CONTACT GEOMETRIC THEORY OF ANOSOV FLOWS IN DIMENSION THREE AND RELATED TOPICS

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To prove itself, the Sun appears.

Rumi

To my mother, Farah.

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SUMMARY

This thesis consists of the author's work on the contact and symplectic geometric theory of Anosov flows in low dimensions, as well as the related topics from Riemannian geometry. This includes the study of the interplay between various geometric, topological and dynamical features of such flows.

After reviewing some basic elements from the theory of contact and symplectic structures in low dimensions, we discuss a characterization of Anosov flows on three dimensional manifolds, purely in terms of those geometric structure. This is based on the previous observations of Mitsumatsu [1] and Eliashberg-Thurston [2] in the mid 90s, and in the context of a larger class of dynamics, namely *projectively Anosov* flows. Our improvement of those observations, which have been left unexplored to a great extent in our view, facilitates employing new geometric tools to the study of questions about (projectively) Anosov flows and vice versa.

We then discuss another characterization of Anosov three flows, in terms of the associated underlying *Reeb dynamics*. Beside the contact topological consequences of this result, it sheds light on contact geometric interpretation of the existence of an invariant volume form for these flows, a condition which is well known to have deep consequences in the dynamics of the flow from the viewpoint of the long term behavior of the flow (*transitivity*) and measure theory (*ergodicity*). The implications of these results on various related theories, namely, *Liouville geometry*, *the theory of contact hyperbolas* and *bi-contact surgery*, are discussed as well. As *contact Anosov flows* are an important and well studied special case of volume preserving Anosov flows, we also make new observation regarding these flows, utilizing the associated *Conley-Zehnder indices* of their periodic orbits, a classical tool from the field of *contact dynamics*.

We finally discuss some Riemannian geometric motivations in the study of contact Anosov flows in dimension three. In particular, this bridges our study to the curvature

properties of Riemannian structures, which are *compatible* with a given contact manifold. Our study of the curvature in this context goes beyond the study of Anosov dynamics, although has implications on the topic. In particular, we investigate a natural curvature realization for compatible Riemannian structures, namely *Ricci-Reeb realization problem*.

The majority of the results in this manuscript, with the exception of some parts of Chapter 5, can be found in the author's previous papers [3, 4, 5, 6].

CHAPTER 1

INTRODUCTION TO CONTACT STRUCTURES IN LOW DIMENSIONS

We start this manuscript with a review of some basic notions from the theory of *contact geometry and topology* in dimension 3, including the connections to their close relatives, *symplectic geometry and topology* in dimension 4. We refer the reader to [7] for a more thorough treatment of these preliminaries, and to [8] for a beautiful survey on the history of the theory.

One can track back the roots of contemporary contact geometry to the early 1870s, and in the introduction of *contact transformations* by Sophus Lie. As geometric structures defined in odd dimensions, *contact structures* were employed during the 20th century in many important areas of research, including *optics*, *thermodynamics*, *control theory*, and most notably, the *Hamiltonian reformulation of Newtonian mechanics*. However, throughout this period, they received considerably less attention than their even dimensional counterpart, *symplectic structures*.

In the 1970s, the study of the topological aspect of the theory of contact structures was started in dimension 3 and in the works of Bennequin, Lutz, Martinet, etc. The introduction of the theories of *J-holomorphic curves* by Gromov in the mid 1980s, and *convex surfaces* by Giroux in the early 1990s were groundbreaking accomplishments, exhibiting the deep interactions between such geometric structures and the topology of the underlying manifold, justifying the term *contact topology*, referring to the study of such connections. By now, other topological theories of contact structures, like *open book decompositions* and *Floer theory*, have been developed since, furthering this active area of research. We remark that some of these techniques are rooted in interplay of contact structures in dimension 3 and symplectic topology in dimension 4, a fact which will be used throughout this manuscript.

It is important for us to mention that throughout the development of the theory of contact geometry and topology, there has been a dynamical approach in the study of these structures, thanks to the *Reeb vector fields* (usually called *characteristic vector fields* in the classical literature) associated to a given contact structures. Starting 1990s, and mainly thanks to the works of Hofer, Wysocki and Zehnder, these vector fields have been used to extract topological information regarding contact structures. For instance, the introduction of various flavors of *contact homologies* as invariants of contact manifolds, relies on such dynamical perspective.

On the other hand, the *Riemannian geometric* aspects of contact structures have been heavily studied in the classical literature, since the mid 20th century. However, the *global* Riemannian geometry of these structures have been left mostly unexplored, until recent years.

In the remainder of this chapter, we recall a few elementary notions and examples about contact structures in dimension 3 and their relation to symplectic topology in dimension 4. Throughout this text, we will use various topological tools, as well as the dynamical and Riemannian geometric theories of these structures. However, we postpone the introduction of these methods and theories to the relevant chapters.

1.1 Elements from contact topology

Convention: During this manuscript, we assume M to be a closed, connected, oriented three manifold, unless stated otherwise.

Definition 1.1.1. *We call the 1-form α a* contact form *on M, if $\alpha \wedge d\alpha$ is a non-vanishing volume form on M. If $\alpha \wedge d\alpha > 0$ (compared to the orientation on M), we call α a* positive *contact form and otherwise, a* negative *contact form. We call $\xi := \ker \alpha$ a (positive or negative) [coorientable]* contact structure *on M. Moreover, we call the pair (M, ξ) a* contact manifold. *When not mentioned, we assume the contact structures to be positive.*

We note that the above gives the definition for a *coorientable* contact structure. A general contact structure is one which is locally defined as such 1-form. However, we assume all contact structures to be coorientable in this text.

Recall that by the Frobenius theorem, the contact structure ξ in the above definition is a *maximally non-integrable* plane field on M. Therefore, in terms of integrability, they are the extreme opposite of *foliations*.

A few important examples of contact structures in dimension 3 are as follows.

Example 1.1.2. *(1) The 1-form $\alpha_{std} = dz - y\,dx$ is a positive contact form on $\mathbb{R}^3$. We call $\xi_{std} = ker\alpha_{std}$ the* standard *positive contact structure on $\mathbb{R}^3$. Similarly, $\ker(dz + y\,dx)$ is the standard negative contact structure on $\mathbb{R}^3$.*

(2) Consider $\mathbb{C}^2$ equipped with J, the standard complex structure on $T\mathbb{C}^2$ and let $\mathbb{S}^3$ be the unit sphere in $\mathbb{C}^2$. It can be seen that the plane field $\xi_{std} := T\mathbb{S}^3 \cap JT\mathbb{S}^3$ is a contact structure on $\mathbb{S}^3$, referred to as standard *contact structure on $\mathbb{S}^3$. Alternatively, ξ_{std} can be defined as the unique complex line tangent to the unit sphere. It is helpful to note that this contact structure is the one point compactification of the standard contact structure on $\mathbb{R}^3$. Similarly, we can construct a negative contact structure on $\mathbb{S}^3$, by considering the conjugate of the complex structure J.*

(3) Consider $\mathbb{T}^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$. It can be seem that for integers $n > 0$ and $n < 0$, the plane fields $\xi_n = \ker\{\cos 2\pi nz dx - \sin 2\pi nz dy\}$ are positive and negative contact structures on $\mathbb{T}^3$, respectively.

(4) (Boothby-Wang fibrations) Let Σ be a closed oriented surface and ω an area form on Σ with $0 \neq [\omega] \in H^2(\Sigma;\mathbb{Z})$. By Kobayashi [9], there exists an $\mathbb{S}^1$-bundle $\pi : M \to \Sigma$, equipped with the connection form α, such that $d\alpha = \pi^*\omega$. It can be easily seen that α is a contact form and we call $(M, \xi := \ker\alpha)$ a Boothby-Wang fibration. Introduced by Boothby and Wang [10], these examples can be generalized to higher dimensions by considering any symplectic manifold (Σ^{2n}, ω).

Gray's theorem states that members of any C^1-family of contact structures, which is constant off of a compact set, are isotopic as contact structures and according to the *Darboux theorem*, all contact structures locally look the same. i.e. around each point in a contact manifold (M, ξ), there exists a neighborhood U and a diffeomorphism of U to $\mathbb{R}^3$, mapping ξ to the standard contact structure (positive or negative one, depending on whether ξ is positive or negative) on $\mathbb{R}^3$. While this means that contact structures lack local invariants, it turns out that understanding their topological properties is more subtle and interesting. The most significant global feature of contact structures is *tightness*, introduced by Eliashberg [11], and determining whether a given contact is tight, as well as classifying such contact manifolds, are a prominent themes in contact topology.

Definition 1.1.3. *The contact manifold (M, ξ) is called* overtwisted*, if M contains an embedded disk that is tangent to ξ along its boundary. Otherwise, (M, ξ) is called* tight*. Moreover, ξ universally tight, if its lift to the universal cover of M is tight as well.*

The significance of the above dichotomy comes from the classification of overtwisted contact structures by Y. Eliashberg [12, 11]. He showed that overtwisted contact structures, up to isotopy, are in one to one correspondence with plane fields, up to homotopy (in particular, they always exist). This means that overtwisted contact structures do not carry more topological information as contact structures, than as plane fields. On the other hand, tight contact structures reveal deeper information about their underlying manifold, and are harder to find, understand and classify.

Remark 1.1.4. *It can be shown that all the contact structures in Example 1.1.2 are (universally) tight. In fact in Example 1.1.2 (1)-(3), they are the only tight contact structures, up to contactomorphism, on their underlying manifolds. Note that all those manifolds admit overtwisted contact structures as well.*

It turns out that one can determine tightness of a contact manifold is based on its relation to four dimensional symplectic topology, the even dimensional sibling of contact topology.

Definition 1.1.5. *Let W be an oriented 4-manifold. We call a 2-form ω on W a symplectic form, if it is closed and $\omega \wedge \omega > 0$. The pair (W, ω) is called a symplectic manifold.*

The 2-form form $\omega_{std} = d(x_1 dy_1 + x_2 dy_2) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ is a symplectic form defined on $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) (known as the *standard* symplectic form), and the Darboux theorem in symplectic geometry states that all symplectic structures are locally equivalent, up to *symplectic deformation*. Using the theory of J-holomorphic curves, Gromov and Eliashberg proved [13, 14] that a contact structure is tight, when it is *symplectically fillable*, even in the weakest sense.

Definition 1.1.6. *Let (M, ξ) be a contact manifold. We call the symplectic manifold (W, ω) a* weak symplectic filling *for (M, ξ), if $\partial W = M$ as oriented manifolds and $\omega|_\xi > 0$. We call (W, ω) a* strong symplectic filling*, if moreover, $\omega = d\alpha$ in a neighborhood of $M = \partial W$, for some 1-form α, such that $\alpha|_{TM}$ is a contact form for ξ. Finally, we call (W, ω) an* exact symplectic filling *for (M, ξ), if such 1-form α can be defined on all of W. We call such (M, ξ)* (weakly, strongly or exactly) symplectically fillable.

Theorem 1.1.7. *(Gromov 85 [13], Eliashberg 90 [14]) If (M, ξ) is (weakly, strongly or exactly) symplectically fillable, then it is tight.*

Remark 1.1.8. *We note that for a 2-form ω to be symplectic, it needs to be at least C^1, because of the closedness condition. However, when ω is exact, i.e. $\omega = d\alpha$ for some 1-form α, this condition is automatically satisfied, assuming the required regularity. So for most purposes, we don't need to assume, for an exact 2-form $\omega = d\alpha$, any regularity more than C^0-regularity, and the methods of symplectic geometry and topology, in particular, the use of J-holomorphic curves and Theorem 1.1.7, can be applied. We can also approximate such $\omega = d\alpha$, by symplectic forms of arbitrary high regularity, using C^1-approximations of α. Therefore, we still call such 2-form ω symplectic, especially in Theorem 2.1.1.*

It is known that not all tight contact structures are weakly symplectically fillable, the set of strongly symplectically fillable contact manifolds is a proper subset of the set of weakly symplectically fillable contact manifolds and the set of exactly symplectically fillable contact manifolds is a proper subset of the set of strongly symplectically fillable contact manifolds. Moreover, if a disconnected contact manifold is strongly or weakly symplectically fillable, each of its components can be shown to be strongly or weakly symplectically fillable, respectively [15, 16].

Example 1.1.9. *(1) The unit ball in* $(\mathbb{R}^4, \omega_{std})$ *is a strong symplectic filling for* $(\mathbb{S}^3, \xi_{std})$*, considered as the unit sphere in* $\mathbb{R}^4$*.*

(2) We want to show that all tight contact structures on $\mathbb{T}^3$*, given in Example 1.1.2 (3), are weakly symplectically fillable. We can observe that after an isotopy and for integers* $n > 0$*, we have* $\xi_n = \ker\big(dz + \epsilon\{\cos 2\pi nz dx - \sin 2\pi nz dy\}\big)$ *for arbitrary small* $\epsilon > 0$*, i.e. we can isotope* ξ_n *to be arbitrary close to the horizontal foliation* $\ker dz$ *on* $\mathbb{T}^3$*. Now consider the symplectic manifold* $(X, \omega) = (\mathbb{T}^2 \times D^2, \omega_1 \oplus \omega_2)$*, where* ω_1 *and* ω_2 *are area forms for* T^2 *and* D^2*, respectively. Clearly,* $\partial X = \mathbb{T}^3$ *and if at the boundary, we consider the coordinates* (x, y) *for* $\mathbb{T}^2$ *and* z *for the angular coordinate of* D^2*, we have* $\omega|_{\ker dz} > 0$*. Since for small* $\epsilon > 0$*,* ξ_n *is a small perturbation of* $\ker dz$*, we also have* $\omega|_{\xi_n} > 0$*. Therefore, all* ξ_n *s are weakly symplectically fillable. It can be seen [17] that except* ξ_1*, none of these contact structures, are strongly symplectically fillable and the canonical symplectic structure on the cotangent bundle* T^*T^2 *provides an exact symplectic filling for* (T^3, ξ_1)*.*

Remark 1.1.10. *The concept of* Giroux torsion *was introduced by Emmanuel Giroux [18]. A contact manifold* (M, ξ) *is said to contain Giroux torsion, if it admits a contact embedding of*

$$\big([0, 2\pi] \times \mathbb{S}^1 \times \mathbb{S}^1 \textit{ with coordinates } (t, \phi_1, \phi_2),\ \ker(\cos t\, d\phi_1 + sint\, d\phi_2)\big) \to (M, \xi).$$

Note that all the tight contact structures on $\mathbb{T}^3$*, discussed in Example 1.1.2 (3) contain Giroux torsion, except for* $n = 1$*. Later in [19], it was proven that contact structures* *containing Giroux torsion do not admit strong symplectic fillings. This notion can be generalized by considering* Giroux π-torsion. *i.e. when the contact manifold contains half of a Giroux torsion:*

$$\big([0,\pi] \times \mathbb{S}^1 \times \mathbb{S}^1 \text{ with coordinates } (t, \phi_1, \phi_2), \ker(\cos t\, d\phi_1 + sint\, d\phi_2)\big) \to (M, \xi).$$

In Example 1.1.2 (3), for all $n > 1$, the contact manifold $(\mathbb{T}^3, \xi_n)$ contains Giroux torsion, while $(\mathbb{T}^3, \xi_1)$ is constructed by gluing two Giroux π-torsions along their boundary.

A special case of exact symplectic fillings was observed by Mitsumatsu, in the presence of smooth volume preserving Anosov flows [1] (see Chapter 2 for more discussion and improvement of the Mitsumatsu's results). Alongside [20], these were the first examples of exact symplectic fillings with disconnected boundaries. Explicit examples of such structures can be found in [1] and they are also studied in [21].

Definition 1.1.11. *We call a pair (α_-, α_+) a* Liouville pair*, if α_- and α_+ are negative and positive contact forms, respectively, whose kernels are transverse and $[-1,1]_t \times M$, equipped with the symplectic structure $d\{(1-t)\alpha_- + (1+t)\alpha_+\})$ is an exact symplectic filling for $(M, \ker \alpha_+) \sqcup (-M, \ker \alpha_-)$, where $-M$ refers to M with the reversed orientation.*

1.2 Associated Reeb vector fields

It turns out that a certain class of vector fields associated to a contact manifold, namely *Reeb vector fields*, gives us a dynamical approach in understanding contact geometry. The relation between the topological aspects of contact structures and these flows has been studied since the early 1990s, mainly in the works of Hofer, Wysocki and Zehnder, and among other things, was used to define novel *Floer theoretic* invariants for contact structures. We will more discussion on this in Chapter 5.

Definition 1.2.1. *Let (M, ξ) be a contact 3-manifold. Any choice of contact form α for ξ defines a unique vector field R_α satisfying*

i) $d\alpha(R_\alpha, .) = 0$*;*

ii) $\alpha(R_\alpha) = 1$*.*

We can easily observe

Proposition 1.2.2. *The Reeb vector field R_α satisfies*

(a) $R_\alpha \pitchfork \xi$*;*

(b) $\mathcal{L}_{R_\alpha}\alpha = 0$ and therefore R_α preserves ξ. Further more, Reeb vector fields are volume preserving, since $\mathcal{L}_{R_\alpha}\alpha \wedge d\alpha = 0$.

(c) On the other hand, any vector field which is transverse to ξ and keeps it invariant is a Reeb vector field for an appropriate choice of contact form.

Example 1.2.3. *The Reeb vector fields for the contact structures given in Example 1.1.2 are*

(1) ∂_z is the Reeb vector field for $(\mathbb{R}^3, \alpha_{std})$.

(2) For an appropriate choice of contact form, the Reeb vector field associated to $(\mathbb{S}^3, \xi_{std})$ is tangent to the Hopf fibration on $\mathbb{S}^3$.

(3) The vector fields orthonormal to ξ_n (considering the flat metric on $\mathbb{T}^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$) are Reeb vector fields.

(4) The integral curves of Reeb vector fields associated to the constructed contact forms on Boothby-Wang fibrations traces the $\mathbb{S}^1$ fibers, described in the construction.

CHAPTER 2

CONTACT AND SYMPLECTIC GEOMETRY OF ANOSOV FLOWS IN DIMENSION THREE

2.1 Introduction to Anosov flows in dimension 3 and their contact and symplectic geometric theory

The main goal of this chapter is to establish the relation between contact and symplectic topology, and the theory of *Anosov flows* in dimension three, leading to a characterization of such flows, purely in terms of such geometric structures.

Anosov flows were introduced by Dimitri Anosov [22, 23] in 1960s as a generalization of geodesic flows of hyperbolic manifolds and were immediately considered an important class of dynamical systems, thanks to their many interesting global properties and structural rigidity. Many tools of dynamical system, including ergodic theory, helped increase our understanding of Anosov flows (see [24] for early developments). But more profound connections to the topology of the underlying manifold, were discovered in dimension 3, thanks to the use of foliation theory. This was initiated by many, including Thurston, Plante and Verjovsky. However, more recent advances in the mid 1990s came from new techniques in foliation theory, introduced by Sergio Fenley, alongside Thierry Barbot, etc (see [25] as the seminal work and [26] for a nice survey of such results).

The main result of this chapter describes Anosov flows in terms of contact and symplectic geometry. See Chapter 1 and Section 2.2 for the related definitions and discussions.

Convention: During this manuscript, unless stated otherwise, we let X to be a non-vanishing C^1 vector field and ϕ^t is the flow generated by X. We also assume the (projectively) Anosov flows to be *orientable*, i.e. the associated *stable and unstable directions* are orientable line fields (assuming the orientability of M, this can be achieved, possibly after going to a double cover of M). Furthermore, we call any geometric quantity which is differentiable along the flow X*-differentiable*.

Theorem 2.1.1. *[5] Let ϕ^t be a flow on the 3-manifold M, generated by the C^1 vector field X. Then ϕ^t is Anosov, if and only if, $\langle X \rangle = \xi_+ \cap \xi_-$, where ξ_+ and ξ_- are transverse positive and negative contact structures, respectively, and there exist contact forms α_+ and α_- for ξ_+ and ξ_-, respectively, such that (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are Liouville pairs.*

Although the relation to contact geometry was observed by Eliashberg-Thurston [2] and more thoroughly by Mitsumatsu [1], Theorem 2.1.1 improves those observations into a full characterization of such flows. More precisely, Mitsumatsu [1] proves the existence of Liouville pairs in the case of *smooth volume preserving* Anosov flows, in order to introduce a large family of *non-Stein Liouville domains* (generalizing a previous work of McDuff [20]). Nevertheless, our goal here is to give a complete characterization of Anosov flows in full generality, which is necessary for developing a contact and symplectic geometric theory of such flows. In order to do so, we use natural geometric quantities, namely *expansion rates* (see Section 2.2), to achieve a refinement of Mitsumatsu's observation, which helps us generalize his result to an arbitrary (possibly non-volume preserving) C^1 Anosov flow, as well as prove the converse. Note that Brunella [27] has shown the abundance of Anosov flows with no invariant volume forms. However, the main technical difficulty is that for smooth volume preserving Anosov flows, the weak stable and unstable bundles are known to be at least C^1 [28], which significantly simplifies the geometry of an Anosov flow (see Chapter 3 and 4 or [1] for this simplified setting). In the absence of such regularity condition in the general setting of Theorem 2.1.1, we introduce approximation techniques (see Section 2.3), tailored for the setting of Anosov flows, which facilitate translating the information of a given C^1 Anosov flow (with possibly C^0 invariant foliations) to the corresponding contact structures (which are at least C^1). We believe that these approximation techniques are independently interesting and can be used regarding other questions in Anosov dynamics (for instance, see Chapter 4 for application in Anosov surgeries). See Remark 2.3.1.

By Theorem 2.1.1, the vector field which generates an Anosov flow lies in the intersection of a pair of positive and negative contact structures, i.e. a *bi-contact structure*. It turns out that this condition has dynamical interpretation and defines a large class of flows, named *projectively Anosov flows* (introduced in [1]). These are flows, which induce, via the projection $\pi : TM \to TM/\langle X\rangle$, a flow with *dominated splitting* on $TM/\langle X\rangle$ (see Section 2.2).

We remark that projectively Anosov flows are previously studied in various contexts, under different names. In the geometry and topology literature, beside projectively Anosov flows, they are referred to as *conformally Anosov flows* and are studied from the perspectives of foliation theory [2, 29, 30, 31], Riemannian geometry of contact structures [32, 33, 4] and Reeb dynamics [3]. This is while, in the dynamical systems literature, the term conformally Anosov is preserved for another dynamical concept (for instance see [34, 35, 36]) and the dynamical aspects of projectively Anosov flows are studied under the titles *flows with dominated splitting* (see [37, 38, 39, 40, 41]) or *eventually relatively pseudo hyperbolic flows* [42].

