L in a contact structure on S^3 . Then $c_A(\xi)$ is an algebraically W-type basic generator. Moreover, if L' is a different non-loose Legendrian representative of the same torus knot with complement (C, ξ') then $c_A(\xi) \neq c_A(\xi')$.*

Proof. Every wing-type Legendrian is obtained from a $tb = pq$ wing-type Legendrian $\tilde{L}$ via a sequence of stabilizations. Now L and $\tilde{L}$ are related to some X- (or V-)type Legendrians via stabilizations, and the path of stabilizations from $\tilde{L}$ to L may be chosen in such a way so as to have no intermediate Legendrians from the edges of the X (or V); that is, through intermediate wing-type Legendrians.

Let us denote the complement of a standard neighborhood of $\tilde{L}$ by $(C, \tilde{\xi})$. From Lemma 6.11, we know that $c_A(\tilde{\xi})$ is a W-type generator. Indeed, from the proof of that Lemma, we know that $c_A(\tilde{\xi})$ is a generator of the form w_j in Figure 16, with $1 \leq j \leq k$. Note that j is precisely the number of stabilizations needed to reach the invariants of an X (or V)-type Legendrian. (In the case of V-type, the conjugate edge may be reached via $k - j$ stabilizations.) Since the path of stabilizations from $\tilde{L}$ to L can be assumed to be just through other wing-type Legendrians, at each step, the hypotheses of Lemma 6.12 will be satisfied, so that in particular, $c_A(\xi)$ will be W-type.

Now suppose L' is a non-loose Legendrian representative of $T_{p,q}$ distinct from L , with complement (C, ξ') . We claim that $c_A(\xi) \neq c_A(\xi')$. There are several cases to consider, in light of the classification of non-loose Legendrian torus knots in [16]. Firstly, we consider X and V type Legendrians. According to Lemma 6.6 if L' were such a Legendrian

FIGURE 18. W-type generators in (S^3, ξ_{std})

complement, then Theorem 6.2 implies $c_A(\xi') = m_2(x, \rho_i)$ where x is inner-type and $i = 1, 3$. In particular, $c_A(\xi')$ would not be W-type so that it could not be equal to $c_A(\xi)$.

Next we consider the case L' is also wing-type. From the proof of Lemma 6.11 and the above argument, we know that if L and L' are not associated to the same X or V, then $c_A(\xi)$ and $c_A(\xi')$ will belong to distinct arcs of the immersed curve and hence cannot be equal. On the other hand, if are associate to the same X or V, then $c_A(\xi) = w_j$ and $c_A(\xi') = w_i$ within the same arc of the immersed curve. However, as we saw in the above argument, $|i - j|$ is precisely the number of stabilizations needed to make them equivalent; therefore, if L' is a distinct Legendrian, $i \neq j$ so that $c_A(\xi) \neq c_A(\xi')$.

□

6.3. Legendrian surgery on non-loose L-space knots. Now we prove the results about Legendrian surgery on non-loose L-space knots. We consider X-type Legendrian knots first.

Proposition 6.14. *Let L be a Legendrian L-space knot with $\text{tb} = n$ in a contact structure on S^3 and (C, ξ) the complement of a standard neighborhood of L . Suppose that (C, ξ) thickens to $(C(\infty), \xi_\infty)$ such that $c_A(\xi_\infty) \in \widehat{\text{CFA}}(-C(\infty, n))$ is algebraically X-type. Then Legendrian surgery on L results in a contact structure ξ_L such that $\widehat{\text{c}}(\xi_L) \neq 0$.*