Although, it is not immediately clear if the class of projectively Anosov flows is larger than Anosov flows, the first examples of such flows on $\mathbb{T}^3$ and Nil manifolds [1, 2], which do not admit any Anosov flows [43], as well as more recent examples of projectively Anosov flows on atoroidal manifolds, which cannot be deformed to Anosov flows [44], proved the properness of the inclusion. In fact, we now know that unlike Anosov flows, projectively Anosov flows are abundant. For instance, there are infinitely many distinct projectively Anosov flows on $\mathbb{S}^3$ and no Anosov flows [45]. Therefore, Theorem 2.1.1 can be seen as a host of geometric and topological rigidity conditions on a projectively Anosov flow. In particular, this enables us to use various contact and symplectic geometric and topological tools in the study of Anosov dynamics. For instance, there are many questions about the knot theory of the periodic orbits of Anosov flows. Thanks to Theorem 2.1.1, such periodic orbits are now *Legendrian knots* for both underlying contact structures and moreover, correspond to *exact Lagrangians* in the constructed Liouville pairs. These are standard and well studied objects in contact and symplectic topology and now, the same techniques can be employed for understanding the periodic orbits of such flows (see Remark 2.3.12).

As discussed in Chapter 1, thanks to the *Darboux theorem*, contact structures have no local invariants and the *Gray's theorem* implies that homotopy through contact structures can be done by an isotopy of the ambient manifold. Therefore, the local structure of contact structures does not carry any information and the subtlety of these structures is hidden in their global topological properties. In fact, we have a hierarchy of topological rigidity conditions on a contact manifold. It is known that all the inclusions below are proper.

$$\left\{\begin{matrix}\text{Stein fillable}\\ \text{contact manifolds}\end{matrix}\right\}\subset\left\{\begin{matrix}\text{Exactly symplectically fillable}\\ \text{contact manifolds}\end{matrix}\right\}\subset\left\{\begin{matrix}\text{Strongly symplectically fillable}\\ \text{contact manifolds}\end{matrix}\right\}$$

$$\subset\left\{\begin{matrix}\text{Weakly symplectically fillable}\\ \text{contact manifolds}\end{matrix}\right\}\subset\left\{\begin{matrix}\text{Tight}\\ \text{contact manifolds}\end{matrix}\right\}\subset\left\{\text{contact manifolds}\right\}.$$

Now, we can naturally apply the hierarchy of contact topology to bi-contact structures and therefore, achieve a filtration of Anosovity concepts (see Section 2.5 for the precise definitions).

$$\left\{\text{Anosov flows}\right\}\subseteq\left\{\begin{matrix}\text{Exactly symplectically bi-fillable}\\ \text{projectively Anosov flows}\end{matrix}\right\}\subseteq\left\{\begin{matrix}\text{Strongly symplectically bi-fillable}\\ \text{projectively Anosov flows}\end{matrix}\right\}$$

$$\subseteq\left\{\begin{matrix}\text{Weakly symplectically bi-fillable}\\ \text{projectively Anosov flows}\end{matrix}\right\}\subseteq\left\{\begin{matrix}\text{Tight}\\ \text{projectively Anosov flows}\end{matrix}\right\}\subset\left\{\text{projectively Anosov flows}\right\}.$$

In the above, we first notice that there are no equivalent of *Stein fillable* contact manifolds for bi-contact structures (or projectively Anosov flows), since Stein fillings can only have connected boundaries [20].

The above hierarchy invokes a general line of questioning, which can help us understand Anosov dynamics, through the lens of contact and symplectic topology.

Question 2.1.2. *What does each bi-contact topological layer imply about the dynamics of the corresponding class of projectively Anosov flows? What bi-contact topological layer is responsible for a given property of Anosov flows?*

One important motivation to study the consequences of these contact geometric conditions on the dynamics of a projectively Anosov flow is the classification of Anosov flows, up to *orbit equivalence* (that is up to homemorphisms mapping the orbits of one flow to another). In the light of Theorem 2.1.1, this can be split into two problems: topological classification of bi-contact structures (a contact topological problem) and understanding the bifurcations of projectively Anosov flows, under *bi-contact homotopy* (a dynamical question). Since many contact topological tools have been successfully developed in the past few decades to address the first problem, understanding the bifurcation problem can lead to important classification results. A very useful perspective is to notice that all the contact topological properties of the above hierarchy are preserved under bi-contact homotopy and therefore, are satisfied for any projectively Anosov flow which is homotopic to some Anosov flow (see Section 2.5 for the related discussions).

Regarding Question 2.1.2, [46] shows that there are no tight projectively Anosov flows on $\mathbb{S}^3$ (generalizing non-existence of Anosov flows) and [45] gives a partial classification of *overtwisted projectively Anosov flows*, i.e. when both contact structures, forming the underlying bi-contact structure, are not tight (are *overtwisted*). More precisely, they show that overtwisted projectively Anosov flows exist, when there are no algebraic obstruction. Although, this is not a full classification, it is worth comparing this with purely algebraic classification of overtwisted contact structures, by Eliashberg [11, 12], reaffirming the parallels in the two theories. This also implies that the class of tight projectively Anosov flows is (considerably) smaller than general projectively Anosov flows.

We will observe the properness of the middle inclusion, by constructing examples on $\mathbb{T}^3$, while the properness of other inclusions remain unknown (Question 2.5.11).

Theorem 2.1.3. *[5] There are (trivially) weakly symplectically bi-fillable projectively Anosov flows, which are not strongly symplectically bi-fillable.*

As mentioned above, all the above contact topological conditions on projectively Anosov flows are purely topological, that is do not depend on the homotopy of any of the two underlying contact structures, with the exception of the first inclusion, i.e. Anosovity of a flow. It turns out that in the study of bi-contact structures (or equivalently, projectively Anosov flows), the local geometry is more subtle than contact structures, due to lack of theorems equivalent to the Darboux and Gray's theorems. Drawing contrast between two notions of *bi-contact homotopy vs. isotopy*, we conclude that bi-contact homotopy is the natural notion from dynamical point of view (see Definition 2.5.1 and the subsequent discussion). The relation between Anosovity and geometry of bi-contact structures is not well understood and we bring related discussions and questions in Section 2.5.

Question 2.1.4. *How does the Anosovity of a flow depend on the geometry of the underlying bi-contact structure, under bi-contact homotopy?*

We show that at least for a fixed projectively Anosov flow, there is a unique *supporting* bi-contact structure, up to bi-contact homotopy.

Theorem 2.1.5. *[5] If (ξ_-, ξ_+) and (ξ'_-, ξ'_+) are two supporting bi-contact structures for a projectively Anosov flow, then they are homotopic through supporting bi-contact structures.*

We also use well known facts in Anosov dynamics, as well as the underlying techniques of Theorem 2.1.5, to derive a family of uniqueness results for the underlying contact structures, reducing the study of the supporting bi-contact structure to only one of the supporting contact structures.

Theorem 2.1.6. *[5] If M is atoroidal and (ξ_-, ξ_+) a supporting bi-contact structure for the Anosov vector field X on M, then for any supporting positive contact structure ξ, ξ is isotopic to ξ_+, through supporting contact structures.*

Theorem 2.1.7. *[5] Let X be an $\mathbb{R}$-covered Anosov vector field, supported by the bi-contact structure (ξ_-, ξ_+) on M, and let ξ be any supporting positive contact structure. Then ξ is isotopic to ξ_+, through supporting contact structures.*

Theorem 2.1.8. *[5] Let X be the suspension of an Anosov diffeomorphism of torus, supported by the bi-contact structure (ξ_-, ξ_+), and ξ a positive supporting contact structure. Then, ξ is isotopic through supporting bi-contact structures to ξ_+, if and only if, ξ is strongly symplectically fillable.*

It turns out that to further investigate the contact topological consequences of Anosovity, we can also use the underlying *Reeb dynamics* of a supporting bi-contact structure. In Chapter 3, we also use the ideas developed in Section 2.2 and proof of Theorem 2.1.1 to give a characterization of Anosovity, based on the *Reeb vector fields*, associated to the underlying contact structures. Reeb vector fields play a very important role in contact geometry and Hamiltonian mechanics and since early 90s, their deep relation to the topology of contact manifolds has been explored (definitions, details and discussions are postponed to Chapter 3).

Theorem 2.1.9. *[5] Let X be a projectively Anosov vector field on M. Then, the followings are equivalent:*

(1) X is Anosov;

(2) There exists a supporting bi-contact structure (ξ_-, ξ_+), such that ξ_+ admits a Reeb vector field, which is dynamically negative everywhere;

(3) There exists a supporting bi-contact structure (ξ_-, ξ_+), such that ξ_- admits a Reeb vector field, which is dynamically positive everywhere.

We draw the following contact topological conclusions from the above characterization:

Theorem 2.1.10. *[5] Let (ξ_-, ξ_+) be a supporting bi-contact structure for an Anosov flow. Then, ξ_- and ξ_+ are* hypertight. *That is, they admit contact forms, whose associated Reeb flows do not have any contractible periodic orbit.*

Consequently, the classical results of Hofer, et al in Reeb dynamics [47, 48, 49] would imply the followings:

Corollary 2.1.11. *Let (ξ_-, ξ_+) be a supporting bi-contact structure for an Anosov flow. Then,*

(1) ξ_- and ξ_+ are universally tight;

(2) M is irreducible;

(3) there are no exact symplectic cobordisms from (M, ξ_+) or $(-M, \xi_-)$ to $(\mathbb{S}^3, \xi_{std})$.

We note that (1) in Corollary 2.1.11 is also concluded from Theorem 2.1.1 and (2) is classical fact from Anosov dynamics, and we are giving new Reeb dynamical proofs for them. On the other hand, part (3) is a symplectic improvement of the main result of [46] and non-existence of Anosov flows on $\mathbb{S}^3$.

2.2 Anosovity and the geometry of expansion

In this section, we review the basic facts about Anosov 3-flows, emphasizing on the expansion behavior of the flows in stable and unstable directions, from a geometric point of view.

Definition 2.2.1. *We call the C^1 flow ϕ^t* Anosov*, if there exists a splitting $TM = E^{ss} \oplus E^{uu} \oplus \langle X \rangle$, such that the splitting is continuous and invariant under ϕ^t_* and*

$$||\phi^t_*(v)|| \geq Ae^{Ct}||v|| \text{ for any } v \in E^{uu},$$

$$||\phi^t_*(u)|| \leq Ae^{-Ct}||u|| \text{ for any } u \in E^{ss},$$

where C and A are positive constants, and $||.||$ is induced from some Riemannian metric on TM. We call E^{uu} ($E^{uu} \oplus \langle X \rangle$) and E^{ss} ($E^{ss} \oplus \langle X \rangle$), the strong (weak) unstable and stable directions (bundles), respectively. Moreover, we call the vector field X, the generator of such flow, an Anosov vector field*.*

As mentioned in the previous section, we assume E^{ss} and E^{uu} to be orientable in what follows. This can be arranged, possibly after going to a double cover of M.

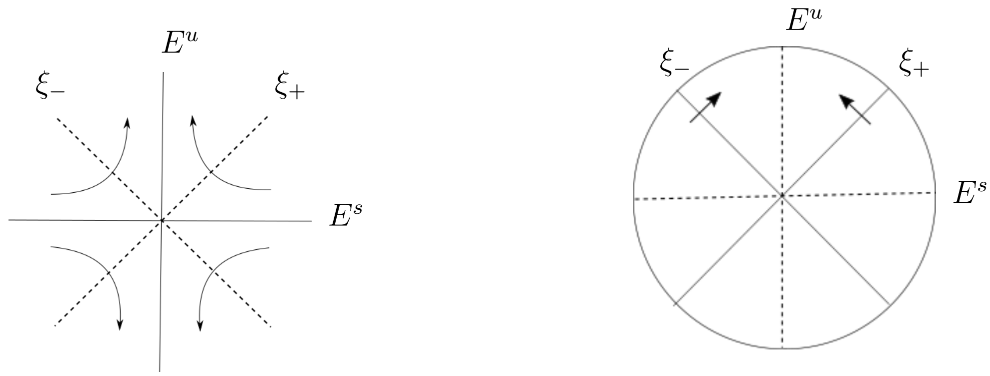
Example 2.2.2. *Classic examples of Anosov flows in dimension 3 include the geodesic flows on the unit tangent space of hyperbolic surfaces and the suspension of Anosov diffeomorphisms of a 2-torus. By now, we know that there are many Anosov flows on hyperbolic manifolds as well [50].*

Here, we note that by [23], a small perturbation of any Anosov flow is Anosov and moreover, is *orbit equivalent* to the original flow, i.e. there exists a homeomorphism mapping the orbits of the perturbed flow to the orbits of the original flow. Therefore, for many problems related to the topological theory of these flows, one can assume higher regularity for the flow. However, for our purposes, it suffices for the generating vector field to be C^1.

In [1] and [2], it is shown that C^1 Anosov vector fields span the intersection of a pair of transverse positive and negative contact structures, i.e. a *bi-contact structure*. However, it is known that the inverse is not true. As a matter of fact, non-zero vector fields in the intersection of a bi-contact structure define a considerably larger class of vector fields, namely *projectively Anosov vector fields*. By [1, 2], this is equivalent to the following definition:

Definition 2.2.3. *We call a flow ϕ^t, generated by the C^1 vector field X,* projectively Anosov*, if its induced flow on $TM/\langle X \rangle$ admits a* dominated splitting*. That is, there exists a splitting $TM/\langle X \rangle = E^s \oplus E^u$, such that the splitting is continuous and invariant under $\tilde{\phi}^t_*$ and*

$$||\tilde{\phi}^t_*(v)||/||\tilde{\phi}^t_*(u)|| \geq Ae^{Ct}||v||/||u||$$



(a) Anosov flows (b) Projectively Anosov flows

Figure 2.1: The local behavior of (projectively) Anosov flows

for any $v \in E^u$ *(*unstable direction*) and* $u \in E^s$ *(*stable direction*), where* C *and* A *are positive constants,* $||.||$ *is induced from some Riemannian metric on* $TM/\langle X\rangle$ *and* $\tilde{\phi}^t_*$ *is the flow induced on* $TM/\langle X\rangle$*, via the projection* $\pi : TM \to TM/\langle X\rangle$*.*

Moreover, we call the vector field X*, a* projectively Anosov vector field*.*

Similar to Anosov flows, we assume the orientability of the stable and unstable directions of projectively Anosov flows in this paper.

In [1, 2], it is shown:

Proposition 2.2.4. *Let* X *be a* C^1 *vector field on* M*. Then,* X *is projectively Anosov, if and only if, there exist positive and negative contact structures,* ξ_+ *and* ξ_- *respectively, which are transverse and* $X \subset \xi_+ \cap \xi_-$*.*

This motivates the following definitions:

Definition 2.2.5. *We call the pair* (ξ_-, ξ_+) *a* bi-contact structure *on* M*, if* ξ_+ *and* ξ_- *are positive and negative contact structures on* M*, respectively, and* $\xi_- \pitchfork \xi_+$*.*

Definition 2.2.6. *Let* X *be a projectively Anosov vector field on* M*. We call a bi-contact structure* (ξ_-, ξ_+) *a* supporting *bi-contact structure for* X*, or the generated projectively Anosov flow, if* $X \subset \xi_- \cap \xi_+$*. We call a positive (negative) contact structure or more generally, any plane field* ξ*, a supporting positive (negative) contact structure or plane field, respectively, for* X *or the generated flow, if* $X \subset \xi$*.*

See Example 2.5.7 for explicit examples of projectively Anosov flows (which are not Anosov). Similar examples can be constructed on Nil manifolds as well [1]. Also, see [44] for using the idea of *hyperbolic plugs* [51] to construct of projectively Anosov flows (which cannot be deformed, through projectively Anosov flows, into Anosov flows) on atoridal manifolds.

Consider the vector bundle $\pi : TM \to TM/\langle X \rangle$ and notice that for any plane field η which is transverse to the flow, there exists a natural vector bundle isomorphism $TM/\langle X \rangle \simeq \eta$, induced by the projection onto η and along X. Therefore, π can be interpreted as such projection as well.

We also notice that in Definition 2.2.3, the line fields $E^u, E^s \subset TM/\langle X \rangle$ do not necessarily lift to invariant line fields $E^{uu}, E^{ss} \subset TM$, respectively (see [29] for the examples of when they do not). However, it is a classical fact from dynamical systems that when the induced flow on $TM/\langle X \rangle$ (usually called the *Linear Poincaré Flow*) admits an invariant continuous *hyperbolic splitting* $E^s \oplus E^u$ (uniformly contracting along E^s and expanding along E^u), such lift does exist and we we will have an invariant splitting as in Definition 2.2.1 (see [52], Proposition 1.1).

Definition 2.2.7. *We call a projectively Anosov flow (vector field)* balanced*, if it preserves a transverse plane field η.*

Proposition 2.2.8. *The flow ϕ^t is a balanced projectively Anosov, if and only if, there exists a splitting $TM = E^{ss} \oplus E^{uu} \oplus \langle X \rangle$, such that the splitting is continuous and invariant under ϕ^t_* and*

$$||\phi^t_*(v)||/||\phi^t_*(u)|| \geq Ae^{Ct}||v||/||u||$$

*for any $v \in E^{uu}$ (*strong unstable direction*) and $u \in E^{ss}$ (*strong stable direction*), where C and A are positive constants, and X is the C^1 generator of the flow.*

Proof. We can easily observe $E^{ss} = \eta \cap \pi^{-1}(E^s)$ and $E^{uu} = \eta \cap \pi^{-1}(E^u)$. □

Remark 2.2.9. *It can be seen that given a projectively Anosov flow, the plane fields $\pi^{-1}(E^s)$ and $\pi^{-1}(E^u)$ are C^0 integrable plane fields, named* stable and unstable foliations*, respectively. In the Anosov case, thanks to ergodic theory, higher regularity of these foliations can be assumed and that is the basis for the use of foliation theory to study Anosov dynamics. Therefore, such tools are not all well transferred to projectively Anosov dynamics in general. However, assuming more regularity for the associated foliations of a projectively Anosov flow, some rigidity results are known [29, 30].*

It is worth to pause and make few observations about the geometry of projectively Anosov flows (the remark is discussed more in depth in [2]).

Remark 2.2.10. *If X is some vector field on M, which is tangent to some plane field ξ, we can measure the contactness of ξ from the rotation of the flow, with respect to ξ, in the following way. Choose some transverse plane field η, which is differentiable in the direction of X (for instance, if X is a balanced projectively Anosov vector field, $E^{ss} \oplus E^{uu}$ can be chosen) and orient it such that X and η induce the chosen orientation of M. Let $\lambda = \xi \cap \eta$ and $\lambda^t_p = \phi^{-t}_*(\xi_{\phi^t(p)}) \cap \eta$ for $x \in M$ and $t \in \mathbb{R}$. Finally, let θ^t_p be the angle between λ_p and $\lambda_p(t)$, for some Riemannian metric, which is differentiable in the direction of X. Then, ξ is a positive or negative contact structure, if and only if,*

$$X \cdot \theta_p(t) < 0 \quad \textit{or} \quad X \cdot \theta_p(t) > 0,$$

respectively, for all p and t.

Now, if X is a projectively Anosov vector field, and (ξ_-, ξ_+) a bi-contact structure such that $X \subset \xi_- \cap \xi_+$, let η be any transverse plane field, and $\lambda_+ = \xi_+ \cap \eta$ and $\lambda_- = \xi_- \cap \eta$. Similar to above, we can define $\lambda^t_{+,p}$ and $\lambda^t_{-,p}$ and observe

$$\lim_{t \to +\infty} \lambda^t_{+,p} = \lim_{t \to +\infty} \lambda^t_{-,p} = \pi^{-1}(E^s) \cap \eta$$

and

$$\lim_{t\to-\infty}\lambda^t_{-,p}=\lim_{t\to-\infty}\lambda^t_{+,p}=\pi^{-1}(E^u)\cap\eta,$$

Equivalently,

$$\lim_{t\to+\infty}\phi^t_*(\xi_+)=\lim_{t\to+\infty}\phi^t_*(\xi_-)=\pi^{-1}(E^u)$$

and

$$\lim_{t\to-\infty}\phi^t_*(\xi_+)=\lim_{t\to-\infty}\phi^t_*(\xi_-)=\pi^{-1}(E^s).$$

Naturally, we can characterize Anosovity of a projectively Anosov vector field by the *expansion rate of its stable and unstable directions*.

We first note that the norm used in the definition of a (projectively) Anosov flow X is in general induced from some C^0 Riemannian structure g. However, if we replace g with $g^T=\frac{1}{T}\int_0^T\phi^{t*}gdt$, where ϕ^t is the flow of X, the resulting Riemannian metric will be differentiable in X-direction, i.e. $\mathcal{L}_Xg^T$ would exist. Moreover, by considering large enough T, with respect to such metric, we can assume $A=1$ in the above definitions, meaning that the expansion or contraction in unstable and stable directions, respectively, for an Anosov flow, or the relative expansion for a projectively Anosov flow, start immediately. Assuming such conditions, we can compute the infinitesimal rate of expansion for vectors in the stable and unstable directions. Remember that any transverse plane field η induces a vector bundle isomorphism to $TM/\langle X\rangle$. Using such isomorphism, the restriction $g|_\eta$ of any Riemannian metric g on TM, defines a metric on $TM/\langle X\rangle$, and conversely, given any Riemannian metric on $TM/\langle X\rangle$, we can define a metric on TM, whose restriction on η is induced from such metric.

Let $\hat{e}_u\in E^u\subset TM/\langle X\rangle$ be the unit vector field (with respect to some Riemannian metric) defined in the neighborhood of a point. Noticing that the linear flow on $TM/\langle X\rangle$

preserves the direction of $\hat{e_u}$, we compute:

$$\mathcal{L}_X \tilde{e}_u = \frac{\partial}{\partial t}\tilde{\phi}_*^{-t}(\tilde{e}_u)\bigg|_{t=0} = \frac{\partial}{\partial t}\frac{\tilde{\phi}_*^{-t}\left(\tilde{\phi}_*^{t}(\tilde{e}_u)\right)}{||\tilde{\phi}_*^{t}(\tilde{e}_u)||}\Bigg|_{t=0} = \left(\frac{\partial}{\partial t}\frac{1}{||\tilde{\phi}_*^{t}(\tilde{e}_u)||}\right)\Bigg|_{t=0}\tilde{e}_u$$

$$= -\left(\frac{\partial}{\partial t}||\tilde{\phi}_*^{t}(\tilde{e}_u)||\right)\Bigg|_{t=0}\tilde{e}_u = -\left(\frac{\partial}{\partial t}\ln||\tilde{\phi}_*^{t}(\tilde{e}_u)||\right)\Bigg|_{t=0}\tilde{e}_u.$$

We can do similar computation for the (locally defined) unit vector field $\tilde{e}_s \in E^s$.

Definition 2.2.11. *Using the above notation, we define* the expansion rate of the (un)stable direction *as*

$$r_s := \frac{\partial}{\partial t}\ln||\tilde{\phi}_*^{t}(\tilde{e}_s)||\bigg|_{t=0} \quad \left(r_u := \frac{\partial}{\partial t}\ln||\tilde{\phi}_*^{t}(\tilde{e}_u)||\bigg|_{t=0}\right).$$

We note that similar notions have been previously used and studied in the literature. For instance, see [53, 54]. Naturally, the positive and negative expansion rates correspond to the expanding and contracting behaviors of the flow in a certain direction, respectively.

Proposition 2.2.12. *The above computation shows:*

$$\mathcal{L}_X \tilde{e}_s = -r_s \tilde{e}_s \ \left(\mathcal{L}_X \tilde{e}_u = -r_u \tilde{e}_u\right),$$

and

$$\tilde{\phi}_*^{T}(\tilde{e}_s) = e^{\int_0^T r_s(t)dt}\tilde{e}_s \ \left(\tilde{\phi}_*^{T}(\tilde{e}_u) = e^{\int_0^T r_u(t)dt}\tilde{e}_u\right).$$

Now consider a transverse plane field $\eta \simeq TM/\langle X\rangle$, equipped with a Riemannian metric $\tilde{g}$ on $TM/\langle X\rangle$, defining the stable and unstable expansion rate of r_s, r_u. From $\tilde{g}$, a Riemannian metric is induced on η, which can be extended to a X-differentiable Riemannian metric g on TM, assuming that $\tilde{g}$ and η are X-differentiable. Let $e_s, e_u \in \eta$ be chosen such that $\pi(e_s) = \tilde{e}_s$ and $\pi(e_u) = \tilde{e}_u$, and notice that $||e_s|| = ||e_u|| = 1$. Let π_η be the projection onto η along X and compute

$$\mathcal{L}_X e_u = \frac{\partial}{\partial t}\phi_*^{-t}(e_u)\bigg|_{t=0} = \frac{\partial}{\partial t}\pi_\eta\big(\phi_*^{-t}(e_u)\big)\bigg|_{t=0} + q_u^\eta X,$$

for some function $q_u^\eta : M \to \mathbb{R}$. Since the metric on η is induced from $\tilde{g}$, this implies

$$\mathcal{L}_X e_u = -r_u e_u + q_u^\eta X.$$

Similarly,

$$\mathcal{L}_X e_s = -r_s e_s + q_s^\eta X,$$

for some function $q_s^\eta : M \to \mathbb{R}$. We have proved:

Proposition 2.2.13. *Let X be a projectively Anosov vector field with r_s, r_u being its expansion rate of the stable and unstable directions (with respect to some metric on $TM/\langle X\rangle$), respectively. Then, for any transverse plane field η, there exists a metric on TM such that for unit vector fields $e_u \in \eta \cap \pi^{-1}(E^u)$ and $e_s \in \eta \cap \pi^{-1}(E^s)$, we have*

$$\mathcal{L}_X e_u = -r_u e_u + q_u^\eta X,$$

and

$$\mathcal{L}_X e_s = -r_u e_s + q_s^\eta X,$$

for appropriate real functions $q_u^\eta, q_s^\eta : M \to \mathbb{R}$.

We also observe the following fact, which we will use in the proof of Theorem 2.1.1:

Proposition 2.2.14. *Let X be a projectively Anosov vector field. When X is balanced (in particular, when X is Anosov), there exists a transverse plane field η as in Proposition 2.2.13, for which $q_u^\eta = q_s^\eta = 0$ everywhere. In this case,*

$$\mathcal{L}_X e_s = -r_s e_s \ \left(\mathcal{L}_X e_u = -r_u e_u\right),$$

and

$$\phi_*^T(e_s) = e^{\int_0^T r_s(t)dt} e_s \ \left(\phi_*^T(e_u) = e^{\int_0^T r_u(t)dt} e_u\right).$$

The definition of projectively Anosov vector fields implies:

Proposition 2.2.15. *Let X be a projectively Anosov vector field and r_s and r_u, the expansion rates of the stable and unstable directions, respectively, with respect to any Riemannian metric, satisfying the metric condition of Definition 2.2.3 with $A = 1$, which is X-differentiable, then*

$$r_u - r_s > 0.$$

Proof. Since X is projectively Anosov, the exists a X-differentiable Riemannian metric g, such that

$$||\tilde{\phi}_*^t(\tilde{e}_u)||/||\tilde{\phi}_*^t(\tilde{e}_s)|| \geq e^{Ct}||\tilde{e}_u||/||\tilde{e}_s||$$

where $\tilde{\phi}^t$ is the flow of X, $||.||$ is the norm on $TM/\langle X\rangle$, induced from g, $\tilde{e}_u \in E^u$ and $\tilde{e}_s \in E^s$ are unit vectors, and C is a positive constants. Therefore,

$$\ln||\tilde{\phi}_*^t(\tilde{e}_u)|| - \ln||\tilde{\phi}_*^t(\tilde{e}_s)|| \geq Ct$$

and

$$r_u - r_s = \frac{\partial}{\partial t}\ln||\tilde{\phi}_*^t(\tilde{e}_u)||\bigg|_{t=0} - \frac{\partial}{\partial t}\ln||\tilde{\phi}_*^t(\tilde{e}_s)||\bigg|_{t=0} \geq C > 0.$$

□

Remark 2.2.16. *In proof of Theorem 2.1.1, we will also see that inverse of the above proposition also holds, in the sense that given a C^1 projectively Anosov vector field, for any Riemannian metric with $r_u - r_s > 0$, the plane fields $\langle X, \pi^{-1}(\frac{e_u+e_s}{2})\rangle$ and $\langle X, \pi^{-1}(\frac{e_u-e_s}{2})\rangle$ define positive and negative contact structures, respectively, possibly after a perturbation to make the plane fields C^1.*

Similar computation, using the definition of Anosov flows and the fact that hyperbolicity of $TM/\langle X\rangle$ implies Anosovity of the flow ([52], Proposition 1.1), yields:

Proposition 2.2.17. *Let X be a projectively Anosov vector field and r_s and r_u. Then X is Anosov, if and only if, with respect to some Riemannian metric, we have*

$$r_u > 0 > r_s.$$

Remark 2.2.18. *The above computation also shows that both Anosovity and projective Anosovity are preserved under reparametrizations of the flow. More precisely, let X is projectively Anosov vector field with expansion rates of r_s and r_u, in the stable and unstable directions, respectively (with respect to some metric). Then, for any positive function $f : M \to \mathbb{R}^{>0}$, the vector field fX has expansion rates of fr_s and fr_u, in the stable and unstable directions, respectively (with respect to the same metric). Therefore, the conditions of both Proposition 2.2.15 and Proposition 2.2.17, are preserved under such transformations.*

2.3 Contact and symplectic geometric characterization of Anosov 3-flows

The goal of this section is to prove Theorem 2.1.1, giving a purely contact and symplectic geometric characterization of Anosov flows in dimension 3.

Remark 2.3.1. *In [1], Mitsumatsu shows that the generator vector field of any smooth volume preserving Anosov flow lies in the intersection of a pair of transverse negative and positive contact structures, admitting contact forms α_- and α_+, respectively, such that (α_-, α_+) is a Liouville pair (see Definition 1.1.11). Beside the symmetry induced by the existence of an invariant volume form (see [5]), the crucial ingredient is the fact that the weak stable and unstable bundles are known to be at least C^1 [28]. Therefore the Anosovity of the flow can be translated easily to the differential geometry, and in particular, the contact geometry of the underlying manifold (note that contact structures are at least*

C^1). We remark that although it is known now that these invariant bundles are C^1 for any smooth Anosov flow in dimension 3 [53], such plane fields are only Hölder continuous, if we want to generalize the result to Anosov flows of lower regularity. In the following, we improve Mitsumatsu's results by showing that the same holds without any regularity assumption on the weak bundles, using careful approximations of these plane fields. We note that to prove the converse of this statement, these approximations are necessary even for smooth flows, since Anosovity of the flow is not assumed (and the weak bundles of smooth projectively Anosov flows are not necessarily C^1). Moreover, we believe that the applications of these approximation techniques can be furthered to other questions about Anosov flows, even when the invariant bundles are C^1, since in that case, they facilitate controlling the second variations of these plane fields (and therefore, the associated Reeb vector fields for the underlying contact structures) along the flow. See Chapter 4 for such application in the surgery theory of Anosov flows.

Theorem 2.3.2. *[5] Let ϕ^t be a flow on the 3-manifold M, generated by the C^1 vector field X. Then ϕ^t is Anosov, if and only if, $\langle X\rangle = \xi_+ \cap \xi_-$, where ξ_+ and ξ_- are transverse positive and negative contact structures, respectively, and there exist contact forms α_+ and α_- for ξ_+ and ξ_-, respectively, such that (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are Liouville pairs.*

Proof. We begin by assuming ϕ^t is Anosov.

Let g be the C^0 Riemannian metric for which the condition of Anosovity is satisfied and $g(X, E^{ss}) = g(X, E^{uu}) = g(E^{ss}, E^{uu}) = 0$. After replacing g with $\frac{1}{T}\int_0^T \phi^{t*}g(t)dt$ for large T, we can assume the same orthogonality conditions hold, $\mathcal{L}_X g$ exists everywhere and the expansion and contraction of $E^u u$ and E^{ss} start immediately (i.e. can assume $A = 1$ in the Definition 2.2.1). This means that if $e_s \in E^{ss}$ and $e_u \in E^{uu}$ are unit vector fields, the stable and unstable expansion rates, r_s and r_u are defined and are negative and positive, respectively (Proposition 2.2.17). Moreover, choose such e_s and e_u so that (e_s, e_u, X) is an oriented basis for M as in Figure 2.2.

Let α_u^{pre} and α_s^{pre} be C^1-approximations for $\hat{e}_u$ and $\hat{e}_s$, g-duals of e_u and e_s, respectively,

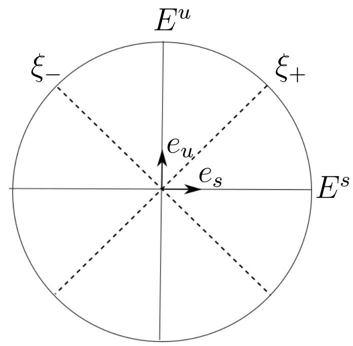


Figure 2.2: The splitting $TM/\langle X\rangle \simeq E^s \oplus E^u$

such that $\alpha_u^{pre}(X) = \alpha_s^{pre}(X) = 0$. To do so, we need to C^1-approximate E^s and E^u as line bundles in $TM/\langle X\rangle$. The direct sum of such line bundles with $\langle X\rangle$ yields the desired C^1 plane fields.

There exist continuous functions f_u and f_s which are differentiable in direction of X and $f_u\alpha_u^{pre}(e_u) = f_s\alpha_s^{pre}(e_s) = 1$. We can C^0-approximate f_u and f_s with smooth functions $\tilde{f}_u$ and $\tilde{f}_s$, such that $|X \cdot f_u - X \cdot \tilde{f}_u|$ and $|X \cdot f_s - X \cdot \tilde{f}_s|$ are arbitrary small. This can be achieved using a delicate partition of unity, which respects the differentiation in the direction of the flow:

Lemma 2.3.3. *Let ϕ^t be any non-singular flow, generated by a C^1 vector field X on a closed manifold M (of arbitrary dimension) and f be any X-differentiable continuous function. For any $\epsilon > 0$, there exists a differentiable function $\tilde{f}$, such that*

$$|f - \tilde{f}| < \epsilon \quad \text{and} \quad |X \cdot f - X \cdot \tilde{f}| < \epsilon.$$

Proof. We can find such function using local solutions and a partition of unity. However, since we want to control the derivative of such function in the direction of the flow, we need to control the parameters of our partition of unity carefully.

Fix $\epsilon > 0$. Consider the collection $\{(U_i, V_i, \Sigma_i, \tau_i)\}_{1\leq i\leq N}$, where Σ_is are C^1 local sections of the flow ϕ^t (which are open disks) and for some $\epsilon_i > 0$, we can define the open flowboxes $U_i := \{\phi^t(x) \text{ s.t. } x \in \Sigma_i, -\tau_i < t < \tau_i\}$ and $V_i := \{\phi^t(x) \text{ s.t. } x \in \Sigma_i, -\frac{\tau_i}{2} < t < \frac{\tau_i}{2}\} \subset U_i$, such that $\{V_i\}_{1\leq i\leq N}$ is a covering for M. Notice, that we can find such covering, since for any $x \in M$, we can find such $(U_x, V_x, \Sigma_x, \tau_x)$, where $x \in \Sigma_x$ and ϵ_x is sufficiently small. Compactness of M implies that finitely many of V_x s cover M. Therefore, we get the desired collection.

Let $\{\psi_i\}_{1\leq i\leq N}$ be a partition of unity with respect to such covering. In particular, we have $supp(\psi_i) \subset V_i$. Note that there exists a compact subset $\tilde{\Sigma}_i \subset \Sigma_i$, such that $supp(\psi_i) \subset \{\phi^t(x) \text{ s.t. } x \in \tilde{\Sigma}_i, -\frac{\tau_i}{2} < t < \frac{\tau_i}{2}\}$. Define $h_i : \mathbb{R} \to \mathbb{R}$ to be a bump function such that $h_i([\frac{\tau_i}{2}, \frac{\tau_i}{2}]) = 1$, $h_i((-\infty, \tau_i] \cup [\tau_i, +\infty)) = 0$ and h_i is monotone elsewhere. Finally, let $C_i := \sup |\frac{dh_i}{dt}|$ and choose some positive real number $\delta_i \in \mathbb{R}$, such that we have $\max\{\delta_i, C_i\delta_i\} < \frac{\epsilon}{N}$.

We can write $f = \sum_{1\leq i\leq N} \psi_i f$. Let g_i be a C^1 function defined on Σ_i, such that $|g_i - \psi_i f| < \delta_i\big|_{\Sigma_i}$ and $g_i = 0$ on $\Sigma_i/\tilde{\Sigma}_i$. Now we can extend g_i to U_i by solving the differential equation $X \cdot g_i = X \cdot (\psi_i f)$ on Σ_i. Note that we have $|g_i - \psi_i f| < \delta_i$, everywhere on U_i. In particular, $|g_i| < \delta_i$ on $U_i \backslash V_i$.

We can then define $\tilde{g}_i$ on M, by letting $\tilde{g}_i(\phi^t(x)) = h_i(t) g_i(\phi^t(x))$ for any $x \in \Sigma_i$ and $\tilde{g}_i = 0$ on $M \backslash U_i$. Note that $|\tilde{g}_i - \psi_i f| < \delta_i$. Moreover, since $X \cdot \tilde{g}_i = (X \cdot h_i) g_i + h_i(X \cdot g_i)$, we have $|X \cdot \tilde{g}_i - X \cdot (\psi_i f)| = |X \cdot g_i - X \cdot (\psi_i f)| = 0$ on $\tilde{V}_i$, $|X \cdot \tilde{g}_i - X \cdot (\psi_i f)| = |(X \cdot h_i) g_i| < C_i \delta_i$ on $U_i \backslash \tilde{V}_i$ and $X \cdot \tilde{g}_i = X \cdot (\psi_i f) = 0$ elsewhere. Therefore, we have $|X \cdot \tilde{g}_i - X \cdot (\psi_i f)| < C_i \delta_i$ everywhere. Now, we can see that $\tilde{f} := \sum_{1\leq i\leq N} \tilde{g}_i$ is the desired function, since it is C^1 by construction and we have

$$|\tilde{f} - f| < \sum_{1\leq i\leq N} |\tilde{g}_i - \psi_i f| < \sum_{1\leq i\leq N} \delta_i < \epsilon$$

and

$$|X \cdot \tilde{f} - X \cdot f| < \sum_{1\leq i\leq N} |X \cdot \tilde{g}_i - X \cdot (\psi_i f)| < \sum_{1\leq i\leq N} C_i \delta_i < \epsilon.$$

$\square$

Using the above lemma, we can find such $\tilde{f}_u$ and $\tilde{f}_s$ for which

$$X \cdot [\tilde{f}_u \alpha_u^{pre}(e_u)] + (\min_{x \in M} r_u)\tilde{f}_u \alpha_u^{pre}(e_u) > 0 \tag{2.1}$$

and

$$X \cdot [\tilde{f}_s \alpha_s^{pre}(e_s)] + (\max_{x \in M} r_s)\tilde{f}_s \alpha_s^{pre}(e_s) < 0. \tag{2.2}$$

Define $\alpha_u^0 := \tilde{f}_u \alpha_u^{pre}$ and $\alpha_s^0 := \tilde{f}_s \alpha_s^{pre}$.

In the following, when there is no confusion, for any point $x \in M$, we refer to $r_s(x)$ by r_s or $r_s(0)$ and to $r_s(\phi^t(x))$ by $r_s(t)$. Similarly, for other functions in this proof.

Now define

$$\alpha_u^T := I_u^T \phi^{T*} \alpha_u^0,$$

$$\alpha_s^T := I_s^{-T} \phi^{-T*} \alpha_s^0;$$

where

$$I_u^T := e^{-\int_0^T r_u(t)dt},$$

$$I_s^T := e^{-\int_0^T r_s(t)dt}.$$

Claim 2.3.4.

$$\alpha_u^T(e_u(0)) = \alpha_u^0(e_u(T)),$$

$$\alpha_s^T(e_s(0)) = \alpha_s^0(e_s(T)).$$

Proof.

$$\alpha_u^T(e_u(0)) = I_u^T \alpha_u^0(\phi_*^T(e_u(0))) = I_u^T \alpha_u^0(\frac{1}{I_u^T} e_u(T)) = \alpha_u^0(e_u(T)),$$

where the middle equality is implied by Proposition 2.2.14. Other implication follows similarly. $\square$

Claim 2.3.5.

$$\lim_{T\to+\infty}\frac{I_u^T}{I_s^T}=\lim_{T\to+\infty}\frac{I_s^{-T}}{I_u^{-T}}=0.$$

Proof.

$$\lim_{T\to+\infty}\frac{I_u^T}{I_s^T}=\lim_{T\to+\infty}e^{\int_0^T r_s(t)-r_u(t)dt}=0.$$

The last equality follows from projective Anosovity of X (Proposition 2.2.15), implying $r_u-r_s>0$.

Similarly,

$$\lim_{T\to+\infty}\frac{I_s^{-T}}{I_u^{-T}}=\lim_{T\to+\infty}e^{\int_{-T}^0 r_s(t)-r_u(t)dt}=0.$$

. □

Claim 2.3.6.

$$X\cdot I_u^T=[r_u(0)-r_u(T)]I_u^T;$$

$$X\cdot I_s^T=[r_s(0)-r_s(T)]I_s^T.$$

Proof.

$$X\cdot I_u^T=\frac{\partial}{\partial h}e^{-\int_0^T r_u(t+h)dt}\bigg|_{h=0}$$

$$=[-\int_0^T r_u'(t)dt]I_u^T=[r_u(0)-r_u(T)]I_u^T.$$

The other implication follows similarly.

□

Now, using the above calculations, we can show that $\ker\alpha_u^T$ and $\ker\alpha_s^T$, C^0-converge to $\pi^{-1}(E^s)=E^{ss}\oplus\langle X\rangle$ and $\pi^{-1}(E^u)=E^{uu}\oplus\langle X\rangle$, respecting certain C^1-quantities.

Lemma 2.3.7. *We have*

$$\lim_{T\to+\infty}\ker\alpha_u^T=\pi^{-1}(E^s)$$

and

$$\lim_{T\to+\infty} \ker \alpha_s^T = \pi^{-1}(E^u).$$

Proof. First compute

$$\lim_{T\to+\infty} \alpha_u^T(e_s(0)) = \lim_{T\to+\infty} I_u^T \alpha_u^0(\phi_*^T e_s(0)) = \lim_{T\to+\infty} \frac{I_u^T}{I_s^T} \alpha_u^0(e_s(T)) = 0.$$

The last equality follows from Claim 2.3.5 and the fact that $\alpha_u^0(e_s)$ is bounded. Similarly,

$$\lim_{T\to+\infty} \alpha_s^T(e_u(0)) = 0.$$

Claim 2.3.4 and the fact that $\alpha_u^T(X) = \alpha_s^T(X) = 0$ finish the proof. □

Now, we see that certain C^1-variations behave nicely under such limiting procedure.

Lemma 2.3.8.

$$\lim_{T\to+\infty} \alpha_u^T \wedge d\alpha_u^T = \lim_{T\to+\infty} \alpha_s^T \wedge d\alpha_s^T = 0.$$

Proof. Using Claim 2.3.4, Claim 2.3.5 and Claim 2.3.6, compute

$$(\alpha_u^T \wedge d\alpha_u^T)(e_s, e_u, X) = \alpha_u^T(e_s)\left[-X \cdot (\alpha_u^T(e_u)) + \alpha_u^T(\mathcal{L}_X e_u)\right]$$

$$-\alpha_u^T(e_u)\left[X \cdot (\alpha_u^T(e_s)) - \alpha_u^T(\mathcal{L}_X e_s)\right]$$

$$= \alpha_u^0(e_s(T))\left[-X \cdot (\alpha_u^0(e_u(T))) - r_u \alpha_u^0(e_u(T))\right] \frac{I_u^T}{I_s^T}$$

$$-\alpha_u^0(e_u(T))\left[(r_u(0) - r_u(T) - r_s(0) + r_s(T))\, \alpha_u^0(e_s(T)) + X \cdot (\alpha_u^0(e_s(T))) + r_s \alpha_u^0(e_s(T))\right] \frac{I_u^T}{I_s^T}$$

$$= A(x)\frac{I_u^T}{I_s^T},$$

where $A(x)$ is a bounded function on M.

Claim 2.3.5 concludes the implication and similar computation for $\alpha_s^T \wedge d\alpha_s^T$ finishes the proof. □

Remark 2.3.9. *Using Proposition 2.2.13, one can easily check that Claim 2.3.4, 2.3.5, 2.3.6 and Lemma 2.3.7, 2.3.8 also hold for similar approximations, when the flow is merely projectively Anosov.*

Lemma 2.3.10. *For large enough T, $\alpha_u^T \wedge d\alpha_s^T$ and $\alpha_s^T \wedge d\alpha_u^T$ are negatively bounded away from 0.*

Proof.

$$(\alpha_u^T \wedge d\alpha_s^T)(e_s, e_u, X)$$

$$= \alpha_u^T(e_s)[-X \cdot (\alpha_s^T(e_u)) - \alpha_s^T(-\mathcal{L}_X e_u)] + \alpha_u^T(e_u)[X \cdot (\alpha_s^T(e_s)) - \alpha_s^T(\mathcal{L}_X e_s)]$$

$$= \frac{I_u^T}{I_s^T}\frac{I_s^{-T}}{I_u^{-T}} A(x) + \alpha_u^0(e_u(T))[X \cdot (\alpha_s^0(e_s(T))) + r_s \alpha_s^0(e_s(T))],$$

where $A(x)$ is a bounded function on M. Using Claim 2.3.5, the first term vanishes in the limit and we will have

$$\alpha_u^T \wedge d\alpha_s^T < 0,$$

since by criteria (2.2) we forced the second term to be negatively bounded away from 0.