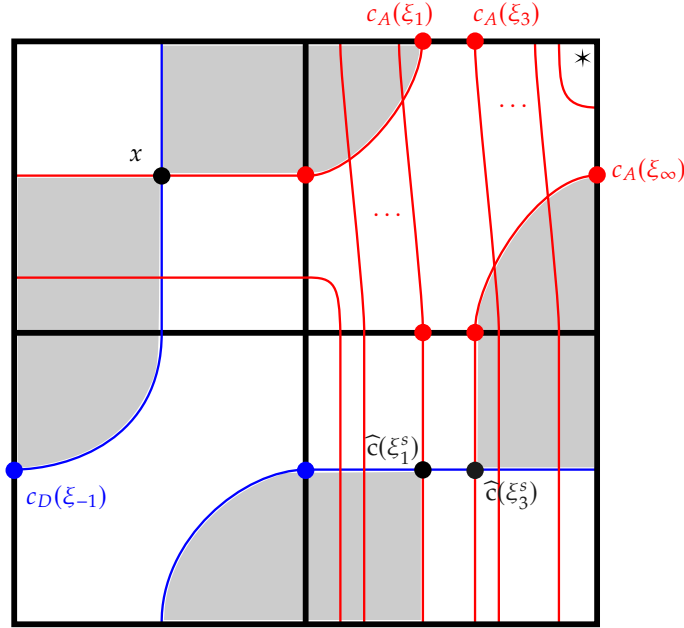


FIGURE 19. Immersed curve computation of contact invariant of Legendrian surgery for X-type contact invariants

Moreover, let L' be another Legendrian whose complement thickens to $(C(\infty), \xi'_\infty)$. If $c_A(\xi_\infty) \neq c_A(\xi'_\infty)$ then $\widehat{c}(\xi_L) \neq \widehat{c}(\xi_{L'})$.

Proof. According to Definition 6.4, we know that $EH(\xi_\infty) \neq 0$ and $EH(\xi) \neq 0$. We also note that $c_A(\xi) = m_2(c_A(\xi_\infty), \rho_1)$ or $c_A(\xi) = m_2(c_A(\xi_\infty), \rho_3)$. We define $\xi_1 = \xi_\infty \cup \xi_+$ and $\xi_3 = \xi_\infty \cup \xi_-$, where $\xi_\pm$ are positive and negative basic slices $B_\pm(\infty, n)$. Thus ξ is one of ξ_1 and ξ_3 . We denote the contact structure obtained by contact Dehn filling on ξ_1 (resp. ξ_3) by ξ_1^s (resp. ξ_3^s). Clearly, ξ_L is one of ξ_1^s or ξ_3^s . We now claim that $\widehat{c}(\xi_1^s) = \widehat{c}(\xi_3^s) \neq 0$. Figure 19 shows the computation with immersed curves. We see that there is a single bigon terminating at $c(\xi_1^s)$, originating at a generator which we denote x . There is another bigon originating at x terminating at the generator $c(\xi_3^s)$. We see that there are no other bigons involving these three generators, so that $\langle x, c(\xi_1^s), c(\xi_3^s) \rangle$ is a summand of $CF(-S^3_{n-1}(L))$ with differential: $\partial(x) = c(\xi_1^s) + c(\xi_3^s)$. Thus, $c(\xi_1^s)$ survives in homology, and is only homologous to $c(\xi_3^s)$. This completes the proof. $\square$

Remark 6.15. It is also possible to perform the above computations using the n -framed solid torus invariants in Figure 12a and interpreting the contact manifold as surgery on a Legendrian of higher tb.

We now turn to the case of wing-type Legendrian torus knots.

Proposition 6.16. *Let L be a Legendrian L-space knot in a contact structure on S^3 , and (C, ξ) the complement of a standard neighborhood of L . If $c_A(\xi)$ is a W-type basic generator, then Legendrian*

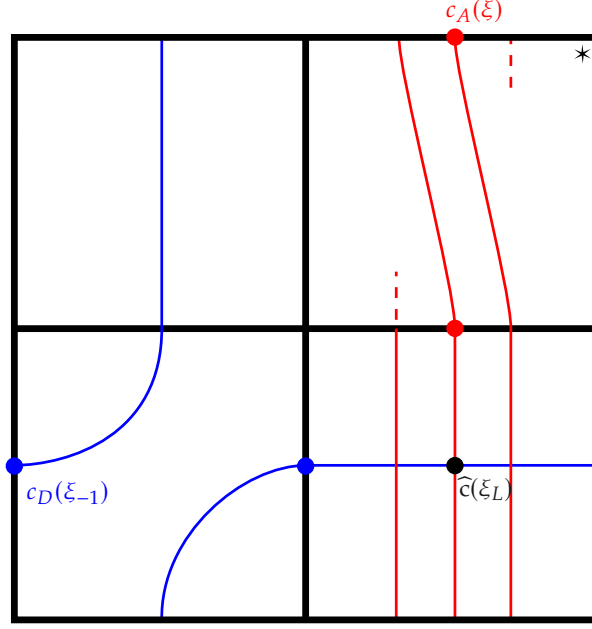


FIGURE 20. Immersed curve computation of Legendrian surgery on a W-type basic generator.

surgery on L results in a contact structure ξ_L with $\widehat{c}(\xi_L) \neq 0$. Moreover if ξ' is a contact structure on the complement of a Legendrian L' such that $c_A(\xi')$ does not include $c_A(\xi)$ in its “pulled-tight” basis decomposition, then $\widehat{c}(\xi_L) \neq \widehat{c}(\xi_{L'})$.