Similar computation and criteria (2.1) implies the other statement. □

Now we have all the ingredients to finish the proof.

Let $\alpha_+^T := \frac{1}{2}(\alpha_u^T - \alpha_s^T)$ and $\alpha_-^T := \frac{1}{2}(\alpha_u^T + \alpha_s^T)$. The goal is to show that (α_-^T, α_+^T) and $(-\alpha_-^T, \alpha_+^T)$ are Liouville pairs, for large T. By Lemma 2.3.8 and 2.3.10, for large T:

$$\alpha_+^T \wedge d\alpha_+^T = \frac{1}{4}\left(\alpha_u^T \wedge d\alpha_u^T - \alpha_u^T \wedge d\alpha_s^T - \alpha_s^T \wedge d\alpha_u^T + \alpha_s^T \wedge d\alpha_s^T\right) > 0.$$

Therefore, α_+^T is a positive contact form for large T. Similar computation shows that α_-^T is a negative contact form for large T.

To show that (α_-^T, α_+^T) is a Liouville pair, we need to show that $\omega^T := d\alpha^T$ is a symplectic form on $[-1,1] \times M$, where $\alpha^T := \{\alpha_t^T\}_{t \in [-1,1]}$ and $\alpha_t^T := (1-t)\alpha_-^T + (1+t)\alpha_+^T = \alpha_u^T - t\alpha_s^T$.

Compute for large T:

$$\omega^T \wedge \omega^T = (d\alpha_u^T - td\alpha_s^T - dt \wedge \alpha_s^T) \wedge (d\alpha_u^T - td\alpha_s^T - dt \wedge \alpha_s^T) =$$

$$= dt \wedge \{-2\alpha_s^T \wedge d\alpha_u^T + 2t\alpha_s^T \wedge d\alpha_s^T\} > 0.$$

Then, Lemma 2.3.8 and 2.3.10 imply that ω^T is symplectic for large T.

To show that $(-\alpha_-^T, \alpha_+^T)$ is a Liouville pair, let $\tilde{\omega}^T := d\tilde{\alpha}^T$, where $\tilde{\alpha}^T := \{\tilde{\alpha}_t^T\}_{t\in[-1,1]}$ and $\tilde{\alpha}_t^T := -(1-t)\alpha_-^T + (1+t)\alpha_+^T = \alpha_s^T - t\alpha_u^T$. Similar computation shows:

$$\tilde{\omega}^T \wedge \tilde{\omega}^T = dt \wedge \{-2\alpha_u^T \wedge d\alpha_s^T + 2t\alpha_u^T \wedge d\alpha_u^T\} > 0,$$

implying that $\tilde{\omega}^T$ is symplectic for large T and finishing the proof of one implication.

We now consider the other implication.

Note that by Proposition 2.2.4, such flow is projectively Anosov and therefore, we have the splitting $TM/\langle X\rangle \simeq E^s \oplus E^u$. Without loss of generality, assume α_+ and α_- induce the same orientation on $\pi^{-1}(E^u)$ and opposite orientations on $\pi^{-1}(E^s)$ (recall that π is the fiberwise projection $TM \to TM/\langle X\rangle$). The idea is to show that for any point in M, when constructing the Liouville form by linearly interpolating α_+ and α_- (or $-\alpha_-$), the symplectic condition at the *time* when the kernel of the interpolation is $\pi^{-1}(E^s)$ (or $\pi^{-1}(E^u)$), implies $r_u > 0$ (or $r_s < 0$). Of course, such *time* is a continuous function on the manifold. But it turns out that, thanks to the openness of the symplectic condition, suitable approximation by a C^1 function suffices.

Orient $\pi^{-1}(E^u)$ such that $\alpha_+(\pi^{-1}(E^u)) > 0$ and $\alpha_-(\pi^{-1}(E^u)) > 0$. Also orient $\pi^{-1}(E^s)$ such that $\alpha_+(\pi^{-1}(E^s)) < 0 < \alpha_-(\pi^{-1}(E^s))$.

Let $\tau_u(x)$ be the (continuous) function such that

$$\ker\{(1-\tau_u)\alpha_- + (1+\tau_u)\alpha_+\} = \pi^{-1}(E^s),$$

and set

$$\alpha_u := (1-\tau_u)\alpha_- + (1+\tau_u)\alpha_+.$$

Consider a transverse plane field η and define $||.|||\big|_{\pi^{-1}(E^u)}$ such that for a unit e_u orienting $\pi^{-1}(E^u)\cap\eta$, we have $\alpha_u(e_u) = 1$. Note that we can rewrite $\alpha_t := (1-t)\alpha_- + (1+t)\alpha_+$ as

$$\alpha_t = \alpha_u - (t-\tau_u)\beta_s;$$

where $\beta_s = \frac{-\alpha_+ + \alpha_-}{2}$ is a C^1 1-form and $\beta_s(\pi^{-1}(E^s)) > 0$.

Similarly, considering the Liouville pair $(-\alpha_-, \alpha_+)$, define $||.|||\big|_{\pi^{-1}(E^s)}$ and let e_s be the unit vector orienting $\pi^{-1}(E^s)\cap\eta$. Note that (e_s, e_u, X) is an oriented basis for M (see Figure 2.2). Using the vector bundle isomorphism $TM/\langle X\rangle \simeq \eta$, we can extend such norm to a Riemannian metric $\hat{g}$ on $TM/\langle X\rangle$ with $\hat{g}(E^s, E^u) = 0$.

Let α_u^T and α_s^T be the C^0-approximations of α_u and α_s, which are C^1, in the same fashion as above and define τ_u^T, such that

$$\ker \alpha_u^T = \ker\{(1-\tau_u^T)\alpha_- + (1+\tau_u^T)\alpha_+\}.$$

Note that τ_u^T is C^1 and we can rewrite

$$\alpha_t = f_u^T\alpha_u^T - (t-\tau_u^T)\beta_s$$

for continuous function $f_u^T = \alpha_t(e_u)\big|_{t=\tau_u^T}$ (but $f_u^T\alpha_u^T$ is C^1, since every other term in the above equation is C^1).

Observe that

$$\lim_{T\to+\infty} \tau_u^T = \tau_u,$$

since $\lim_{T\to+\infty} \ker \alpha_u^T = \ker \alpha_u$ (Lemma 2.3.7). Plug in e_u into α_t at $t = \tau_u^T$ to get

$$f_u^T \alpha_u^T(e_u) = 1 + (\tau_u - \tau_u^T)\beta_s(e_u)$$

and in particular,

$$\lim_{T\to+\infty} f_u^T \alpha_u^T(e_u) = 1. \tag{2.3}$$

Similarly, plug in e_s into α_t at $t = \tau_u^T$ to get

$$\alpha_u^T(e_s) = \frac{(\tau_u - \tau_u^T)\beta_s(e_s)}{f_u^T}.$$

Compute

$$\begin{aligned} &X \cdot (\alpha_u^T(e_s)) = \\ &\frac{[X\cdot(\tau_u - \tau_u^T)\beta_s(e_s) + (\tau_u - \tau_u^T)X\cdot(\beta_s(e_s))]f_u^T\alpha_u^T(e_u)}{(f_u^T\alpha_u^T(e_u))^2} \\ &-\frac{[X\cdot(\tau_u - \tau_u^T)\beta_s(e_u) + (\tau_u - \tau_u^T)X\cdot(\beta_s(e_u))](\tau_u - \tau_u^T)\beta_s(e_s)\alpha_u^T(e_u)}{(f_u^T\alpha_u^T(e_u))^2} \\ &= \frac{A(x)(\tau_u - \tau_u^T) + B(x)X\cdot(\tau_u - \tau_u^T)}{(f_u^T\alpha_u^T(e_u))^2} \end{aligned}$$

for bounded functions A and $B = f_u^T\beta_s(e_s)\alpha_u^T(e_u) + \beta_s(e_u)\beta_s(e_s)\alpha_u^T(e_u)(\tau_u - \tau_u^T)$.

Since $\lim_{T\to+\infty} X\cdot(\alpha_u^T(e_s)) = 0$ and B is non-zero for large T, we have

$$\lim_{T\to+\infty} X\cdot(\tau_u - \tau_u^T) = 0,$$

implying

$$\lim_{T\to+\infty} X\cdot[f_u^T\alpha_u^T(e_u)] = \lim_{T\to+\infty}\{X\cdot(\tau_u - \tau_u^T)\beta_s(e_u) + (\tau_u - \tau_u^T)X\cdot(\beta_s(e_u))\} = 0. \tag{2.4}$$

Also note that

$$\lim_{T\to+\infty} X\cdot[f_u^T\alpha_u^T(e_s)] = \lim_{T\to+\infty}\{X\cdot(\tau_u-\tau_u^T)\beta_s(e_s)+(\tau_u-\tau_u^T)X\cdot(\beta_s(e_s))\} = 0. \quad (2.5)$$

Now if $\alpha := \{\alpha_t\}_{t\in[-1,1]}$ and $\omega := d\alpha$, compute

$$\omega = d(f_u^T\alpha_u^T) - [dt - d\tau_u^T]\beta_s - (t-\tau_u^T)d\beta_s;$$

$$0 < \omega\wedge\omega\big|_{t=\tau_u^T} = dt\wedge 2\{-\beta_s\wedge d(f_u^T\alpha_u^T)+(t-\tau_u^T)\beta_s\wedge d\beta_s\}\big|_{t=\tau_u^T} = dt\wedge\{-2\beta_s\wedge d(f_u^T\alpha_u^T)\}.$$

Compute

$$[\beta_s\wedge d(f_u^T\alpha_u^T)](e_s,e_u,X) =$$

$$= \beta_s(e_s)[-X.(f_u^T\alpha_u^T(e_u)) - f_u^T\alpha_u^T(-\mathcal{L}_X e_u)] - \beta_s(e_u)[X.(f_u^T\alpha_u^T(e_s)) - f_u^T\alpha_u^T(-\mathcal{L}_X e_s)]$$

Now by (3.1), (3.2), (3.3) and Proposition 2.2.13:

$$0 < \omega\wedge\omega\big|_{t=\tau_u} = \lim_{T\to+\infty}\omega\wedge\omega\big|_{t=\tau_u^T} =$$

$$= \lim_{T\to+\infty} -dt\wedge\beta_s\wedge d(f_u^T\alpha_u^T) = \beta_s(e_s)r_u\, dt\wedge\hat{e}_s\wedge\hat{e}_u\wedge\hat{X}.$$

Therefore, $r_u > 0$.

Similarly, it can be shown that $r_s < 0$. This shows hyperbolicity of the splitting $TM/\langle X\rangle = E^s\oplus E^u$. By Proposition 1.1 of [52], this is equivalent to the flow being Anosov. $\square$

Remark 2.3.11. *By Theorem 2.1.1, if (ξ_-,ξ_+) is a supporting bi-contact structure for an Anosov flow, then ξ_- and ξ_+ are tight (see Theorem 1.1.7), strongly symplectically fillable [15, 16] and contain no Giroux torsion[19]. Furthermore, although in general universal tightness is not achieved from symplectic fillability, since any lift of an Anosov flow to any cover, is also Anosov, ξ_- and ξ_+ are universally tight in this case.*

Remark 2.3.12. *We note that Theorem 2.1.1 provides new geometric tools for understanding the periodic orbits of Anosov flows, in particular regarding the knot theory of such periodic orbits, which there are many unanswered questions about [55]. More precisely, if γ is a periodic orbit of an Anosov flow with supporting bi-contact structure (ξ_-, ξ_+), then γ is a* Legendrian knot *for both ξ_- and ξ_+. Furthermore, $I \times \gamma$ is an* exact Lagrangian *in both Liouville pairs, constructed on $I \times M$. These are standard and well-studied objects in contact and symplectic topology and now, those methods can be transferred to the study of such periodic orbits.*

2.4 Uniqueness of the underlying (bi-)contact structures

In this section, we want to establish various uniqueness theorems, about the (bi)-contact structures underlying a given (projectively) Anosov flow. Let X be the C^1 vector field generating such flow and ξ be any oriented plane field such that $X \subset \xi$. In particular, we want to establish the uniqueness, up to bi-contact homotopy (see Definition 2.5.1), of the supporting bi-contact structure, as well as explore the conditions under which, we can retrieve the information of such bi-contact structure, from only one of the contact structures. First, we need a definition.

Definition 2.4.1. *We call a vector $v \in T_pM$* dynamically positive (negative)*, if the plane $\langle v\rangle \oplus \langle X\rangle$ can be extended to a positive (negative) contact structure ξ, such that for some ξ_- (ξ_+), there exists a supporting bi-contact structure (ξ_-, ξ) ((ξ, ξ_+)) for X. We call a vector field v* dynamically positive (negative) *on the set $U \subset M$, if it is dynamically positive (negative) at every $p \in U$. Finally, we call a plane field ξ* dynamically positive (negative) *on the set $U \subset M$, if $\xi = \langle v\rangle \oplus \langle X\rangle$ for some dynamically positive (negative) vector field v on U.*

This is basically a mathematical way of saying that a vector (or vector field or a plane field) is dynamically positive (or negative) at a point, if it lies in the interior of the first or third region (the second or forth region) of Figure 2.1 (b). Note that if ξ_+ is a positive contact structure coming from a supporting bi-contact structure (ξ_-, ξ_+), by Remark 2.2.10, ξ_+ is dynamically positive everywhere. But this is not true in general. That is, a general supporting positive contact structure can be dynamically negative on a subset of the manifold. However, as we will shortly discuss, the behavior of the contact structure can be easily understood in such regions.

In particular, note that if (ξ_-, ξ_+) is a supporting bi-contact structure for X, then ξ_+ (ξ_-) is dynamically positive (negative) on M.

Next, we see that when a supporting positive (negative) contact structure is dynamically positive (negative) everywhere on M, it is in fact isotopic, through supporting positive (negative) contact structures, to a positive (negative) contact structure, coming from any given supporting bi-contact structure. In particular, such a contact structure is part of a supporting bi-contact structure.

Lemma 2.4.2. *Let (ξ_-, ξ_+) be a supporting bi-contact structure for the projectively Anosov flow, generated by X, and ξ any supporting positive contact structure which is dynamically positive everywhere. Then, ξ is isotopic to ξ_+, through supporting positive contact structures which are dynamically positive everywhere.*

Proof. It suffices to show that linear interpolation of ξ and ξ_+ is through positive contact structures and Gray's theorem guarantees the existence of isotopy. For simplicity, we assume $\pi^{-1}(E^s)$ and $\pi^{-1}(E^u)$ are C^1 plane fields. Otherwise, we can use the approximations used in the proof of Theorem 2.1.1 and the fact that both projective Anosovity and contactness are open conditions.

Choose C^1 1-forms α_s and α_u such that $\ker \alpha_s = \pi^{-1}(E^u)$, $\ker \alpha_u = \pi^{-1}(E^s)$ and $\xi_+ = \ker \alpha_+$, where

$$\alpha_+ := \frac{\alpha_u - \alpha_s}{2}.$$

Then, there exists C^1 function f, such that

$$\xi = \ker \frac{f\alpha_u - \alpha_s}{2}.$$

Letting $\alpha'_+ := \frac{f\alpha_u - \alpha_S}{2}$, we show that for all $t \in [0,1]$,

$$\alpha_t := (1-t)\alpha_+ + t\alpha'_+$$

is a positive contact structure.

Choose some transverse plane field η and assume $e_s \in \pi^{-1}(E^s) \cap \eta$ and $e_u \in \pi^{-1}(E^u) \cap \eta$ are the vector fields defined by $\alpha_s(e_s) = \alpha_u(e_u) = 1$, and r_s and r_u are the corresponding expansion rates of stable and unstable directions, respectively, i.e.

$$-\mathcal{L}_X e_s = r_s e_s - q_s^\eta\, X \text{ and } -\mathcal{L}_X e_u = r_u e_u - q_s^\eta\, X,$$

for some real functions q_s^η, q_u^η (see Proposition 2.2.13).

We can easily compute (as in proof of Theorem 2.1.1 and using Proposition 2.2.13):

$$4(\alpha_0 \wedge d\alpha_0)(e_s, e_u, X) = \big(\alpha_u \wedge d\alpha_u - \alpha_u \wedge d\alpha_s - \alpha_s \wedge d\alpha_u + \alpha_s \wedge d\alpha_s\big)(e_s, e_u, X)$$

$$= -\alpha_u(e_u)\alpha_s([e_s, X]) + \alpha_s(e_s)\alpha_u([e_u, X]) = r_u - r_s > 0;$$

$$4(\alpha_1 \wedge d\alpha_1)(e_s, e_u, X) = \big(f\alpha_u \wedge d(f\alpha_u) - f\alpha_u \wedge d\alpha_s - \alpha_s \wedge d(f\alpha_u) + \alpha_s \wedge d\alpha_s\big)(e_s, e_u, X)$$

$$= -f\alpha_u(e_u)\alpha_s([e_s, X]) + \alpha_s(e_s)\left[X \cdot (f\alpha_u(e_u)) + f\alpha_u([e_u, X])\right] = fr_u - fr_s + X \cdot f > 0;$$

$$4(\alpha_0 \wedge d\alpha_1)(e_s, e_u, X) = \big(\alpha_u \wedge d(f\alpha_u) - \alpha_u \wedge d\alpha_s - \alpha_s \wedge d(f\alpha_u) + \alpha_s \wedge d\alpha_s\big)(e_s, e_u, X)$$

$$= -\alpha_u(e_u)\alpha_s([e_s, X]) + \alpha_s(e_s)\left[X \cdot (f\alpha_u(e_u)) + f\alpha_u([e_u, X])\right] = fr_u - r_s + X \cdot f;$$

$$4(\alpha_1 \wedge d\alpha_0)(e_s, e_u, X) = \big(f\alpha_u \wedge d\alpha_u - f\alpha_u \wedge d\alpha_s - \alpha_s \wedge d\alpha_u + \alpha_s \wedge d\alpha_s\big)(e_s, e_u, X)$$

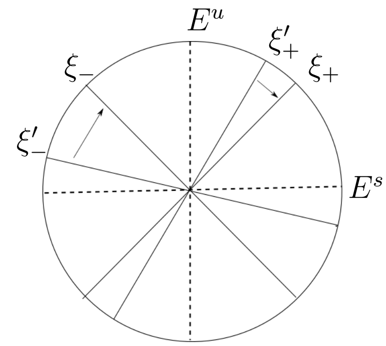


Figure 2.3: Uniqueness of the supporting bi-contact structure

$$= -f\alpha_u(e_u)\alpha_s([e_s, X]) + \alpha_s(e_s)\alpha_u([e_u, X]) = r_u - fr_s.$$

It yields

$$(\alpha_t \wedge d\alpha_t)(e_s, e_u, X)$$

$$= t^2(fr_u - fr_s + X \cdot f) + (1-t)^2(r_u - r_s) + t(1-t)(r_u - r_s + fr_u - fr_s + X \cdot f) > 0,$$

completing the proof. □

This, in particular, implies that the supporting bi-contact structure for any projectively Anosov flow is unique, up to homotopy through supporting bi-contact structures.

Theorem 2.4.3. *[5] If (ξ_-, ξ_+) and (ξ'_-, ξ'_+) are two supporting bi-contact structures for a projectively Anosov flow, generated by X, then they are homotopic through supporting bi-contact structures.*

Proof. By Remark 2.2.10, ξ'_+ and ξ'_- are dynamically positive and negative everywhere, respectively. The proof of Lemma 2.4.2 finishes the proof (See Figure 2.3). □

It is important to understand how a positive contact structure ξ with projectively Anosov vector field $X \subset \xi$ behaves in a region, where it is dynamically negative (similarly, we can describe the behavior of a negative contact structure in a region, where it is dynamically positive). Consider a X-differentiable transverse plane field η, and using any Riemannian metric as described above, define the function

$$\theta_\xi : M \to [0, 2\pi),$$

which measures the angle between $\xi \cap \eta$ and the bi-sector of $\eta \cap \pi^{-1}(E^s)$ and $\eta \cap \pi^{-1}(E^u)$ in the positive region. Note that this function is continuous and differentiable with respect to X, where ξ is dynamically negative, $\xi = \pi^{-1}(E^s)$ or $\xi = \pi^{-1}(E^u)$. Remark 2.2.10 guarantees that at such points

$$X \cdot \theta_\xi < 0,$$

since at those points, the flow rotates ξ clockwise in those regions (see Figure 2.1 (b)) and by Frobenius theorem, ξ needs rotate faster in a clockwise fashion, to stay a positive contact structure.

Now consider the family of plane fields

$$\eta_\theta := \langle X \rangle \oplus l_\theta,$$

for $\theta \in I_- := [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]$, where $l_\theta \subset \eta$ is the oriented line field which has angle θ with the dynamically positive bi-sector of $\eta \cap \pi^{-1}(E^s)$ and $\eta \cap \pi^{-1}(E^u)$. Note that such l_θ is either dynamically negative, or the same as E^s or E^u (ignoring the orientation). After a generic smooth perturbation of ξ, we can assume the set $\Sigma_\theta := \{x \in M \text{ s.t. } \xi = \eta_\theta\}$ is a differentiable manifold, which is transverse to X, since $X \cdot \theta_\xi < 0$ (and using *the implicit function theorem*). Hence, such solution set is a union of tori, since the splitting $TM/\langle x \rangle \simeq E^s \oplus E^u$ would trivialize the tangent space of such surface. Therefore, if $N \subset M$ is the set on which ξ is dynamically negative, then

$$\bar{N} = \Sigma := \bigcup_{\theta \in I_-} \Sigma_\theta \simeq \bigcup_{1 \leq i \leq k} T_i \times [0, 1],$$

for some integer k, where T_i s are tori and for each $1 \leq i \leq k$ and $\tau \in [0, 1]$, X is transverse to $T_i \times \{\tau\}$.

From the above observations and what we know about Anosov flows, we can derive a host of uniqueness theorems about ξ.

Lemma 2.4.4. *Using the above notations, let X be an Anosov flows and $N \subset M$, the subset of M on which the positive contact structure ξ is dynamically negative.*

a) $\bar{N} \simeq \bigcup_{1 \leq i \leq k} T_i \times [0, 1]$, where T_i s are incompressible tori;

b) $\partial \bar{N} = \{x \in M$ s.t. $\xi = \pi^{-1}(E^s)\} \cup \{x \in M$ s.t. $\xi = \pi^{-1}(E^u)\}$;

c) If T_1, T_2, ..., T_j of part (a) are parallel through transverse tori, then there exists a map

$$(\mathbb{S}^1 \times \mathbb{S}^1 \times [0, (j-1)\pi] \textit{ with coordinates } (s, t, \theta), \ker\{\cos\theta \, dt + \sin\theta \, ds\}) \to (M, \xi),$$

which is a contact embedding on $(\mathbb{S}^1 \times \mathbb{S}^1 \times [0, (j-1)\pi))$.

d) If we only assume X to be projectively Anosov (not necessarily Anosov), we can conclude all the above, except T_i might not be incompressible.

Proof. Part (a) and (b) follow from the above discussion and the fact that any surface which is transverse to an Anosov flow is an incompressible torus [27, 56, 57]. For Part (c), notice that if we consider the two tori bounding a connected component of $\bar{N}$, we have a half-twist of the flow (a Giroux π-torsion) in between (see Remark 1.1.10). More precisely, we can reparametrize the angle θ of the above discussion, by the flowlines, when in the region between any two adjacent tori, where the flow is dynamically positive (and where the flow is dynamically negative, we automatically have $X \cdot \theta_\xi < 0$). Therefore, we get a contact embedding of

$$\big([0, \pi] \times \mathbb{S}^1 \times \mathbb{S}^1 \text{ with coordinates } (t, \phi_1, \phi_2), \ker\{\cos t \, d\phi_1 + sint \, d\phi_2\big\}) \to (M, \xi).$$

$\square$

Theorem 2.4.5. *[5] If M is atoroidal and (ξ_-, ξ_+) a supporting bi-contact structure for the Anosov vector field X on M, then for any supporting positive contact structure ξ, ξ is isotopic to ξ_+, through supporting contact structures.*

Proof. By Lemma 2.4.4 ξ is dynamically positive everywhere, and Lemma 2.4.2 finishes the proof. □

An Anosov flow is called $\mathbb{R}$-*covered*, if the lift of its stable (or unstable) foliation to the universal cover is the product foliation of $\mathbb{R}^3$ by planes. This is an important class of Anosov flows and is studied in depth, in the works of Fenley, Bartbot, Barthelmé, etc. In particular, it is shown [58, 25] that there is no embedded surface transverse to such flows. Hence,

Theorem 2.4.6. *[5] Let X be an $\mathbb{R}$-covered Anosov vector field, supported by the bi-contact structure (ξ_-, ξ_+) on M, and let ξ be any supporting positive contact structure. Then, ξ is isotopic to ξ_+, through supporting contact structures.*

In the case of M being a torus bundle, the underlying contact structures can be characterized by having the minimum torsion (see Remark 1.1.10). Although, similar phenomena can be observed in the case of projective Anosov flows, we state the theorem for Anosov flows, for which the relation of torsion and symplectic fillability is established in [19, 21]. The proof relies on the classification of contact structures on torus bundles and $\mathbb{T}^2 \times I$ by Ko Honda and one should consult [59] and [60] for more details and precise definitions.

Theorem 2.4.7. *[5] Let X be the suspension of an Anosov diffeomorphism of torus, supported by the bi-contact structure (ξ_-, ξ_+), and ξ a positive supporting contact structure. Then, ξ is isotopic through supporting bi-contact structures to ξ_+, if and only if, ξ is strongly symplecitcally fillable.*

Proof. If ξ is dynamically positive everywhere, Lemma 2.4.2 yields the isotopy. Otherwise, for any incompressible torus T_i in Lemma 2.4.4, the flow is a suspension flow for an appropriate Anosov diffeomorphism of T_i. Notice that there is at least two of such T_i, since ξ is coorientable. The idea is that, in this case, ξ rotates at least 2π more than ξ_+, as we move in the $\mathbb{S}^1$-direction (see Lemma 2.4.4) and since ξ_+ rotates some itself, that means that ξ rotates more than 2π. Therefore, ξ contains Giroux torsion and is not strongly symplectically fillable.

Let $\tilde{M} := \overline{M\backslash T_1} \simeq \mathbb{T}^2 \times I$, where we have compactified $M\backslash T_1$, by gluing two copies of T_1 along the boundary. i.e. T_1^1 and T_1^2, such that $\partial\tilde{M} = -T_1^1 \sqcup T_1^2$ (abusing notation, we call the induced contact structures, ξ_+ and ξ). After a choice of basis for $\mathbb{T}^2$, let s_+^i (s^i), $i = 1, 2$, be the slope of the *characteristic foliation* of ξ_+ (ξ) on T_1^i, respectively. That is the foliation of T_1^i by $TT_1^i \cap \xi_+$ ($TT_1^i \cap \xi$).

Note that since ξ_+ is universally tight, by [60], ξ_+ has *nonnegative twisting* as it goes from T_1^1 to T_1^2. Furthermore, since ξ_+ does not contain Giroux torsion, such twisting is less than 2π. We claim that $s_+^1 \neq s_+^2$ and $s^1 \neq s^2$. That is because an Anosov diffeomorphism of the 2-torus preserves exactly two slopes of the torus and those are the intersections of $\pi^{-1}(E^s)$ and $\pi^{-1}(E^u)$ with the boundary.

Now by Lemma 2.4.4 c), there exist at least a π-twisting between T_1^1 and T_2, as well as between T_2 and T_1^2. That is a total of at least 2π-twisting. i.e. a contact embedding

$$\big([0, 2\pi] \times \mathbb{S}^1 \times \mathbb{S}^1 \text{ with coordinates } (t, \phi_1, \phi_2), \ker\{\cos t\, d\phi_1 + sint\, d\phi_2\}\big) \to (\tilde{M}, \xi),$$

with $\{0\} \times \mathbb{S}^1 \times \mathbb{S}^1 \to T_1^1$. Since $Im(\{1\} \times \mathbb{S}^1 \times \mathbb{S}^1)$ has the same slope s_1 as T_1^1 (after a 2π-twist), this implies that $Im(\{1\} \times \mathbb{S}^1 \times \mathbb{S}^1) \cap T_1^2 = \emptyset$ and therefore, we will achieve an embedding

$$\big([0, 2\pi] \times \mathbb{S}^1 \times \mathbb{S}^1 \text{ with coordinates } (t, \phi_1, \phi_2), \ker\{\cos t\, d\phi_1 + sint\, d\phi_2\}\big) \to (M, \xi),$$

meaning that ξ contains Giroux torsion. $\square$

2.5 Bi-contact topology and Anosov dynamics: Remarks and questions

In Theorem 2.1.1, we proved that the Anosovity of a flow is equivalent to a host of contact and symplectic geometric conditions. This bridge naturally creates a hierarchy of geometric conditions on the flow and therefore, a new filtration of Anosov dynamics, starting with projectively Anosov flows and ending with Anosov flows. It is of general interest to understand which layer of geometric conditions is responsible for properties of Anosov flows and introduces new geometric and topological tools to study questions in Anosov dynamics. In this section, we want to establish such a hierarchy, make some remarks and formalize a platform for such study. Noting that such contact topological properties are preserved under homotopy of a projectively Anosov flow, this study also sheds light on the classical problem of classifying Anosov flows, by exploring the dynamical phenomena which can appear under *bi-contact homotopy*.

In Theorem 2.1.1, we observed that underlying any Anosov flow is a bi-contact structure, corresponding to the projective Anosovity of the flow. In order to reduce the questions about Anosov dynamics to contact topological questions, we first need to understand the dependence of Anosovity on the geometry of the supporting bi-contact structure. First, we define two notions of equivalence for bi-contact structure, which can describe deformation of a projectively Anosov flow.

Definition 2.5.1. *We call two bi-contact structures* (ξ_-,ξ_+) *and* (ξ'_-,ξ'_+) bi-contact homotopic*, if there exists a homotopy of bi-contact structures* $(\xi^t_-,\xi^t_+), t\in[0,1]$ *with* $(\xi^0_-,\xi^0_+)=(\xi_-,\xi_+)$ *and* $(\xi^1_-,\xi^1_+)=(\xi'_-,\xi'_+)$*. We call the two bi-contact structures* isotopic*, if such homotopy is induced by an isotopy of the underlying manifold.*

Note that a bi-contact homotopy of (ξ_-,ξ_+) and (ξ'_-,ξ'_+) is equivalent to the supported projectively Anosov flows to be homotopic through projectively Anosov flows. Also in this case, by Gray's theorem, ξ_- and ξ'_-, as well as ξ_+ and ξ'_+, are isotopic, but not necessary through the same isotopy.

The notion of bi-contact isotopy seems to be too rigid for the study of many geometric and topological aspects of Anosov dynamics, since it is not even preserved under general perturbations of a (projectively) Anosov flow. Note that even for a fixed projectively Anosov flow, the supporting bi-contact structure is not a priori unique up to isotopy (see Theorem 2.4.3). On the other hand, bi-contact homotopies are more natural for many problems of a topological and geometric nature. That is partly due to the *structural stability* of Anosov flows. That means, the perturbation of an Anosov flow is an Anosov flow which is orbit equivalent to the original flow. It is also known that the same holds for a generic projectively Anosov flow [41]. Moreover, Theorem 2.4.3 shows that for a fixed projectively Anosov flow, the supporting bi-contact structure is unique up to bi-contact homotopy.

However, the dependence of Anosovity on bi-contact homotopy is yet to be understood.

Question 2.5.2. *Let* (ξ_-, ξ_+) *be a bi-contact structure, supporting an Anosov flow, and* (ξ'_-, ξ'_+) *another bi-contact structure which is bi-contact homotopic to* (ξ_-, ξ_+)*. Is a projectively Anosov flow which is supported by* (ξ'_-, ξ'_+) *Anosov?*

While an affirmative answer to the above question might be too optimistic (although we are not aware of an explicit counterexample), confirming the following more modest conjecture can still reduce many problems in Anosov dynamics, to a great extent, to contact topological problems. That includes problems concerning the orbit structures, their periodic orbits and the classification of Anosov flows up to orbit equivalence.

Conjecture 2.5.3. *Two Anosov flows which are supported by bi-contact homotopic bi-contact structures are orbit equivalent. Equivalently, two Anosov flows which are homotopic through projectively Anosov flows are orbit equivalent.*

A weaker notion than bi-contact homotopy, is when given two bi-contact structures, the positive contact structures, as well as the negative contact structures, are isotopic. But the transversality of the two might be violated during the homotopy. In other words, we can ask whether a pair of positive and negative contact structures be transverse in two distinct ways?

Question 2.5.4. *Let (ξ_-, ξ_+) and (ξ'_-, ξ'_+) be two bi-contact structures, such that ξ_- and ξ'_-, as well as ξ_+ and ξ'_+, are isotopic. Are (ξ_-, ξ_+) and (ξ'_-, ξ'_+) bi-contact homotopic?*

After questions regarding the relation of Anosovity and the geometry of the supporting bi-contact structure, we can ask about the relation to the topology of bi-contact structures. More precisely, Anosovity implies rigid contact and symplectic topological properties of the underlying contact structures (see Theorem 2.1.1 and Remark 2.3.11). It is natural to ask about the degree to which these topological properties are responsible for the dynamical properties of Anosov flows.

Definition 2.5.5. *We call a bi-contact (ξ_-, ξ_+) structure* tight*, if both ξ_- and ξ_+ are tight, and we call a projectively Anosov flow* tight*, if it is supported by a tight bi-contact structure.*

To the best of our knowledge, it is not known whether the tightness of one of the supporting contact structures imply the same property for the other.

Question 2.5.6. *If (ξ_-, ξ_+) is a bi-contact structure, such that ξ_- is tight. Can ξ_+ be overtwisted?*

In [1], Mitsumatsu introduces a criteria of making a pair positive and negative contact structures transverse, which yields tight bi-contact structures on $\mathbb{T}^3$ and nil manifolds (note that these projectively Anosov flows are not Anosov [43]). Here, we put down the explicit examples on $\mathbb{T}^3$.

Example 2.5.7. *After an isotopy and considering the integers $m, n > 0$, we consider the positive and negative tight contact structures of Example 1.1.2 (3) on $\mathbb{T}^3$,*

$\xi_n = \ker dz + \epsilon\{\cos 2\pi nz dx - \sin 2\pi nz dy\}$ *and* $\xi_{-m} = \ker dz + \epsilon'\{\cos 2\pi mz dx + \sin 2\pi mz dy\}$,

respectively. It is easy to observe that if $\epsilon \neq \epsilon'$, then ξ_n and ξ_{-m} are transverse everywhere, and therefore, their intersection contains tight projectively Anosov vector fields.

Definition 2.5.8. *We call a bi-contact structure* (ξ_-, ξ_+) *on* M weakly, strongly or exactly symplectically bi-fillable*, if there exists* (W, ω)*, which is a weak, strong or exact symplectically filling for* $(M, \xi_+) \sqcup (-M, \xi_-)$*, respectively, where* $-M$ *is* M *with reversed orientation. We call a projectively Anosov flow* weakly, strongly or exactly symplectically bi-fillable*, respectively, if the associated bi-contact structure is weakly, strongly or exactly symplectically bi-fillable. Furthermore, we call the symplectic bi-filling* trivial*, if* $W \simeq M \times [0, 1]$.

Note that any Anosov flow is trivially exactly symplectically bi-fillable. Furthermore, any exactly bi-fillable projectively Anosov flow is strongly bi-fillable and any strongly bi-fillable projectively Anosov flow is weakly bi-fillable.

Using the idea of Example 1.1.9 (2), we can show that these projectively Anosov flows are in fact, weakly bi-fillable.

Theorem 2.5.9. *[5] The tight projectively Anosov flows of Example 2.5.7 are trivially weakly symplectically bi-fillable.*

Proof. Let $X = \mathbb{T}^2 \times A$, where A is an annulus. Consider the coordinates (x, y) for $\mathbb{T}^2$ and let z be the angular coordinate of A, near its boundary. If ω_1 and ω_2 are some area forms on $\mathbb{T}^2$ and A, respectively, then $\omega = \omega_1 \oplus \omega_2$ will be a symplectic form on X, such that $\omega|_{\ker dz} > 0$. Choosing small enough $\epsilon, \epsilon' > 0$ in Example 2.5.7, ξ_n and ξ_{-m} would be arbitrary close to $\ker dz$ and therefore, $\omega|_{\xi_n}, \omega|_{\xi_{-m}} > 0$, implying that (ξ_{-m}, ξ_n) is weakly symplectically bi-fillable, for any pair of integers $m, n > 0$. $\square$

Using [15, 16], we know such tight projectively Anosov flows are not strongly symplectically bi-fillable (that would conclude ξ_n and ξ_{-m} to be strongly symplectically bi-fillable, which is not the case [17]).

Corollary 2.5.10. *There are (trivially) weakly symplectically bi-fillable projectively Anosov flows, which are not strongly symplectically bi-fillable.*

The properness of other inclusions in the described hierarchy of projectively Anosov flows remains an open problem.

Question 2.5.11. *Are there tight projectively Anosov flows, which are not weakly symplectically bi-fillable? Are there strongly bi-fillable projectively Anosov flows, which are not exactly bi-fillable? Are there exactly bi-fillable projectively Anosov flows, which are not Anosov?*

Remark 2.5.12. *Here, we remark that our filtration of contact and symplectic conditions on a projectively Anosov flow is what we found more natural and can be refined and modified in other ways and using other conditions, for instance on the topology of the symplectic fillings, etc. In particular, we note that in our definition of symplectic bi-fillings for a projectively Anosov flow, supported by a bi-contact structure* (ξ_-, ξ_+)*, we did not consider bi-fillability for both* (ξ_-, ξ_+) *and* $(-\xi_-, \xi_+)$*, a condition which is satisfied for Anosov flows by Theorem 2.1.1. Note that symplectic fillability, for a contact manifold* (M, ξ) *with connected boundary, does not depend on the coorientation of the contact structure, since if* (W, ω) *is a symplectic filling for such contact manifold, then* $(W, -\omega)$ *would be a symplectic filling for* $(M, -\xi)$*. But when we have disconnected boundary, like in the case of bi-contact structures, fillability properties might change if we flip the orientation of only one of the contact structures.*

CHAPTER 3

INTERPLAY WITH REEB DYNAMICS: TOWARDS INVARIANT VOLUME FORMS

An immediate interaction between Anosov flows and the *dynamics* of contact structure, i.e. its associated Reeb vector fields (see Section 1.2), happens when a Reeb vector field is Anosov. These are called *contact Anosov flows* and constitute a very important and well studied class of Anosov flows. While the geodesic flow on the unit tangent space of hyperbolic surface provides a classic example of such flows (see Example 2.2.2), Thurston-Handel [61] provided the first *non classical* example of these flow, and by now, we know the abundance of contact Anosov flows in dimension 3, thanks to the surgery operation introduced by Foulon-Hasselblatt [50].

Among other features, contact Anosov flows can be easily seen to be *volume preserving* (see Proposition 1.2.2). It turns out that the existence of such invariant volume form has deep consequences on various aspects of Anosov flows. In this chapter, we discuss the implications of being volume preserving for an Anosov flow, from the viewpoint of contact geometry, as the characterization of Theorem 2.1.1 suggests that at least in principle, any feature of Anosov flows, such as admitting an invariant volume form, can be translated into contact geometry.

However, we begin this chapter with making observations regarding the Reeb flows associated with a supporting bi-contact structure for an Anosov 3-flow. These observations will naturally lead to the proof of Theorem 2.1.9, 2.1.10 and Corollary 2.1.11. This exhibits an interplay between Anosov and Reeb dynamics, beyond the class of contact Anosov flows. We will later see that similar computation shows that these underlying Reeb vector field can also determine when an Anosov flow is volume preserving. The applications to the theory of *bi-contact surgery* will be discussed in Chapter 4.

3.1 A characterization of Anosovity based on Reeb flows and consequences

In this section, we use ideas developed in Chapter 2 and the proof of Theorem 2.1.1 to give a characterization of Anosovity, based on the *Reeb flows*, associated to the underlying contact structures of a projectively Anosov flow, as well as discuss its contact topological implications in the Anosov case.

Theorem 3.1.1. *[5] Let X be a projectively Anosov vector field on M. Then, the followings are equivalent:*

(1) X is Anosov;

(2) There exists a supporting bi-contact structure (ξ_-, ξ_+), such that ξ_+ admits a Reeb vector field, which is dynamically negative everywhere;

(3) There exists a supporting bi-contact structure (ξ_-, ξ_+), such that ξ_- admits a Reeb vector field, which is dynamically positive everywhere.

Proof. For simplicity assume $\pi^{-1}(E^s)$ and $\pi^{-1}(E^u)$ are C^1 plane fields. The general case follows from the approximations described in the proof of Theorem 2.1.1 and the fact that Anosovity, as well as being dynamically positive (negative) everywhere, are open conditions.

Assuming (1), we now show (2).

Choose a transverse plane field η and let $e_s \in \pi^{-1}(E^s) \cap \eta$ and $e_u \in \pi^{-1}(E^u) \cap \eta$ be the unit vector fields with respect to the Riemannian metric satisfying $r_s < 0 < r_u$, and α_s defined by $\alpha_s(e_s) = 1$ and $\alpha_s(\pi^{-1}(E^u)) = 0$ (see proof of Theorem 2.1.1 for notation). Similarly, define α_u. Define $\alpha_+ := \frac{1}{2}(\alpha_u - \alpha_s)$. Note that $(\xi_-, \xi_+ := \ker \alpha_+)$ is a supporting bi-contact structure, for an appropriate choice of ξ_-. The span of the Reeb vector field, R_{α_+}, is determined by the two equations

$$d\alpha_+(X, R_{\alpha_+}) = 0 = d\alpha_+(e_+, R_{\alpha_+}),$$

where $e_+ \in \xi_+$ is a vector field such that that $\langle X \rangle \oplus \langle e_+ \rangle = \xi_+$.

Consider the vector $v := -r_s e_u - r_u e_s$ and note that since $r_s < 0 < r_u$, such a vector is dynamically negative. Compute

$$d\alpha_+(X, v) = -r_s d\alpha_+(X, e_u) - r_u d\alpha_+(X, e_s) = -r_s r_u + r_s r_u = 0.$$

This implies $R_{\alpha_+} \subset \langle X, v \rangle$ and therefore, R_{α_+} is dynamically negative everywhere.

Now assume (2) and we establish (1).

Let α_+ be such contact form for ξ_+. Define α_u and α_s such that $\alpha_+ = \frac{1}{2}(\alpha_u - \alpha_s)$ and $\alpha_s(\pi^{-1}(E^u)) = \alpha_u(\pi^{-1}(E^s)) = 0$. Finally, choose a transverse plane field η and define the Riemannian metric such that for unit vectors $e_s \in \pi^{-1}(E^s) \cap \eta$ and $e_u \in \pi^{-1}(E^u) \cap \eta$, we have $\alpha_s(e_s) = \alpha_u(e_u) = 1$. By the above computation, we observe $R_{\alpha_+} \subset \langle X, -r_s e_u - r_u e_s \rangle$. Since such vector is dynamically negative, this implies

$$r_s < 0 < r_u,$$

and therefore, X is Anosov.

Equivalence of (1) and (3) is similar. □

We record out the following simple but very useful observation from the above proof.

Corollary of Proof 3.1.2. *In the above setting, we have*

$$R_{\alpha_+} \subset \langle X, -r_s e_u - r_u e_s \rangle.$$

The above characterization can be used to show that the contact structures, underlying an Anosov flow, are *hypertight*. That is, they admit contact forms, whose associated Reeb flows do not have any contractible periodic orbits. The importance of hypertightness is due to the celebrated works of Helmut Hofer, et al in Reeb dynamics (see [47, 48, 49]) which show that if (M, ξ) is a hypertight contact manifold, then, ξ is tight, M is irreducible and

(M, ξ) does not admit an exact symplectic cobordism to $(\mathbb{S}^3, \xi_{std})$. Note that in our case, such properties hold for any covering of (M, ξ) as well, since we can lift the Anosov flow on M to an Anosov flow on the covering.

Theorem 3.1.3. *[5] Let (ξ_-, ξ_+) be a supporting bi-contact structure for an Anosov flow. Then, ξ_- and ξ_+ are hypertight.*

Proof. Consider ξ_+ (the ξ_- case is similar). By Theorem 3.1.1, ξ_+ admits a Reeb vector field, which is dynamically negative everywhere. In particular, it is transverse to both stable and unstable foliations. However, it is a well known fact that there are no contractible closed transversals for stable or unstable foliations, associated to an Anosov flow (see Lemma 3.1 in [26]). More precisely, by [62], if a codimension 1 foliation on a 3-manifold admits a contractible closed transversal, then there exists a closed loop in one of the leaves of the foliation, whose holonomy is trivial from one side and non-trivial from the other. That is impossible for the stable (unstable) foliation of an Anosov flow, since such holonomy needs to be contracting (expanding) on both sides. □

Corollary 3.1.4. *Let (ξ_-, ξ_+) be a supporting bi-contact structure for an Anosov flow. Then,*

(1) ξ_- and ξ_+ are universally tight;

(2) M is irreducible;

(3) there are no exact symplectic cobordisms from (M, ξ_+) or $(-M, \xi_-)$ to $(\mathbb{S}^3, \xi_{std})$.