Proof. Figure 20 shows the computation of $\widehat{c}(\xi_L) = c_A(\xi) \boxtimes c_D(\xi_{-1})$ with immersed curves. We see that there are no bigons at all involving the generator $\widehat{c}(\xi_L)$. Hence, it survives in homology. Furthermore, it is homologically linearly independent of any other generator of the form $x \boxtimes c_D(\xi_{-1})$, which implies the second claim. $\square$

Lastly, we are ready to prove the main results.

Proof of Proposition 3.7. We first prove that a contact structure on $S^3_r(T_{p,q})$ obtained from negative contact $(r - \lfloor r + 1 \rfloor)$ -surgery on L with $\text{tb} = \lfloor r + 1 \rfloor$ is tight, where L is either a non-loose torus knot without half convex torsion in its complement or a Legendrian knot in (S^3, ξ_{std}) . Recall that there are three types of such non-loose torus knots, infinite X, V, and wings. Since $-1 < r - \lfloor r + 1 \rfloor < 0$, this contact surgery can be converted into a sequence of Legendrian surgeries on L and its (stabilized) push-offs by [8]. By Proposition 6.14, if L is in an infinite X (except for the one obtained from the convex Lutz twist on (S^3, ξ_{std})) then Legendrian surgery on L results in a contact structure ξ_L with non-vanishing contact invariant. If L is in the infinite V or the infinite X obtained from the convex Lutz twist on (S^3, ξ_{std}) , then negative contact surgery on L is equivalent to some positive contact surgery on a Legendrian representative in (S^3, ξ_{std}) , whose tightness was shown by the third author and Mark [47]. Lastly, by Proposition 6.16, if L is in a wing, then Legendrian

surgery on L results in a contact structure ξ_L with non-vanishing contact invariant. Clearly the remaining sequence of Legendrian surgeries also preserve non-vanishing of contact invariant.

Now suppose we have two different representatives L and L' . We first consider the case that L is non-loose and L' is in (S^3, ξ_{std}) . Then ξ_L and $\xi_{L'}$ have distinct contact invariant because of the (contravariant) functoriality of the contact invariant under Legendrian surgery. Next, if L is in a wing, then by Proposition 6.16, ξ_L and $\xi_{L'}$ have distinct contact invariant. Next, if L is in an infinite X or V , then ξ_L and $\xi_{L'}$ have distinct contact invariants by Proposition 6.14, unless the complements L and L' both thicken to an identical ξ_∞ . This exactly gives the overcounts from Proposition 3.5. Now by the (contravariant) functoriality of contact invariant under Legendrian surgery, the final contact structures obtained by a sequence of Legendrian surgery also have different contact invariants.

Lastly, suppose we choose two different stabilizations of a fixed L and its push-offs. In this case, a sequence of Legendrian surgeries constructs a Weinstein cobordism and the different choices of stabilizations induce different Spin^c structures on the Weinstein cobordisms. According to results of Plamenevskaya [61] and Simone [62], different Spin^c structures on Weinstein cobordisms from a fixed contact structure induce distinct contact invariants on the other end. This implies that distinct choices of stabilizations yields non-isotopic contact structures. $\square$

Proof of Theorem 1.28. According to Proposition 3.4, every tight contact structure on $S_r^3(T_{p,q})$ can be obtained by either contact surgery or by Dehn filling a non-thickenable torus. In the first case, the proof follows from Proposition 3.7. In the second case, the contact structures are strongly fillable by Theorems 1.23 and 1.24 except when $(p, q) = (2, \pm 3)$ and $r = 0$, so they have non-vanishing contact invariants by [24]. $\square$