3.2 Towards volume preserving Anosov 3-flows

For almost two decades since the introduction of Anosov flows in the early 1960s [22, 23], the only known examples of Anosov flows on three dimensional closed manifolds were based on either the suspension of Anosov diffeomorphisms of 2-torus, or the geodesic flows on the unit tangent space of hyperbolic surfaces. All such examples are orbit equivalent to an algebraic volume preserving flow, by their natural construction and a lot of interesting properties of Anosov flows were derived, assuming the existence of such invariant volume forms. However, the first examples of Anosov flows, which are not orbit equivalent to a volume preserving one, were constructed in 1980 by Franks and Williams [63]. Since then, understanding the relation between the existence of an invariant volume form and various aspects of Anosov dynamics has been studied from different viewpoints. In particular, from a topological viewpoint, such property is associated with the *transitivity* of an Anosov flow [64] and from a measure theoretic viewpoint, they correspond to ergodic Anosov flows [22, 24]. Moreover, many other dynamical aspects of such flows, including the regularity theoretical aspects, are well-studied in the literature (for instance, see [65, 28]).

Our goal in the remainder of this chapter, is to study the relation between the divergence of a flow and Anosovity, in the context of a larger class of dynamics, namely the class of projectively Anosov flows, and using the notion of expansion rates of the invariant bundles (see Chapter 2). As we saw, these quantities measure the infinitesimal change of the length of vectors in the stable and unstable directions, and facilitate geometric understanding of Anosov flows. In particular, they play a significant role in the more recent contact and symplectic geometric theory of Anosov flows [1, 5, 66]. Therefore, our study provides new perspective on the class of volume preserving Anosov flows, in terms of those geometries.

We will begin our study with a natural description of the divergence of a projectively Anosov flow in terms of its associated expansion rates of the invariant bundles, encapsulated in the following two theorems:

Theorem 3.2.1. *[6] Let X be the generator of a projectively Anosov flow on M and Ω be some volume form which is X-differentiable. There exists a metric on M, such that $div_X\Omega = r_s + r_u$, where r_s and r_u are the expansion rates of the stable and unstable directions, respectively, measured by such metric.*

Theorem 3.2.2. *Let X be a projectively Anosov flow with r_s and r_u being its stable and unstable expansion rates, measured by some metric. Then,*

(a) There exists a volume form Ω on M, which is X-differentiable and $div_X\Omega = r_s+r_u$.

(b) For any $\epsilon > 0$, there exists a C^1 volume form Ω^ϵ, such that $|div_X \Omega^\epsilon - (r_s + r_u)| < \epsilon$.

Although the above description of the divergence is hardly surprising, it accommodates the use of such relation from the viewpoint of differential and contact geometry. One immediate corollary is

Corollary 3.2.3. *Any projectively Anosov flow preserving some C^0 volume form is Anosov. In particular, any* contact *projectively Anosov flow (that is when a projectively Anosov flow preserves a transverse contact structure) is Anosov.*

Although the above corollary is well known in the dynamical systems literature (for instance see [67]), it seems that this fact is unexpectedly left obscured in some other areas of research, most importantly when such flows appear in the Riemannian geometry literature. While contributing meaningfully to the related subjects, one can find many interesting results on the Riemannian geometry of contact projectively Anosov flows, ignoring that they are in fact Anosov (for instance, see [68, 32]).

It is well known that many important properties of (projectively) Anosov flows are independent of the norm involved in their definition. On the other hand, there are natural volume forms for such setting, induced from the underlying contact structures of these flows (see Section 2.2). It turns out that we can characterize the Anosovity of a projectively Anosov flow, in terms of the divergence of the flow being bounded by these volume forms in an appropriate sense (see Remark 3.4.1).

Theorem 3.2.4. *[6] Let X be a projectively Anosov flow. Then, the followings are equivalent:*

(1) X is Anosov.

(2) There exists a positive contact form α_+, such that for some ξ_-, $(\xi_-, \xi_+ := \ker \alpha_+)$ is a supporting bi-contact structure and $-\alpha_+ \wedge d\alpha_+ < (div_X \Omega^{\alpha_+})\Omega^{\alpha_+} < \alpha_+ \wedge d\alpha_+$.

(3) There exists a negative contact form α_-, such that for some ξ_+, $(\xi_- := \ker \alpha_-, \xi_+)$ is a supporting bi-contact structure and $\alpha_- \wedge d\alpha_- < (div_X \Omega^{\alpha_-})\Omega^{\alpha_-} < -\alpha_- \wedge d\alpha_-$.

Using our description of the divergence of an Anosov flow, we next study the geometric consequences of the existence of an invariant volume form for a smooth Anosov flow, from various viewpoints. More precisely, Theorem 3.2.2 shows the symmetry of expansion and contraction in the unstable and stable directions, respectively, in the case of volume preserving Anosov flows and furthermore, thanks to the differentiability of the weak stable and unstable bundles in this case [28, 53], such symmetry is well-behaved in the approximation techniques we use, when translating the metric description of Anosov flows to the contact geometric one. We study such symmetry from the view point of *theory of contact hyperbolas*, *Reeb dynamics* and *Liouville geometry*, giving various characterizations of volume preserving Anosov flows.

To begin with, we study volume preserving Anosov flows in terms of the theory of contact hyperbolas and (-1)-Cartan structures, developed by Perrone [69] (see Section 3.5 for definitions) as an analogue of the theory of *contact circles* by Geiges-Gonzalo [70, 71]. Moreover, we will see that these conditions are, in fact, equivalent to a purely Reeb dynamical description of volume preserving Anosov flows.

Theorem 3.2.5. *[6] Let ϕ^t be a smooth projectively Anosov flow on M. Then, the followings are equivalent:*

(1) The flow ϕ^t is a volume preserving Anosov flow.

(2) There exists a supporting bi-contact structure (ξ_-, ξ_+) and contact forms α_- and α_+ for ξ_- and ξ_+, respectively, such that (α_-, α_+) is a (-1)-Cartan structure.

(3) There exists a supporting bi-contact structure (ξ_-, ξ_+) and Reeb vector fields R_{α_-} and R_{α_+} for ξ_- and ξ_+, respectively, such that $R_{\alpha_-} \subset \xi_+$ and $R_{\alpha_+} \subset \xi_-$.

To the best of our knowledge, the only known examples of taut contact hyperbolas, except an explicit example constructed on $\mathbb{T}^3$, are achieved using the symmetries of Lie manifolds, giving examples which are compatible with algebraic Anosov flows [69]. However, Theorem 3.2.5 shows that we can also construct examples of taut contact hyperbolas on hyperbolic manifolds, thanks to the construction of an infinite family of contact Anosov flows on hyperbolic manifolds by Foulon-Hasselblatt [50]. We note that it is not known if any specific manifold can admit infinitely many distinct Anosov flows, while there are at most finitely many contact Anosov flows on any manifold up to orbit equivalence [72]. This gives a partial answer to the classification problem posed in the final remark of [69].

Corollary 3.2.6. *There exist infinitely many hyperbolic manifolds which admit a (-1)-Cartan structure (and in particular, a taut contact hyperbola).*

In this Chapter, we also study smooth volume preserving Anosov flows from the perspective of Liouville geometry. The construction of exact symplectic 4-manifold for a general Anosov 3-flow is done in Chapter 2. However, we observe that such a construction is significantly simplified in the presence of an invariant volume form (the case previously studied by Mitsumatus [1]). In fact, after a canonical reparametrization of a smooth volume preserving Anosov flow, we show that we can improve the relation between such a flow and both the underlying Reeb dynamics of Theorem 3.2.5, as well as the Liouville geometry associated with the corresponding exact symplectic 4-manifold. We call such reparametrization *the Liouville reparametrization of a smooth volume preserving Anosov flow*.

Theorem 3.2.7. *[6] Let X be a smooth volume preserving Anosov vector field. The Liouville reparametrization X_L of X satisfies the following:*

(1) The flow of X_L preserves the transverse plane field $\langle R_{\alpha_-}, R_{\alpha_+}\rangle$, where R_{α_-} and R_{α_+} are the Reeb vector fields of Theorem 3.2.5 (2);

(2) The pair (M, X_L) can be extended to a Liouville structure $([-1,1]\times M, Y)$, such that $([-1,1]\times M, Y)\big|_{M\times\{0\}} = (M, X_L)$.

3.3 Divergence and the expansion rates

In this section, we show that the divergence of a projectively Anosov flow with respect to the (a priori C^0) volume form, which is induced from any norm satisfying the relevant definition, can be naturally characterized in terms of the expansion rates of the stable and unstable directions. We then give approximation results for volume forms with higher regularity.

Theorem 3.3.1. *[6] Let X be the generator of a projectively Anosov flow on M and Ω be some volume form which is X-differentiable. There exists a metric on M, such that $div_X\Omega = r_s + r_u$, where r_s and r_u are the expansion rates of the stable and unstable directions, respectively, measured by the metric.*

Proof. Choose a smooth transverse plane field η and let α_X be a C^1 1-form such that $\alpha_X(\eta) = 0$ and $\alpha_X(X) = 1$. Furthermore, choose the contact form $\tilde{\alpha}_+$, so that $(\xi_-, \xi_+ := \ker \tilde{\alpha}_+)$ is a supporting bi-contact structure for X, for some negative contact structure ξ_-.

We can write $\tilde{\alpha}_+ = \tilde{\alpha}_u - \tilde{\alpha}_s$, where $\tilde{\alpha}_u\big|_{E^s} = \tilde{\alpha}_s\big|_{E^u} = 0$. Notice that $\tilde{\alpha}_u$ and $\tilde{\alpha}_s$ are C^0 1-forms, which are X-differentiable, since $\tilde{\alpha}_+$ is C^1 and the projection resulting in such decomposition is X-differentiable.

Since $\tilde{\alpha}_s \wedge \tilde{\alpha}_u \wedge \alpha_X$ is a positive volume form on M, there exists a positive function $f : M \to \mathbb{R}^+$, such that $\Omega = \alpha_s \wedge \alpha_u \wedge \alpha_X$, where $\alpha_s = f\tilde{\alpha}_s$ and $\alpha_u = f\tilde{\alpha}_u$.

Finally, we can define the norm $||.||$ with $||X|| = ||e_s|| = ||e_u|| = 1$, where $e_s \in E^s \cap \eta$, $e_u \in E^u \cap \eta$ and $\alpha_s(e_s) = \alpha_u(e_u) = 1$. Notice that by construction $||.||$ is X-differentiable.

Letting r_s and r_u be the expansion rates of the stable and unstable directions, respectively, we can compute

$$(\mathcal{L}_X\Omega)\,(e_s, e_u, X) = -\Omega([X, e_s], e_u, X) - \Omega(e_s, [X, e_u], X) = (r_s + r_u)\,\Omega(e_s, e_u, X),$$

completing the proof. □

Corollary 3.3.2. *Any projectively Anosov flow preserving some C^0 volume form is Anosov. In particular, any contact projectively Anosov flow is Anosov.*

Proof. Note that any preserved C^0 volume form is X-differentiable ($\mathcal{L}_X\Omega = 0$). Therefore, Theorem 3.3.1 and Proposition 2.2.15 imply $r_s < 0 < r_u$, which guarantees Anosovity. □

Theorem 3.3.3. *[6] Let X be a projectively Anosov flow with r_s and r_u being its stable and unstable expansion rates, measured by some metric. Then,*

(a) there exists a volume form Ω on M, which is X-differentiable and $div_X \Omega = r_s + r_u$,

(b) for any $\epsilon > 0$, there exists a C^1 volume form Ω^ϵ, such that $|div_X \Omega^\epsilon - (r_s + r_u)| < \epsilon$.

Proof. (a) Choose a smooth transverse plane field η. Via η, the norm involved in the definition of the expansion rates will induce a norm $||.||$ on TM. Define $e_s \in E^s \cap \eta$ and $e_u \in E^u \cap \eta$, so that $||e_s|| = ||e_u|| = 1$ and (e_s, e_u, X) is an oriented basis for TM. Finally, define the 1-forms α_s, α_u and α_X so that $\alpha_s\big|_{E^u} = \alpha_u\big|_{E^s} = \alpha_X\big|_\eta = 0$ and $\alpha_s(e_s) = \alpha_u(e_u) = \alpha_X(X) = 1$.

Letting $\Omega := \alpha_s \wedge \alpha_u \wedge \alpha_X$, it is easy to see $div_X \Omega = r_s + r_u$, as in Theorem 3.3.1.

(b) Let Ω be the volume form constructed in part (a) and Ω^∞ be any smooth volume form on M. There exist a X-differentiable function $f : M \to \mathbb{R}^+$, such that $\Omega = f\Omega^\infty$. Notice that

$$\mathcal{L}_X \Omega = (X \cdot f)\Omega^\infty + f\mathcal{L}_X \Omega^\infty = (div_X \Omega)\Omega.$$

By Lemma 2.3.3, there exists a C^1 function f^ϵ such that $|f^\epsilon - f|$ and $|X \cdot f^\epsilon - X \cdot f|$ are arbitrary small. Therefore, letting $\Omega^\epsilon := f^\epsilon \Omega^\infty$ and computing

$$\mathcal{L}_X \Omega^\epsilon = (X \cdot f^\epsilon)\Omega^\infty + f^\epsilon \mathcal{L}_X \Omega^\infty = (div_X \Omega^\epsilon)\Omega^\epsilon,$$

we confirm that $div_X \Omega^\epsilon$ can be taken to be arbitrary close to $div_X \Omega = r_s + r_u$. □

3.4 A contact geometric characterization of Anosovity based on divergence

In this section, we show that we can use the volume forms, naturally coming from the underlying contact structures, to give necessary and sufficient conditions for Anosovity of a projectively Anosov flow, which is independent of the metric and utilizes the expansion rates. We note that to go from the natural setting of projectively Anosov flows, with an

C^0 splitting which is a priori solely differentiable along the flow, to the contact geometric setting, which involves C^1 geometric objects, we need subtle approximation techniques as in the proof of Theorem 2.1.1.

The following remark shows that a given contact form for one of the underlying contact structures of a projectively Anosov flow, induces a natural volume form, as well as a norm, with respect to which we can compute the expansion rates.

Remark 3.4.1. *Notice that if* $(\xi_- := \ker \alpha_-, \xi_+ := \ker \alpha_+)$ *is a supporting bi-contact structure for a projectively Anosov flow,* α_+ *(*α_-*) naturally define two volume forms on* M*. One being the* contact volume form*, i.e.* $\alpha_+ \wedge d\alpha_+$ *(*$\alpha_- \wedge d\alpha_-$*). Additionally, we can uniquely write* $\alpha_+ = \alpha_u - \alpha_s$ *(*$\alpha_- = \alpha_u + \alpha_s$*), where* α_u *and* α_s *are continuous 1-forms, such that* $\ker \alpha_u = E^s$, $\ker \alpha_s = E^u$, $\alpha_s(e_s) > 0$ *and* $\alpha_u(e_u) > 0$*. This induces the positive volume form* $\Omega^{\alpha_+} := \alpha_s \wedge \alpha_u \wedge \alpha_X$ *(*$\Omega^{\alpha_-} := \alpha_s \wedge \alpha_u \wedge \alpha_X$*), where* α_X *is any 1-form satisfying* $\alpha_X(X) = 1$.

Furthermore, α_+ *(*α_-*) define a norm on* $TM/\langle X \rangle$*, by letting* $||\tilde{e}||_s = ||\tilde{e}_u|| = 1$*, where* $e_s \in E^s$ *and* $e_u \in E^u$ *are vectors in* $TM/\langle X \rangle$*, satisfying* $\pi^*\alpha_s(\tilde{e}_s) = \pi^*\alpha_u(\tilde{e}_u) = 1$*. With respect to such norm, we can measure the expansion rates of the underlying flow.*

Here, we bring two lemmas, which will simplify the computation in the proof of Theorem 3.4.4.

Lemma 3.4.2. *Let* α_+ *and* α_- *be positive and negative contact forms, such that* $(\xi_- := \ker \alpha_-, \xi_+ := \ker \alpha_+)$ *is a supporting bi-contact structure for the projectively Anosov flow generated by* X*. Moreover, let* Ω^{α_+} *(*Ω^{α_-}*) be the volume form, and* r_u^+ *and* r_s^+ *(*r_u^- *and* r_s^-*) be the expansion rates, induced by* α_+ *(*α_-*) as in Remark 3.4.1. Then,*

$$\alpha_+ \wedge d\alpha_+ = (r_u^+ - r_s^+)\Omega^{\alpha_+} \left(\alpha_- \wedge d\alpha_- = -(r_u^- - r_s^-)\Omega^{\alpha_-} \right).$$

Proof. Let $e_s \in E^s$ and $e_u \in E^u$ be the unit vector fields on TM, as in Remark 3.4.1.

$$\alpha_+ \wedge d\alpha_+ = \Big\{\alpha_+(e_s)d\alpha_+(e_u, X) - \alpha_+(e_u)d\alpha_+(e_s, X)\Big\}\Omega^{\alpha_+}$$

$$= \Big\{ - \alpha_+(e_s)\alpha_+([e_u, X]) + \alpha_+(e_u)\alpha_+([e_s, X])\Big\}\Omega^{\alpha_+} = (r_u^+ - r_s^+)\Omega^{\alpha_+}.$$

Similar computation for α_- finishes the proof. □

Note that Theorem 3.3.1 also yields:

Lemma 3.4.3. *With the notation of Lemma 3.4.2,*

$$\mathcal{L}_X \Omega^{\alpha_+} = (r_u^+ + r_s^+)\Omega^{\alpha_+} \left(\mathcal{L}_X \Omega^{\alpha_-} = (r_u^- + r_s^-)\Omega^{\alpha_-}\right).$$

In other words,

$$div_X \Omega^{\alpha_+} = r_u^+ + r_s^+ \left(div_X \Omega^{\alpha_-} = r_u^- + r_s^-\right).$$

In the following, the flow being C^1 suffices.

Theorem 3.4.4. *[6] Let X be a C^1 projectively Anosov flow. Then, the followings are equivalent:*

(1) X is Anosov.

(2) There exists a positive contact form α_+, such that for some ξ_-, $(\xi_-, \xi_+ := \ker \alpha_+)$ is a supporting bi-contact structure and $-\alpha_+ \wedge d\alpha_+ < (div_X \Omega^{\alpha_+})\Omega^{\alpha_+} < \alpha_+ \wedge d\alpha_+$.

(3) There exists a negative contact form α_-, such that for some ξ_+, $(\xi_- := \ker \alpha_-, \xi_+)$ is a supporting bi-contact structure and $\alpha_- \wedge d\alpha_- < (div_X \Omega^{\alpha_-})\Omega^{\alpha_-} < -\alpha_- \wedge d\alpha_-$.

Proof. We prove the equivalence of (1) and (2). Showing the equivalence of (1) and (3) is similar.

Assume (2) and let r_u and r_s be the associated expansion rates, for some projectively Anosov flow supported by (ξ_-, ξ_+), induced by α_+ as in Remark 3.4.1. Using Lemma 3.4.2

and Lemma 3.4.3, we can translate the condition on α_+ to

$$r_s - r_u < r_s + r_u < r_u - r_s.$$

This yields $r_s < 0$ and $r_u > 0$, implying the Anosovity of X.

Now, we prove the other implication, utilizing a similar idea as above. However, the main subtlety is to use the X-differentiable norm satisfying the Anosovity condition $r_s < 0 < r_u$, to construct C^1 contact forms α_+ and α_- whose induced norms also satisfy such condition.

Define the 1-forms $\tilde{\alpha}_u$ and $\tilde{\alpha}_s$ by letting $\tilde{\alpha}_u\big|_{E^s} = \tilde{\alpha}_s\big|_{E^u} = 0$ and $\tilde{\alpha}_u(e_u) = \tilde{\alpha}_s(e_s) = 1$, where $e_s \in E^s$ and $e_u \in E^u$ are the unit vector fields, induced from the norm. If these 1-forms were C^1, then $\tilde{\alpha}_u - \tilde{\alpha}_s$ and $\tilde{\alpha}_u + \tilde{\alpha}_s$ would have been the desired positive and negative contact forms. However, $\tilde{\alpha}_u$ and $\tilde{\alpha}_s$ are only C^0 in general (in contrast to the smooth Anosov case, where the weak stable and unstable bundles are known to be C^1 [53]) and we need to approximate them with C^1 1-forms in a way that their induced expansion rates still satisfy the Anosovity condition. This has been done carefully in the proof of the main theorem of [5]. Here, we describe the construction, leaving the details to the reader.

We C^0-approximate $\tilde{\alpha}_s$ and $\tilde{\alpha}_u$ by C^1 1-forms $\bar{\alpha}_s$ and $\bar{\alpha}_u$, such that $\bar{\alpha}_s(X) = \bar{\alpha}_u(X) = 0$. There exist X-differentiable functions $\bar{f}_s$ and $\bar{f}_u$, such that $\bar{f}_s\bar{\alpha}_s(e_s) = \bar{f}_u\bar{\alpha}_u(e_u) = 1$. We can approximate $\bar{f}_s$ and $\bar{f}_u$ by C^1 functions f_s and f_u, assuming that $|\bar{f}_s - f_s|$ and $|X \cdot \bar{f}_s - X \cdot f_s|$, as well as $|\bar{f}_u - f_u|$ and $|X \cdot \bar{f}_u - X \cdot f_u|$, are arbitrary small ([5], Lemma 4.2). In particular, since we have $r_s < 0 < r_u$ everywhere, we can assume that

$$X \cdot [f_u\bar{\alpha}_u(e_u)] + (\min_{x\in M} r_u) f_u\bar{\alpha}_u(e_u) \quad \text{and} \quad X \cdot [f_s\bar{\alpha}_s(e_s)] + (\max_{x\in M} r_s) f_s\bar{\alpha}_s(e_s)$$

are everywhere positive and negative, respectively.