6.4. Examples: $T_{5,8}$ and $T_{5,-8}$. Here, we consider $T_{5,\pm 8}$ as an example showing how non-loose torus knots and bordered invariants are related. The relevant immersed curve is shown in Figure 21a. Note that only the top half is shown; by the symmetry of Heegaard Floer invariants for knot complements, the bottom half is just a 180° rotation of the curve shown. We locate the inner-type generators on the vertical midline, marked as red and gray dots: the top generator is V-type (by symmetry there is another one at the bottom), and there are four others with nonnegative Alexander grading, characterized by the fact that the immersed curve locally curves up and right as well as down and left. One of these is not realized as a contact class (gray dot), while the rest are realized (red dots). Together with their conjugates, we have 6 X-type contact invariants. Following the red curves up-right or down-left, we obtain invariants for complements of Legendrians of X-type with $\text{tb} = 0$ and no half convex torsion. There are seven in the figure, marked as black dots; together with their conjugates, we obtain fourteen, and along with the two V-type, a total of 16, as expected from [16, Section 5]. Additionally, we see the vertical segment between height 9.5 and height 12.5 has two algebraically W-type generators, possibly corresponding to wing-type Legendrians attached to the X-type Legendrian located at height 9.5 in the

figure. Indeed, there is one such wing, and its contact class is the black dot at level 10.5, whereas the other generator directly above it (gray dot) is not realized. Finally, we see that along the non-vertical segment are contact classes for Legendrians in the tight S^3 with $tb = 0$ (blue dots); since we are in the positive torus knot case, all are realized as invariants of Legendrian complements.

Now we turn to the case of $T_{5,-8}$. The relevant immersed curve is shown in Figure 21b. We locate the inner-type generators on the vertical midline, marked as red and gray dots: There are five inner-type generators (with nonnegative Alexander grading), which are algebraically X-type due to their local behavior. It turns out three of these are not realized as contact classes (gray dots), while the other two are realized (red dots). Following the red curves up-right or down-left, we obtain invariants for complements of Legendrians of X-type with $tb = 0$ and no half convex torsion. There are four in the figure, marked as black dots; together with their conjugates, we obtain 8. Once again, we see the vertical segment between height 9.5 and height 12.5 has two algebraically W-type generators, possibly corresponding to wing-type Legendrians attached to the X-type Legendrian located at height 9.5 in the figure. However, neither is actually realized as there are no wings for at the Seifert framing, namely $tb = 0$ (recall that $pq < 0$), so they are marked as gray dots. Similarly, there are no Legendrians in the tight S^3 due to the $tb = 0$ being too high.

7. Symplectic fillability

In this section we will prove our results about symplectic fillability of contact structures on $S_r^3(T_{p,q})$. Specifically, in the first section, we will disprove a conjecture about the non-Stein fillability of two contact structures on $-\Sigma(2, 3, 23)$. In the second section, we will largely determine when contact structures on $S_r^3(T_{p,q})$ are Stein fillable, symplectically fillable, or not symplectically fillable.

7.1. Stein fillability and surgery on cables. The goal here is to prove Theorem 1.27 which regards the Stein fillability of all but one contact structure on $-\Sigma(2, 3, 23)$. The proof relies on our understanding of non-thickenable tori and surgeries on cables. We begin with the latter. Given a knot K its (p, q) -cable is the knot $K_{p,q}$ on the boundary of a regular neighborhood N of K that represents $p\lambda + q\mu$ in $H_1(\partial N)$ where μ is the unique curve on ∂N that bounds a disk in N and λ is the unique curve that bounds a surface in $S^3 \setminus N$. Certain surgeries on $K_{p,q}$ correspond to surgeries on K . Specifically, we have the following.

Lemma 7.1. *Given a knot K in S^3 , let $K_{p,q}$ be its (p, q) -cable and let $\mathcal{F}$ be the framing on $K_{p,q}$ coming from the torus on which it sits. Then we have*

$$S_{(pq \pm 1)/p^2}^3(K) \cong S_{\mathcal{F} \pm 1}^3(K_{p,q}).$$

This has been known, for example [31, Corollary 7.3.], but is also easy to prove by recalling that performing surgery on a knot contained in a surface with surgery coefficient differing from the surface framing by ± 1 is equivalent to cutting the manifold along the

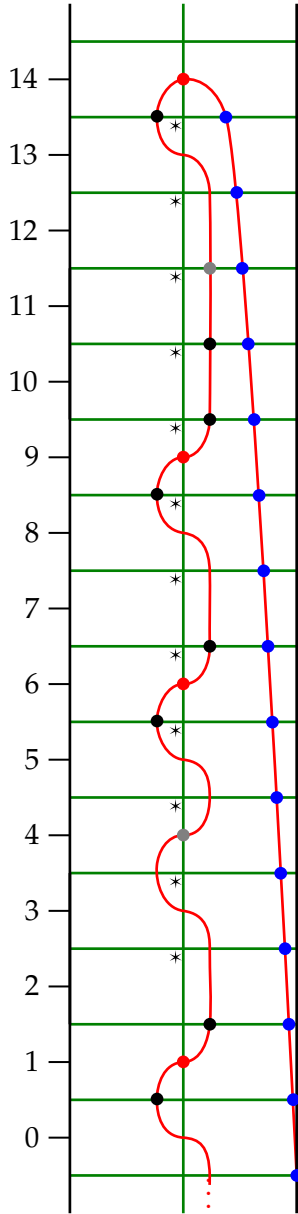
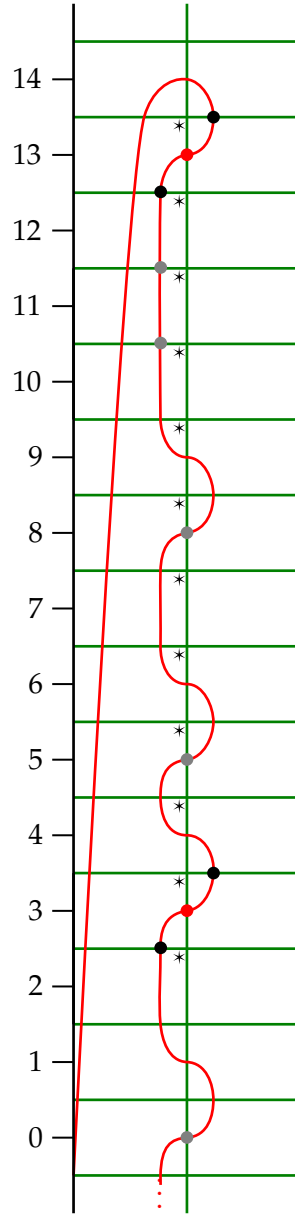
(A) Half of the immersed curve for $-T_{5,8}$ (B) Half of the immersed curve for $-T_{5,-8}$

FIGURE 21. Immersed curves for $T_{5,-8}$ and $T_{5,8}$. Red dots: innermost contact structures. Gray dots: algebraically possible contact classes but not realized. Black dots: non-loose representatives. Blue dots: tight representatives.

surface and regluing by a Dehn twist along the curve. Specifically, if T is the boundary of a solid torus S and γ is a curve of slope q/p on a torus in S isotopic to ∂S , then the meridional slope after the $\mathcal{F} \pm 1$ surgery is obtained from the meridional slope ∞ before

surgery, by applying the change of coordinates

$$\begin{pmatrix} 1 \mp pq & \pm p^2 \\ \mp q^2 & 1 \pm pq \end{pmatrix}.$$

The contact geometric analog of this result was explored in [1, 10]. There, the following result was established; see [10, Lemma 2.4]

Lemma 7.2. *Given a tight contact structure ξ on a solid torus S_r with convex boundary, having two dividing curves of slope s , let T be a convex torus in S_r isotopic to the boundary with two dividing curves of slope s' . The contact structure on S_r is described by a decorated path in the Farey graph from r clockwise to s . We can break this path into two paths, P_1 from r to s' and a path P_2 from s' to s , with decorations induced from the original path. Let L be a Legendrian divide on T . The result of Legendrian surgery on L is a solid torus $S_{r'}$ where r' is obtained from r by applying the above change of coordinates, and the path describing the contact structure is $P_1 \cup P'_2$ where P'_2 is obtained from applying the above change of coordinates to P_2 .*

We are now ready to prove Theorem 1.27.

Proof of Theorem 1.27. We first recall the classification of contact structures on $-\Sigma(2, 3, 23)$. This manifold is the result of $1/4$ surgery on the right-handed trefoil. In [29] it was shown that there are 6 contact structures, which, in the introduction, were denoted $\xi_{0,-2}$, $\xi_{0,0}$, $\xi_{0,2}$, $\xi_{1,-1}$, $\xi_{1,1}$, and $\xi_{2,0}$. In terms of our classification, the first three come from Dehn filling (C, ξ_1) where C is the complement of the $T_{2,3}$ torus knot, and ξ_n is the contact structure coming from the complement of the non-thickenable tori of slope $1/n$. The next two come from Dehn filling (C, ξ_2) , and the last one comes from Dehn filling (C, ξ_3) . The Dehn fillings of (C, ξ_1) correspond to contact surgery on the maximal Thurston-Bennequin invariant Legendrian $(2, 3)$ -torus knot.