Now, letting $\alpha_u^0 := f_u\bar{\alpha}_u$ and $\alpha_s^0 := f_s\bar{\alpha}_s$, we define

$$\alpha_u^T := I_u^T \phi^{T*}\alpha_u^0 \quad \text{and} \quad \alpha_s^T := I_s^{-T}\phi^{-T*}\alpha_s^0;$$

where

$$I_u^T := e^{-\int_0^T r_u(t)dt} \quad \text{and} \quad I_s^T := e^{-\int_0^T r_s(t)dt}.$$

We use the following properties (which are proved in Section 2.3 and we bring them here for convenience):

Proposition 3.4.5. *One can compute*

(1) $\alpha_u^T(e_u) = \alpha_u^0(e_u) = f_u\bar{\alpha}_u(e_u)$ *and* $\alpha_s^T(e_s) = \alpha_s^0(e_s) = f_u\bar{\alpha}_s(e_s)$*,*

(2) $\lim_{T\to+\infty} \alpha_u^T(e_s) = \lim_{T\to+\infty} \alpha_s^T(e_u) = 0$*,*

(3) $\lim_{T\to+\infty} X \cdot \alpha_u^T(e_s) = \lim_{T\to+\infty} X \cdot \alpha_s^T(e_u) = 0$

We define $\alpha_+^T := \alpha_u^T - \alpha_s^T$ and claim that this is the desired negative contact form. Let $e_s^T \in E^s$ and $e_u^T \in E^u$ be the unit vector fields, and r_s^T and r_u^T be the expansion rates, with respect to the norm induced from α_+^T. Similar to Lemma 3.4.2, we can compute (A_s and A_u are positive functions satisfying $e_s^T = A_s e_s$ and $e_u^T = A_s e_u$):

$$r_s^T = \alpha_+^T(e_u^T)d\alpha_+^T(e_s^T, X) = A_uA_s\alpha_+^T(e_u)d\alpha_+^T(e_s, X) = A_uA_s\alpha_+^T(e_u)\big\{-X\cdot\alpha_+^T(e_s) - \alpha_+^T([e_s, X])\big\}$$

$$= A_uA_s\Big\{\alpha_u^T(e_u) - \alpha_s^T(e_u)\Big\}\Big\{-X\cdot\alpha_u^T(e_s) + X\cdot\alpha_s^T(e_s) - r_u\alpha_u^T(e_s) + r_s\alpha_s^T(e_s)\Big\}$$

$$\approx A_uA_s\alpha_u^T(e_u)\big\{X\cdot[f_s\bar{\alpha}_s(e_s)] + r_sf_s\bar{\alpha}_s(e_s)\big\} < 0.$$

Note that in the computation above, T is assumed to be sufficiently large (yielding the approximation) and we have used Proposition 3.4.5.

Similar computation for the unstable expansion rate shows that such α_+^T satisfies the conditions of (2), finishing the proof. $\square$

3.5 Invariant volume forms and (-1)-Cartan structures

In this section, we study the symmetries that the existence of an invariant volume form implies on the geometry of a smooth volume preserving Anosov flow. This gives us various characterizations of an Anosov flow being volume preserving, in terms of the theory of contact hyperbolas, the Reeb dynamics of the supporting contact structures, and Liouville geometry.

In what follows, by a *volume preserving* Anosov flow, we mean one which preserves a smooth volume form. We note that if the flow is smooth (including in all the results of this section), there is no ambiguity about the regularity of the preserved volume form. That is since, by Corollary 2.1 of [73], when a *smooth* Anosov flow preserves a continuous volume form, such volume form is in fact smooth.

We first note that by [28, 53], E^s and E^u are C^1 plane fields, when X is a smooth Anosov flow. This is an important fact in what follows, since it helps us preserve the metric symmetries of a volume preserving Anosov flow, when translating to the framework of contact geometry (which for Anosov flows of lower regularity requires approximation techniques as in Theorem 3.4.4, which do not a priori respect such symmetry).

Let $\tilde{e}_u \in E^u$ be unit vector field with respect to a metric, satisfying the Anosovity condition and let $\tilde{\alpha}_u$ be a 1-form, such that $\tilde{\alpha}_u\big|_{E^s} = 0$ and $\tilde{\alpha}_u(\tilde{e}_u) = 1$. $\tilde{\alpha}_u$ can be approximated by a C^1 1-form α_u, such that $\alpha_u\big|_{E^s} = 0$ and for the vector field e_u with $\alpha_u(e_u) = 1$, we have $r_u := \alpha_u([e_u, X]) > 0$. That is, the induced expansion rate of the unstable direction is positive.

Let Ω be a smooth volume form which is invariant under the flow, and define $\alpha_s := \Omega(., e_u, X)$. Note that α_s is a C^1 1-form, whose kernel is E^s. Since $div_X \Omega = 0$, by Theorem 3.3.1 we have $r_s := \alpha_s([e_s, X]) = -r_u < 0$.

Now, define $\alpha_+ := \alpha_u - \alpha_s$ and notice that α_+ is a positive contact form, since its induced expansion rates satisfy $r_u - r_s = 2r_u = -2r_s > 0$. Similarly, define the negative contact structure $\alpha_- = \alpha_u + \alpha_s$. Therefore, for any smooth volume preserving Anosov flow, we have a supporting bi-contact structure ($\xi_- := \ker \alpha_- = \ker \alpha_u + \alpha_s, \xi_+ := \ker \alpha_+ = \ker \alpha_u - \alpha_s$), which captures the symmetry of an invariant volume form.

Furthermore, by solving $d\alpha_+(R_{\alpha_+}, X) = d\alpha_-(R_{\alpha_-}, X) = 0$ and , one can easily show that

$$R_{\alpha_+} \subset \langle e_u - e_s \rangle = \xi_- \quad \text{and} \quad R_{\alpha_-} \subset \langle e_u + e_s \rangle = \xi_-.$$

As indicated in the discussion above, additional geometric symmetry can be observed in the case of volume preserving Anosov flows. In this section, we describe this extra structure, in terms of the theory of *contact hyperbolas*, developed by Perrone [69], following the similar theory of *contact circles* by Geiges-Gonzalo [70, 71].

A *contact hyperbola* on M is a pair of positive and negative contact forms (α_1, α_2), such that $\alpha_a := a_1\alpha_1 + a_2\alpha_2$ is also a contact form, for any $a := (a_1, a_2) \in \mathbb{H}^1_r$, where $H^1_r = \{(a_1, a_2) | a_1^2 - a_2^2 = r\}$ for $r \in \{-1, 1\}$. Furthermore, a contact hyperbola is called *taut*, if $\alpha_a \wedge d\alpha_a = r\alpha_1 \wedge d\alpha_1$ (or equivalently, $\alpha_a \wedge d\alpha_a = -r\alpha_2 \wedge d\alpha_2$), for any $a \in \mathbb{H}^1_r$. It is easy to show [69] that (α_1, α_2) is a taut contact hyperbola, if and only if,

$$\alpha_1 \wedge d\alpha_1 = -\alpha_2 \wedge d\alpha_2 \quad \text{and} \quad \alpha_1 \wedge d\alpha_2 = -\alpha_2 \wedge d\alpha_1.$$

Notice that if (α_1, α_2) is a taut contact hyperbola, then $\ker \alpha_1$ and $\ker \alpha_2$ form a bi-contact structure if transverse, with any flow directing the intersection of them being projectively Anosov. It is known that the converse is not true, i.e. there are projectively Anosov flows which do not come form a contact hyperbola [69].

As it is seen above, for a smooth volume preserving Anosov flow, the supporting bi-contact structure $(\ker \alpha_- = \ker \{\alpha_u + \alpha_s\}, \ker \alpha_+ := \ker \{\alpha_u - \alpha_s\})$ exists, where $\alpha_u\big|_{E^s} = \alpha_s\big|_{E^u} = 0$, and the induced volume forms and the expansion rates satisfy

$\Omega^{\alpha_-} = \Omega^{\alpha_+}$ and $r_s^+ = r_s^- = -r_u^+ = -r_u^-$, respectively. By Lemma 3.4.2, we have

$$\alpha_+ \wedge d\alpha_+ = (r_u^+ - r_s^+)\Omega^{\alpha_+} = (r_u^- - r_s^-)\Omega^{\alpha_-} = -\alpha_- \wedge d\alpha_-.$$

Moreover, the discussion in the beginning remarks of this section shows that $R_{\alpha_+} \subset \xi_-$ and $R_{\alpha_-} \subset \xi_+$, yielding

$$\alpha_+ \wedge d\alpha_- = -\alpha_- \wedge d\alpha_+ = 0.$$

Therefore, (α_-, α_+) is a taut contact hyperbola in this case. In fact, it satisfies the stronger condition of $\alpha_+ \wedge d\alpha_- = -\alpha_- \wedge d\alpha_+ = 0$. A taut contact hyperbola with this property is called a (-1)-*Cartan structure [69]*. It turns out that, not only this can be improved to a geometric characterization of volume preserving flows, it will also give us a characterization, purely in terms of the underlying Reeb flows.

Theorem 3.5.1. *[6] Let ϕ^t be a smooth projectively Anosov flow on M. Then, the followings are equivalent:*

(1) The flow ϕ^t is a volume preserving Anosov flow.

(2) There exist a supporting bi-contact structure (ξ_-, ξ_+) and contact forms α_- and α_+ for ξ_- and ξ_+, respectively, such that (α_-, α_+) is a (-1)-Cartan structure.

(3) There exist a supporting bi-contact structure (ξ_-, ξ_+) and Reeb vector fields R_{α_-} and R_{α_+} for ξ_- and ξ_+, respectively, such that $R_{\alpha_-} \subset \xi_+$ and $R_{\alpha_+} \subset \xi_-$.

Proof. The above discussion shows that (1) implies (2). We can also conclude (3) from (2), by noticing that $\alpha_- \wedge d\alpha_+ = -\alpha_+ \wedge d\alpha_- = 0$ yields $R_{\alpha_-} \subset \xi_+$ and $R_{\alpha_+} \subset \xi_-$. Therefore, the only remaining part is to show that a projectively Anosov flow is volume preserving, with the assumptions of (3).

We first note that any projectively Anosov flow with $R_{\alpha_-} \subset \xi_+$ and $R_{\alpha_+} \subset \xi_-$ is Anosov, thanks to Theorem 6.3 of [5]. Let α_- and α_+ be the contact forms in (3). As in Remark 3.4.1, we consider the expansion rates r_s and r_u, induced from the decomposition $\alpha_+ = \alpha_u - \alpha_s$. Notice that we can write $\alpha_- = f\alpha_u + g\alpha_s$, for positive X-differentiable functions $f, g > 0$. One can easily check by solving $d\alpha_+(R_{\alpha_+}, X) = 0$ and using the fact that $\alpha_+(R_{\alpha_+}) = 1$ (or as it is done in Section 3.1), we can write (q_+ being a real function)

$$R_{\alpha_+} = \frac{1}{r_u - r_s}(-r_s e_u - r_u e_s) + q_+ X. \tag{3.1}$$

and from $\alpha_-(R_{\alpha_+}) = 0$, we get

$$g = -\frac{f r_s}{r_u}. \tag{3.2}$$

Also, using $\alpha_+(R_{\alpha-}) = 0$ and $\alpha_-(R_{\alpha_-}) = 1$, we can write ($q_-$ being a real function)

$$R_{\alpha_-} = \frac{1}{f+g}(e_u + e_s) + q_- X. \tag{3.3}$$

On the other hand, we have

$$0 = d\alpha_-(R_{\alpha_-}, X) = d\alpha_-(e_s + e_u, X) = -X \cdot \alpha_-(e_s, e_u) - \alpha_-([e_s + e_u, X])$$

$$\Rightarrow X \cdot f + X \cdot g + g r_s + f r_u = 0 \tag{3.4}$$

Using Equation 3.2 in the above, we get

$$X \cdot f + X \cdot g + (f + g)(r_s + r_u) = 0,$$

which yields

$$r_s + r_u = -X \cdot (f + g). \tag{3.5}$$

Finally, in order to show that the flow is volume preserving, it suffices to define $\Omega := e^{f+g}\Omega^{\alpha_+}$ and use Lemma 3.4.2 Equation 3.5 and to compute

$$div_X \Omega = X \cdot (f + g)e^{f+g}\Omega^{\alpha_+} + e^{f+g}(div_X \Omega^{\alpha_+})\Omega^{\alpha_+}$$

$$= e^{f+g}\big(X \cdot (f+g) + r_s + r_u\big)\Omega^{\alpha_+} = 0.$$

□

It is noteworthy that the above proof shows that in fact, for a volume preserving Anosov flow, we have a function worth of pairs of Reeb vector fields satisfying (3) of Theorem 3.5.1. This is useful in particular, when we require higher regularity of the underlying contact geometry. More precisely, the contact forms in Theorem 3.5.1 (2) are, except in the case of algebraic Anosov flows, only C^1. Therefore, their Reeb vector fields can be only assumed to be C^0 in general (this is due to the result of Ghys [74], which asserts that except in the case of algebraic Anosov flows, the weak stable and unstable bundles cannot be C^2). However, if we let go of the *symmetry* of the contact forms in Theorem 3.5.1 (2), we can achieve higher regularity of the underlying contact geometry in the following sense.

Let ϕ^t be a volume preserving Anosov flow and α_+ a smooth (i.e. C^∞) contact form, which C^1-approximates $\alpha_u - \alpha_s$ and $\alpha_+(X) = 0$ (α_u and α_s are as in the above theorem). Note that R_{α_+} is smooth and $\xi_- := \langle R_{\alpha_+}, X\rangle$ is a smooth negative contact structure, if the C^1-approximation of $\alpha_u - \alpha_s$ is small enough. Now, a negative contact form for ξ_- is of the form $\alpha_- := f\alpha_u + g\alpha_s$, and in order to have $R_{\alpha_-} \subset \xi_+ := \ker \alpha_+$, it suffices to choose any f and g satisfying Equation 3.2, where r_s and r_u are the expansion rates associated with α_+ (as in Remark 3.4.1). We note that although ξ_- is smooth, the contact form α_- can only be assumed to be C^1 in general.

Corollary of Proof 3.5.2. *In Theorem 3.5.1 (3), the supporting bi-contact structure (ξ_-, ξ_+), as well as at least one of R_{α_+} or R_{α_-} can be chosen to be to be smooth.*

Theorem 3.5.1 gives examples of (-1)-Cartan structures and taut contact hyperbolas, whenever we have volume preserving flows, including the case of algebraic Anosov flows and more interestingly, we get examples on hyperbolic manifolds, using the examples of contact Anosov flows on those manifolds [50].

Corollary 3.5.3. *There exist infinitely many hyperbolic manifolds which admit a* (-1)*-Cartan structure (and in particular, a taut contact hyperbola).*

3.6 From the viewpoint of Liouville geometry

In Chapter 2, we have shown how from an Anosov flow X on a 3-manifold M, we can construct two *Liouville pairs*, (α_-, α_+) and $(-\alpha_-, \alpha_+)$, where $(\xi_- := \ker\alpha_-, \xi_+ := \ker\alpha_+)$ is supporting bi-contact structure for X. That is, $\omega_1 := d\alpha_1$ and $\omega_2 := d\alpha_2$ are exact symplectic structures on $[-1,1]_t \times M$, where $\alpha_1 := (1-t)\alpha_- + (1+t)\alpha_+$ and $\alpha_2 := (1-t)\alpha_- - (1+t)\alpha_+$.

Recall that $(W, d\alpha)$ is an *exact symplectic 4-manifold*, if W is an oriented 4-manifold (with boundary) and $d\alpha$ is an exact symplectic structure on W, i.e. $d\alpha \wedge d\alpha > 0$. For any exact symplectic manifold $(W, d\alpha)$, there exists a unique vector field Y, such that $\iota_Y d\alpha = \alpha$, or equivalently $\mathcal{L}_Y d\alpha = d\alpha$. Such vector field is call a *Liouville vector field*, if it points in the outward direction on ∂W, and the pair (W, Y) is called a *Liouville structure*. We note that this is the case for an exact symplectic manifold constructed from an Anosov 3-flows above, since it is a *symplectic filling* for the contact manifold $(M, \xi_+) \cup (-M, \xi_-)$.

The relation between the associated Liouville vector field and the underlying Anosov vector field is more subtle in the general case of the above construction. But for smooth volume preserving Anosov flows, such connection becomes very straightforward, thanks to the symmetries implied by the existence of an invariant volume form and the fact that in this case, the weak stable and unstable bundles are C^1 [28, 53] and as in Theorem 3.5.1, we do not need approximation techniques (like in Theorem 3.4.4), which might not respect such symmetry.

In what follows, consider X to be a smooth volume preserving Anosov flow on the 3-manifold M and let $(\alpha_- = \alpha_u + \alpha_s, \alpha_+ = \alpha_u - \alpha_s)$ be the (-1)-Cartan structure of Theorem 3.5.1. We have the 1-form $\alpha_1 = (1-t)\alpha_- + (1+t)\alpha_+ = 2\alpha_u - 2t\alpha_s$ on

$[-1,1]_t \times M$, and compute

$$d\alpha_1 \wedge d\alpha_1 = \left\{2d\alpha_u - 2dt \wedge \alpha_s - 2td\alpha_s\right\} \wedge \left\{2d\alpha_u - 2dt \wedge \alpha_s - 2td\alpha_s\right\}$$

$$= -4dt \wedge \alpha_s \wedge d\alpha_u = 4r_u dt \wedge \Omega^{\alpha_+},$$

implying that $([-1,1]_t \times M, d\alpha_1)$ is an exact symplectic manifold. Similarly, we can show $d\alpha_2 \wedge d\alpha_2 = -r_s dt \wedge \Omega^{\alpha_-}$, yielding another exact symplectic structure on $[-1,1]_t \times M$. Notice that with the above assumptions, we have $r_u = -r_s > 0$ and $\Omega^{\alpha_+} = \Omega^{\alpha_-}$. But the fact that the weak stable and unstable bundles are C^1 for a smooth Anosov flow plays a crucial role in preserving the symmetry, when going from a metric description of the underlying Anosov flow to a contact geometric one, and hence, significantly simplifying the construction of the above Liouville pairs, compared to Anosov flows of lower regularity studied in Chapter 2. We also remark that, unless X is an algebraic Anosov flow [74], the (-1)-Cartan structure is a priori only C^1, and therefore, the 2-forms $d\alpha_1$ and $d\alpha_2$ above are exact symplectic structures, a priori only in the C^0 sense.

Now, if we define the vector field $Y_1 := \frac{1}{r_u}X + 2t\partial_t$, we can compute

$$\iota_{Y_1} d\alpha_1 = \frac{2}{r_u}\iota_X d\alpha_u - 4t\alpha_s - \frac{2t}{r_u}\iota_X d\alpha_s = 2\alpha_u - 4t\alpha_s + 2t\alpha_s = \alpha_1.$$

Therefore, Y_1 is the Liouville vector field for $([-1,1]_t \times M, d\alpha_1)$. Notice that a similar computation helps us compute the Liouville vector field of $([-1,1]_t \times M, d\alpha_2)$. It is noteworthy and surprising that although the constructed symplectic structures above are a priori only C^0, their corresponding Liouville vector fields are C^1, if we make appropriate choices of the 1-forms (Simic [54] shows that if an Anosov flow is at least C^2, the 1-forms α_S and α_u can be chosen such that their corresponding expansion rates r_S and r_u are C^1).

Now, we can consider the vector field $X_L := \frac{1}{r_u}X$, which generates a reparametrization of the original flow. Its associated expansion rates are $r'_u = \frac{r_u}{r_u} = 1$ and $r'_s = \frac{r_s}{r_u} = -1$, respectively (see Remark 3.18 of [5]) and we have $Y_1\big|_{\{0\}\times M} = X_L$. We call such X_L

the Liouville reparametrization of a smooth volume preserving Anosov 3-flow (in the sense of [54], this is the *synchronization* of the flow with respect to both stable and unstable directions, simultaneously).

The following theorem proves that the Liouville reparametrization of a smooth volume preserving Anosov flow has even a closer relation to the underlying Reeb dynamics of Theorem 3.5.1

Theorem 3.6.1. *[6] Let X be a smooth volume preserving Anosov vector field. The Liouville reparametrization X_L of X satisfies the following:*

(1) The flow of X_L preserves the transverse plane field $\langle R_{\alpha_-}, R_{\alpha_+}\rangle$, where R_{α_-} and R_{α_+} are the Reeb vector fields of Theorem 3.5.1 (2);

(2) The pair (M, X_L) can be extended to a Liouville structure $([-1,1]\times M, Y)$, such that $([-1,1]\times M, Y)\big|_{M\times\{0\}} = (M, X_L)$.

Proof. The above argument yields (2). In order to prove (1), let $(\alpha_- = \alpha_u + \alpha_s, \alpha_+ = \alpha_u - \alpha_s)$ be the (-1)-Cartan structure of part (2) in Theorem 3.5.1. We have

$$\mathcal{L}_{X_L}\alpha_+ = \mathcal{L}_{X_L}(\alpha_u - \alpha_s) = \alpha_u + \alpha_s = \alpha_-.$$

Similarly, one can show $\mathcal{L}_{X_L}\alpha_- = \alpha_+$. Define the 1-form α_{X_L} by letting $\alpha_{X_L}(X_L) = 1$ and $\alpha_{X_L}(\langle R_{\alpha_-}, R_{\alpha_+}\rangle) = 0$. The goal is prove $\mathcal{L}_{X_L}\alpha_{X_L} = 0$. Note that by construction, α_{X_L} is differentiable along the flow and $(\mathcal{L}_{X_L}\alpha_{X_L})\wedge\alpha_{X_L} = 0$.

Also, by plugging the basis $(R_{\alpha_-}, R_{\alpha_+}, X_L)$, we can observe

$$d\alpha_+ = \alpha_{X_L}\wedge\alpha_- \quad \text{and} \quad d\alpha_- = \alpha_{X_L}\wedge\alpha_+,$$

which implies

$$(\mathcal{L}_{X_L}\alpha_{X_L})\wedge\alpha_- = \mathcal{L}_{X_L}(\alpha_{X_L}\wedge\alpha_-) - \alpha_{X_L}\wedge\mathcal{L}_{X_L}\alpha_-$$

$$= \mathcal{L}_{X_L} d\alpha_+ - \alpha_{X_L} \wedge \alpha_+ = d(\mathcal{L}_{X_L}\alpha_+) - \alpha_{X_L} \wedge \alpha_+ = d\alpha_- - d\alpha_- = 0.$$

Similarly, we have $(\mathcal{L}_{X_L}\alpha_{X_L}) \wedge \alpha_+ = 0$. This yields $\mathcal{L}_{X_L}\alpha_{X_L} = 0$, completing the proof.

□

CHAPTER 4

APPLICATIONS TO ANOSOV SURGERIES

In this chapter, we discuss the implications of our study of divergence and invariant volume forms in Chapter 3 in surgery theory.

In this chapter, we investigate the applications of our approach to the surgery theory of Anosov flows. Surgery theory has been a very important part of the geometric theory of Anosov flows from the early days. Various Dehn-type surgery operations, including Handel-Thurston [61], Goodman [75] or Foulon-Hasselblat [50] surgeries, have helped construction of new examples of Anosov flows, answering historically important questions. These include the first examples of Anosov flows on hyperbolic manifolds [75], the construction of infinitely many *contact* Anosov flows on hyperbolic manifolds [50] or the first (non-trivial) classification of Anosov flows on hyperbolic manifolds [76].

Recently, Salmoiraghi [66, 77] has introduced two novel bi-contact geometric surgery operations of (projectively) Anosov flows, which contribute towards the contact geometric theory of Anosov flows (see [1, 5, 44] for instance) and the related surgery theory, reconstructing previously known surgery operations of Foulon-Hasselblat and Thurston-Handel. These surgeries are applied in the neighborhood of a *Legendrian-transverse knot*, i.e. a knot which is *Legendrian* (tangent) for one of the underlying contact structures in the *supporting bi-contact* and transverse for the other one. One of these surgery operations is done by cutting the manifold along an annulus tangent to the flow and the other one on is based on a transverse annulus. However, the relation to Goodman surgery, which is one of the most significant surgery operations on Anosov flows, and is applied in the neighborhood of a periodic orbit of such flow, relies on one condition. That requires being able to push a periodic orbit to a Legendrian-transverse knot. Salmoiraghi observes that this is possible for the unit tangent space of hyperbolic surfaces [66] and and furthermore, shows that if such condition is satisfied, the Goodman surgery can be reconstructed, using the bi-contact surgery on a transverse annulus (in fact, he generalizes such operation to projectively Anosov flows) [77]. We show that such condition can be satisfied for any C^{1+} Anosov flow, by choosing a norm which yields constant divergence on a given periodic orbit of the flow, giving an affirmative answer to the question posed in [66]. This takes us one step closer to a contact geometric surgery of Anosov flows, unifying the previously introduced operations.