Recall there are two non-thickenable tori $S^\pm$ in (S^3, ξ_{std}) with complement $(C_{2,3}, \xi_2)$. These come by attaching a solid torus with lower meridian ∞ and convex boundary with dividing slope $1/2$ to $(C_{2,3}, \xi_1)$. There are two such tori, distinguished by the sign of the basic slice that goes from 0 to $1/2$. Now, inside of $S^\pm$, take a Legendrian divide $L^\pm$ on a convex torus contact isotopic to $\partial S^\pm$. Notice that $L^\pm$ is a $(2, 1)$ -cable of the right-handed trefoil. Applying Lemma 7.2 to $L^\pm$ will change $S^\pm$ into a solid torus with lower meridian $1/4$ given by a path in the Farey graph from $1/4$ to $1/2$. In other words, this Legendrian surgery corresponds to a Dehn filling of $(C_{2,3}, \xi_2)$. Thus, we see that $\xi_{1,-1}$ and $\xi_{1,1}$ are Stein fillable, as claimed. $\square$

7.2. Obstructions to fillability. We finish this paper by determining fillability of tight contact structures on $S_r^3(T_{p,q})$. We first begin with the case $r \geq pq - 1$.

Proof of Theorem 1.23. We first consider the case $n \leq r < n + 1$ for each $n \geq pq$. According to Proposition 3.4, each tight contact structure on $S_r^3(T_{p,q})$, except for the ones from non-thickenable tori, is obtained by $(r - n - 1)$ -contact surgery on a non-loose Legendrian representative of $T_{p,q}$ with $tb = n + 1$. The contact surgery diagram for these contact

structures are shown in [16, Figure 1]. Here, $(r - n - 1)$ -contact surgery on $L_{\pm}$ cancels the component with $(+1)$ -surgery coefficient and hence we obtain a Stein handlebody diagram. If $pq - 1 \leq r < pq$, then the contact surgery diagram is shown in [16, Figure 2] and the $(r - pq)$ -surgery on L cancels the $(+1)$ -component and hence we obtain a Stein handlebody diagram.

The ones from non-thickenable tori for $r \in J \cup J_1$ can be obtained by attaching symplectic handles to the tight contact structures on $S_0^3(T_{p,q})$, which are all weakly fillable, and hence they are also weakly fillable. Since the resulting manifolds are rational homology spheres, they are symplectically fillable.

When $(p, q) = (2, -3)$ and $r = 0$, we know that $S_r(T_{p,q})$ is a torus bundle. According to Honda [36] and Giroux [30], there are infinitely many tight contact structures and all but one of these contact structures contain Giroux torsion. For the exceptional one, we can apply the argument above to find a Stein handlebody diagram. The result now follows from Gay [21] which shows that the presence of Giroux torsion obstructs strong fillability, together with the fact that attaching symplectic handles preserves weak fillability. $\square$

We now turn to the case when $r < \eta(p, q)$ where

$$\eta(p, q) = pq - \max \left\{ \frac{p}{p - p'}, \frac{q}{q'} \right\}.$$

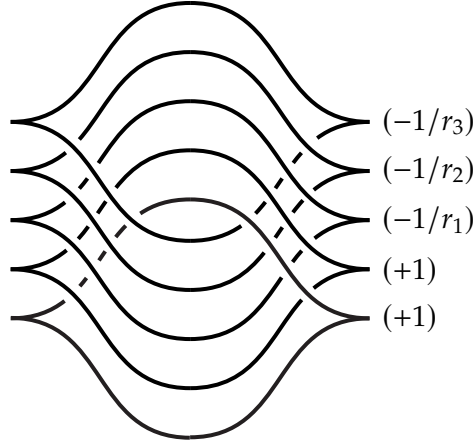
We note that in this case $e_0 = -1$, see Figure 3.

Matkovič [54] characterized the fillability of zero twisting contact structures on small Seifert fibrations with $e_0 = -1$, based on the work of Wendl [66]. This characterization starts with the fact that any such contact structure on $Y(-1; r_1, r_2, r_3)$ has a contact surgery diagram given in Figure 22, see [45, Proposition 6.1]. We now consider the continued fraction expansion

$$-1/r_i = [a_0^i, \dots, a_{k_i}^i].$$

If we denote the Legendrian unknot in Figure 22 with contact framing $(-1/r_i)$ by L_0^i and then attach a chain of unknots $L_1^i, \dots, L_{k_i}^i$ to L_0^i , then the contact surgeries in the diagram can be achieved by realizing L_j^i by a Legendrian unknot with $\text{tb} = a_j^i + 1$ for $j > 0$ and with $\text{tb} = a_0^i$ for $j = 0$, and then performing Legendrian surgery on this link. We call the link $L_0^i \cup \dots \cup L_{k_i}^i$ a *leg* of the diagram.

We now define the numbers s_i^j by $-1/s_i^j = [a_0^i, \dots, a_j^i]$ and say that $Y(-1; r_1, r_2, r_3)$ has a *pair of complementary sub-legs* if there is a pair i' and i'' and $j' \leq k_{i'}$ and $j'' \leq k_{i''}$ such that $s_{i'}^{j'} + s_{i''}^{j''} = 1$. (We note that if any pair of the r_i add up to at least 1, then there will be a pair of complementary sub-legs; see [40, Lemma 3.2]). This condition was called a *circular sublink* in [54]. We say a contact structure on $Y(-1; r_1, r_2, r_3)$, given by a surgery as described above, is *balanced* if it has a pair of complementary sub-legs, and one of the sub-legs has been stabilized with a consistent sign, while the other sub-leg has been stabilized with only the other sign.

FIGURE 22. A contact surgery diagram for $Y(-1; r_1, r_2, r_3)$.

Theorem 7.3 (Matković [54]). *If ξ is a contact structure on $Y(-1, r_1, r_2, r_3)$ given by some contact surgery on the link in Figure 22, then ξ is symplectically fillable if and only if it is balanced.*

Proof of Theorem 1.24. If one sets $-1/r_1 = -\frac{p}{p'}$ and $-1/r_2 = -\frac{q}{q-q'}$, where $pq' - qp' = 1$, and does not perform surgery on the top unknot in Figure 22, then the top unknot becomes a non-loose (p, q) -torus knot with $\text{tb} = pq$, and all such torus knots come from some such surgery [16, Theorem 1.3]. Moreover, any non-loose Legendrian representative of a (p, q) -torus knot with $\text{tb} < pq$ destabilizes to one of these. Thus, any negative contact surgery on a non-loose (p, q) -torus knot with smooth surgery coefficient $r < pq$ can be achieved by performing contact surgery with $-1/r_3 = r + pq$.

Notice that $r_1 + r_2 = \frac{pq-1}{pq} < 1$. Thus there can be no complementary sub-links involving the first two legs. Similarly, if $r < \eta(p, q)$ then the third leg cannot have a sub-link that is complementary to any of the other two. So there are no balanced contact structures coming from Legendrian surgeries on these non-loose knots, and we see that no contact surgeries on non-loose knots produce symplectically fillable contact structures by the above theorem of Matković.

Other possible contact structures on $S_r^3(T_{p,q})$ come from negative contact surgery on a Legendrian representative in (S^3, ξ_{std}) , which are obviously Stein fillable, and fillings of non-thickenable tori for $r \in J$. These latter contact structures can be obtained by attaching symplectic handles to $S_0^3(T_{p,q})$ and hence they are weakly fillable. Since the resulting manifolds are rational homology spheres, they are symplectically fillable.

When $(p, q) = (2, 3)$ and $r = 0$, we know that $S_r(T_{p,q})$ is a torus bundle. According to Honda [36] and Giroux [30], there are infinitely many tight contact structures and all but one of these contact structures contain Giroux torsion. The exceptional one can be obtained by Legendrian surgery on a Legendrian representative of $T_{p,q}$ in (S^3, ξ_{std}) and hence it is Stein fillable. The result now follows from Gay [21] which shows that the presence of

Giroux torsion obstructs strong fillability, together with the fact that attaching symplectic handles preserves weak fillability. $\square$

We note that when $r \in (\eta(p, q), pq - 1)$ we can still apply Matkovič theorem to determine when a contact structure obtained by contact surgery on a loose Legendrian knot is Stein fillable or not. The only other tight contact structures on $S_r^3(T_{p,q})$ come from surgery on a Legendrian representative in (S^3, ξ_{std}) or filling the complement of a non-thickenable torus. In the former case, we know the contact structures are Stein fillable, and in the latter they are symplectically fillable and in some (possibly rare) cases Stein fillable (see Section 7.1).

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