Theorem 3.5.1 shows that for volume preserving Anosov flows, the Reeb vector fields associated with the supporting bi-contact structure $(\xi_- = \ker \alpha_-, \xi_+ = \ker \alpha_+)$ can be contained in one another. In this case, if we flow a periodic orbit of the flow γ_0, which is a Legendrian knot for both ξ_- and ξ_+, along one of these Reeb vector fields, say R_{α_+}, it stays Legendrian for ξ_+ (since R_{α_+} preserves ξ_+) and it immediately becomes transverse to ξ_- (since R_{α_+} is a Legendrian vector field for ξ_-). We call such a knot a *Legendrian-transverse* knot.

Salmoiraghi [77] shows that using the coordinates coming from the above argument on the Reeb vector field, one can reconstruct the classical *Goodman surgery* in the neighborhood of a periodic orbit of an Anosov flow, using a bi-contact surgery. However, notice that in the above argument, it suffices for the Reeb vector field of just one of the contact structures to be contained in the other contact structure, only in a small neighborhood of the periodic orbit which one wants to apply the Goodman surgery on. In the following, we show that this is possible for *any* (possibly non volume preserving) C^{1+} Anosov flow, where by a C^{1+} flow, we mean a C^1 flow with Hölder continuous derivatives (that includes any C^2 flow). The main idea is to show that one can assume that the flow has constant divergence along a fixed periodic orbit.

Theorem 4.0.1. *[6] Let X be a C^{1+} Anosov flow. Given any periodic orbit γ_0, there exists a supporting bi-contact structure $(\xi_-, \xi_+ = \ker \alpha_+)$, such that we have $R_{\alpha_+} \subset \xi_-$ in a regular neighborhood of γ_0. Therefore, there exists an isotopy $\{\gamma_t\}_{t\in[0,1]}$, which is supported in an arbitrary small neighborhood of γ_0, and γ_t is a Legendrian-transverse* *knot for any* $0 < t \leq 1$.

Proof. As discussed above, it is enough to show that there exists a tubular neighborhood $N(\gamma_0)$ and a pair of contact forms α_+ and α_-, such that $(\ker\alpha_-, \ker\alpha_+)$ is a supporting bi-contact structure for X and $\alpha_-(R_{\alpha_+}) = 0$. It is easy to show that this is would have been possible, if the associated expansion rates were constant. The idea of the proof is to find an appropriate norm, which satisfies this condition on γ_0 and use the openness of the contact condition. To do so, we need an approximation technique similar to Theorem 3.4.4. The only caveat is that we need our approximation not to affect the preassigned norm on γ_0.

Let T be the period of γ_0 and $\lambda_u^{\gamma_0}$ and $\lambda_s^{\gamma_0}$ be the eigenvalues of the return map along γ_0, corresponding to the unstable and stable directions, respectively. We can choose a X-differentiable norm on $TM/\langle X\rangle\big|_{\gamma_0}$, such that the induced expansion rates $r_u\big|_{\gamma_0}$ and $r_s\big|_{\gamma_0}$ are constants satisfying $e^{r_uT} = \lambda_u^{\gamma_0}$ and $e^{r_sT} = \lambda_s^{\gamma_0}$. We can then extend such norm to a C^1 norm on $TM/\langle X\rangle \simeq E^s \oplus E^u$ in a neighborhood of γ_0. Let $N(\gamma_0)$ be a possibly smaller neighborhood, on which $r_s < 0 < r_u$.

We define the C^0 1-forms $\tilde{\alpha}_u$ and $\tilde{\alpha}_u$, by letting $\tilde{\alpha}_u(E^s) = \tilde{\alpha}_s(E^u) = 0$ and $\tilde{\alpha}_u(e_u) = \tilde{\alpha}_s(e_s) = 1$, where $e_s \in E^s$ and $e_u \in E^u$ are the unit vectors with respect to our norm. We can C^0-approximate $\tilde{\alpha}_u$ and $\tilde{\alpha}_u$ by smooth 1-forms $\bar{\alpha}_u$ and $\bar{\alpha}_u$ and find X-differentiable functions f_u and f_s such that $f_u\bar{\alpha}_u(e_u) = f_s\bar{\alpha}_s(e_s) = 1$. Using the following lemma, we can approximate these functions with appropriate C^1 functions to serve our goal.

Lemma 4.0.2. *If f is X-differentiable and η-Hölder continuous and γ is a periodic orbit of X (a C^1 flow on n-dimensional closed manifold M). Then, for any $\epsilon > 0$, there exists a C^1 function $\bar{f}$, such that $f\big|_\gamma = \bar{f}\big|_\gamma$ and we have $|f - \bar{f}| < \epsilon$ and $|X \cdot f - X \cdot \bar{f}| < \epsilon$.*

Proof. Let $N_\delta(\gamma)$ be a sufficiently small tubular neighborhood of γ, on which the function $d(x)$, measuring the distance of $x \in M$ from γ, is C^1, i.e. $N_\delta = \{x \in M | d(x) < \delta\}$. Let $\bar{d}(x)$ be any C^1 function on M, where $\bar{d}(x) = d(x)$ on $N_{\frac{\delta}{2}}(\gamma) \subset N_\delta(\gamma)$ and $\bar{d}(x) \neq 0$ everywhere.

Now, we can write $f(x) = f^{\gamma}(x) + \bar{d}^{\frac{\eta}{2}}(x)g(x)$, where $f^{\gamma}(x)$ is any C^1 extension of $f\big|_{\gamma}$ on M and $g(x)$ is well-defined, continuous and X-differentiable function on $M\backslash\gamma$. We extend g to M by letting $g(\gamma) = 0$.

Claim 4.0.3. *The function g is continuous and X-differentiable on $N_\delta(\gamma)$.*

Proof.

$$\lim_{d(x)\to 0} g(x) = \lim_{d(x)\to 0} \frac{f(x) - f^{\gamma}(x)}{d^{\frac{\eta}{2}}(x)} = \lim_{d(x)\to 0} \frac{f(x) - f^{\gamma}(x)}{d^{\eta}(x)} d^{\frac{\eta}{2}}(x) = 0.$$

The last equality follows from f being η-Hölder continuous. Therefore, g is a continuous function on $N_\delta(\gamma)$ (in fact it is $\frac{\eta}{2}$-Hölder continuous). Moreover, g is X-differentiable in this neighborhood, since we have $X \cdot g\big|_{\gamma} = 0$. □

Now, we use Lemma 4.2 of [5] to find a C^1 function $\bar{g}$, where $|g - \bar{g}|$ and $|X \cdot g - X \cdot \bar{g}|$ are arbitrary small. In fact, if we define $\bar{f} := f^{\gamma_0} + \bar{d}^{\frac{\eta}{2}}\bar{g}$, we can find an approximation of g, such that $\bar{f}$ is the desired C^1 function. This completes the proof of Lemma 4.0.2.

□

Let $\bar{f}_s$ and $\bar{f}_u$ be the approximations of f_s and f_u as in Lemma 4.0.2. As in [5], we can define the C^1 contact forms α_+ and α_- with $e'_s \in E^s$, $e'_u \in E^u$, r'_u and r'_s induced by α_+ as in Remark 3.4.1, such that on γ_0, we have $\alpha_+(e_u) = \bar{f}_u\bar{\alpha}_u(e_u) = f_u\bar{\alpha}_u(e_u) = 1$ and similarly $\alpha_+(e_s) = 1$, which yields $e_s = e'_s$, $e_u = e'_u$, $r_s = r'_s$ and $r_u = r'_u$, when restricted to γ_0 (we refer the reader to [5] for the the technical details of the approximations used in the definition. It is enough for us to know that the induced unit vectors and expansion rates from these approximating contact forms are arbitrary close to the ones we started with, while agreeing on γ_0). As in Equation 3.1, we have $R_{\alpha'_+} = \frac{1}{r'_u - r'_s}\{-r'_u e'_s - r'_s e'_u\} + q_+ X$, for some real function q_+. Let $\xi'_- := \langle R_{\alpha_+}, X\rangle$.

Claim 4.0.4. *There exists a regular neighborhood $N(\gamma_0)$, on which ξ'_- is a negative contact structure.*

Proof. Choose a 1-form α_- such that $\xi'_- := \ker \alpha_-$ and $\alpha_(e'_s + e'_u) > 0$. Compute

$$d\alpha_-(R_{\alpha_+}, X) = \frac{1}{r'_u - r'_s}\alpha([X, -r'_u e'_s - r'_s e'_u])$$
$$\Rightarrow d\alpha_-(R_{\alpha_+}, X)\big|_{\gamma_0} = \frac{r'_s r'_u}{r'_u - r'_s}\alpha(e'_s + e'_u) < 0,$$

where in the last equality, we have used the fact that by construction, we have $X \cdot r'_s = X \cdot r_s = X \cdot r'_u = X \cdot r_u = 0$ on γ_0. Thanks to the openness of the contact condition, ξ'_- is a negative contact structure in some small regular neighborhood $N(\gamma_0)$. $\square$

We can extend $\xi'\big|_{N(\gamma_0)}$ to some negative contact structure ξ_- on M, such that the supporting bi-contact structure $(\xi_-, \xi_+ = \ker \alpha_+)$ has the desired properties. $\square$

Corollary 4.0.5. *[6, 77] The bi-contact surgeries of Salmoiraghi [66, 77] can be applied in an arbitrary small neighborhood of a periodic orbit of any C^{1+} Anosov flow. In particular, the bi-contact surgery of [77] reconstructs the Goodman surgery.*

CHAPTER 5

CONLEY-ZEHNDER INDICES AND DYNAMICS OF CONTACT ANOSOV FLOWS

In this Chapter, we discuss some contact and symplectic topological aspects of *contact Anosov 3-flows*. In particular, we will use *Conley-Zehnder indices*, an important tool in *contact dynamics* (see [78]), to study the case when a Reeb vector field is Anosov.

For the reference, we put down the following.

Definition 5.0.1. *Let X be an Anosov vector field, such that $E^{ss} oplus E^{uu}$ is a C^1 contact structure. We call X a* contact Anosov flow. *Similarly, if ξ is a contact structure admitting some Anosov Reeb vector field, we call ξ a* Anosov contact structure.

5.1 Elements from contact dynamics: Conley-Zehnder indices

One way to study the topological properties of contact manifolds is through the dynamics of such objects, more precisely, through the dynamics of the associated Reeb vector fields.

It is well known that these vector fields play a significant role in the theory of contact geometry, comparable to the role of Hamiltonian vector fields in symplectic geometry. As a matter of fact, Reeb vector fields are Hamiltonian vector fields on the so called *symplectization* of a contact manifold.

However, it was not till early 1990s that dynamics of Reeb vector fields were used to study the topology of contact manifolds, thanks to many, but first and foremost, Hofer, Wysocki, and Zehnder. Here we introduce an index associated to the closed orbits of Reeb vector fields, which plays a significant role in the theory..

Recall that the group of symplectic linear maps reduces to the group of area preserving linear maps in dimension 2. i.e.

$$Sp(1_{\mathbb{C}}) := \{A \in \mathbb{R}^{2\times 2} | A^T J A = Id\} = SL(2;\mathbb{R}),$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the standard complex structure on $\mathbb{R}^2$. We can uniquely write any $A \in Sp(1_{\mathbb{C}})$ as $A = MU$, where $U \in SO(2)$ and M is a symmetric positive definite matrix. Since the space of positive definite matrices is contractible, we conclude that $Sp(1_{\mathbb{C}})$ is homotopy equivalent to $SO(2)$ and therefore $\pi_1(Sp(1_{\mathbb{C}})) = \mathbb{Z}$.

Now considering a path of symplectic maps, starting from Id, we want to measure its rotation around the generator of $\pi_1(Sp(1_{\mathbb{C}}))$ (notice that we are not assuming that such a path is closed). It is not necessary, but for the sake of simplicity, we restrict our definition to paths ending in the subspace of $Sp(1_{\mathbb{C}})$ given by

$$Sp^*(1_{\mathbb{C}}) := \{A \in Sp(1_{\mathbb{C}}) | det(A - Id) \neq 0\}.$$

We call such paths of symplectic maps, *non-degenerate* and denote the space of such paths by $\Sigma^*(1_{\mathbb{C}})$.

For non-degenerate paths of symplectic maps

$$\Phi : [0, T] \to Sp(1_{\mathbb{C}}) \text{ with } \Phi(0) = Id \text{ and } \Phi(T) \in Sp^*(1_{\mathbb{C}})$$

we can define a unique index map, defined by the following axioms, see [78]:

Theorem 5.1.1. *There exists a unique map, called Conley-Zehnder index,*

$$\mu_{CZ} : \Sigma^*(1_{\mathbb{C}}) \to \mathbb{Z}$$

such that

(1) Homotopy Invariance: μ_{CZ} is invariant under homotopy through non-degenerate paths.

(2) <u>Maslov Compatibility</u>: Let $L : [0, T] \to Sp(1_{\mathbb{C}})$ be a continuous closed loop, then

$$\mu_{CZ}(L\Phi) - \mu_{CZ}(\Phi) = 2\mu(L);$$

where $\mu(L)$ is the Maslov index of L.

(3) <u>Invertibility</u>:

$$\mu(\Phi^{-1}) = -\mu(\Phi)$$

(4) <u>Normalization</u>:

$$\mu_{CZ}(e^{i\pi t}|_{t\in[0,1]}) = 1$$

Remark 5.1.2. *(1) Notice that Maslov index assigns to any closed path in $\Sigma^*(1_{\mathbb{C}})$, the degree of the map. Therefore Axiom (2) means that for any full round around the generator of $\pi_1(Sp(1_{\mathbb{C}}))$, we are adding 2 to the Conley-Zehnder index.*

(2) From the above axioms, we can prove that since for any $A \in Sp(1_{\mathbb{C}})$ we have $det(A) = 1$, we can determine the parity of $\mu_{CZ}(\Phi)$ by merely looking at the eigenvalues of $\Phi(T)$. More precisely, $\mu_{CZ}(\Phi)$ is even if $\Phi(T)$ has real positive eigenvalues λ and $\frac{1}{\lambda}$ (we call Φ positively hyperbolic in this case) and $\mu_{CZ}(\Phi)$ is odd if either $\Phi(T)$ has real negative eigenvalues λ and $\frac{1}{\lambda}$ (we call Φ negatively hyperbolic in this case) or $\Phi(T)$ has complex conjugate eigenvalues $e^{\pm i\phi}$ (we call Φ elliptic in this case).

(3) There is a sophisticated iteration theory relating $\mu_{CZ}(\Phi)$ to $\mu_{CZ}(\Phi^m)$ for $m \in \mathbb{N}$, see [79] for instance. In this paper, we only deal with the easiest case which is when Φ is (positively or negatively) hyperbolic. In this case, for any $m \in \mathbb{N}$:

$$\mu_{CZ}(\Phi^m) = m \cdot \mu_{CZ}(\Phi).$$

Remark 5.1.3. *(1) There are other equivalent definitions of Conley-Zehnder indices. In particular, we can define $\mu_{CZ}(\Phi)$ as the algebraic intersection of Φ and the subvariety given by*

$$Sp(1_{\mathbb{C}}) \setminus Sp^*(1_{\mathbb{C}}) = \{A \in Sp(1_{\mathbb{C}}) | det(A - Id) = 0\}.$$

We can formulate the proof of our main theorem using this definition. But we find the axiomatic definition more intuitive for a non-expert.

(2) One can extend the above definition to higher dimensions and also, there are different generalizations for degenerate paths of symplectic linear maps, including generalizations by Robbin-Salamon, C. Viterbo and Y. Long (see [79][80]). However, considering non-degenerate paths is enough for our purpose.

Given a periodic Reeb orbit γ of X_α for a contact manifold, fixing a symplectic trivialization ν of $(\xi|_\gamma, d\alpha)$ along γ and picking a point $p \in \gamma$, the flow of X_α defines a path of symplectic linear maps

$$\Phi : [0, T] \to Sp(1_{\mathbb{C}})$$

such that $\Phi(0) : \xi|_p \to \xi|_p = Id$ and T is the period of γ. Notice that we used the fact that X_α preserves $d\alpha$ and therefore $d\alpha|_\xi$. Now we can use all the terminology of being non-degenerate, (positively or negatively) hyperbolic and elliptic, directly for the period orbit γ and we call a Reeb flow non-degenerate, if all of its periodic orbits are non-degenerate. Abusing notation and assuming that $det(\phi(T) - Id) \neq 0$, we define the Conley-Zehnder index of γ with respect to ν to be the Conley-Zehnder index of the induced path of symplectic maps and write it as $\mu^\nu_{CZ}(\gamma)$.

Assume $[\gamma] = 0 \in H_1(M)$ and let Σ be a 2-chain such that $\partial\Sigma = \gamma$. By obstruction theory, Σ defines a trivialization of ξ along γ which is unique up to homotopy. The fact that the endpoint of the induced path is independent of such trivialization (it is simply the Poincaré return map $\Phi(T) : \xi|_p \to \xi|_p$) and homotopy invariance property of Conley-Zehnder indices guarantee that Conley-Zehnder index of γ only depends on Σ and hence, we use the notation $\mu^\Sigma_{CZ}(\gamma)$. Moreover, it is well known (for instance see [81]) that if

Σ_1 and Σ_2 are two of such 2-chains for γ, we can compute the difference of the induced Conley-Zehnder indices by

$$\mu_{CZ}^{\Sigma_1}(\gamma) - \mu_{CZ}^{\Sigma_2}(\gamma) = \langle 2e(\xi), \Sigma_1 \sqcup -\Sigma_2 \rangle$$

where $-\Sigma_2$ refers to Σ_2 with reversed orientation and $e(\xi)$ is the Euler class of ξ. Note that $\Sigma_1 \sqcup -\Sigma_2$ is a cycle and therefore, represents and element of $H_2(M)$. In particular, $\mu_{CZ}(\gamma)$ is well-defined if $2e(\xi) = 0$ or $H_2(M) = 0$.

Remark 5.1.4. *For any periodic Reeb orbit γ and $m \in \mathbb{N}$, going m rounds around γ is also a periodic Reeb orbit which we denote by γ^m and similar iteration theory mentioned as in Remark 5.1.2 part 3) shows that fixing a symplectic trivialization of $(\xi|_\gamma, d\alpha)$, for a (positively or negatively) hyperbolic periodic Reeb orbit γ:*

$$\mu_{CZ}(\gamma^m) = m \cdot \mu_{CZ}(\gamma).$$

We can finally state celebrated result of Hofer [47] and Hofer-Wysocki-Zehnder [48] ([49] can be helpful as well), deriving contact topological information from such indices.

Theorem 5.1.5. *[47, 48, 49] Let (M, ξ) be a contact manifold which satisfies one of the followings:*

(1) ξ is overtwisted;

(2) M is reducible;

(3) there exists an exact symplectic cobordism from (M, ξ) to $(\mathbb{S}^3, \xi_{std})$.

Then any non-degenerate associated Reeb vector field for (M, ξ) admits a contractible unknotted periodic orbit γ with

$$\mu_{CZ}^D(\gamma) = 2$$

in the first two cases and

$$\mu_{CZ}^D(\gamma) \in \{2, 3\}$$

in the third case, where D is the contraction disk for γ.

Remark 5.1.6. *An exact symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) is (X, ω) such that $\partial X = (-M_-) \sqcup (M_+)$ and there exists a global Liouville vector field Y with $\alpha := \iota_Y \omega$ being a contact form for ξ_- and ξ_+, when restricted to the boundary of X. It is worth mentioning that the third case above is a very large class of contact manifolds, including all overtwisted contact manifolds (i.e. it includes case (1)) [82].*

We end this section by noticing for contact Anosov flows, by Theorem 5.1.5 and thanks to the fact that Anosov flows do not have any contractible periodic (i.e. are hypertight), the invariant transverse contact structure satisfies similar properties to the supporting contact structure of Anosov 3-flows in Corollary 2.1.11.

Corollary 5.1.7. *Let X be a contact Anosov flow on M and ξ the invariant transverse contact structure. Then, ξ is universally tight, M is irreducible and there is no exact symplectic cobordism from (M, ξ) to $(\mathbb{S}^3, \xi_{std})$.*

5.2 A brief introduction to cylinderical contact homology

One of the significant roles that Conley-Zehnder indices play in contact dynamics is the definition of a variety of Floer theoretical invariants for contact manifolds, called *contact homology*. Introduced by Eliashberg-Hofer-Givental [81], these are (in principle) invariants of contact structures derived from the study of *J-holomorphic curves* in the *symplectization* of a contact manifold. Although the theory is famously not well defined in full generality, these invariants are shown to be mathematically accurate in a wide class of cases, which is sufficient for our purpose. In particular, [83, 84] shows that such theory is well defined, when we restrict to hypertight contact structures on 3-manifolds and only consider J-holomorphic cylinders (notice the invariant contact structure for a contact Anosov flow is hypertight. See Corollary 5.1.7).

Here, we bring a very brief and incomplete overview of *cylinderical contact homology*. One should consult [83, 84] for more details and thorough discussions.

Consider the contact 3-manifold (M, ξ) and choose a contact form α for ξ. The *symplectization* of (M, α) is a the exact symplectic 4-manifold $(\mathbb{R}_s \times M, d(e^s\alpha)$ and we choose an almost complex structure J on this symplectic manifold which preserves $\ker e^s\alpha\big|_{\{t\}\times M}$ for any $t \in \mathbb{R}$ and $J(\partial_s) = X_\alpha$, where X_α is the Reeb vector field associated with α. The *cylinderical contact homology* is (ideally) defined in this setting, by an appropriate *count* of certain *J-holomorphic cylinders* in the above symplectic manifold. Those are maps $u : \mathbb{R}_s \times \mathbb{S}^1_t \to \mathbb{R} \times M$, such that

$$\partial_s u + J\partial_t u = 0,$$

$\lim_{s\to\pm\infty} \pi_{\mathbb{R}}(u(s,t)) = \pm\infty$ and $\lim_{s\to\pm\infty} \pi_M(u(s,.))$ is (possibly a reparametrization of) $\gamma_\pm$, for some periodic orbits $\gamma_\pm$ of X_α. We call γ_+ (γ_-) the positive (negative) end of such J-holomorphic cylinder.

It turns out that if we let $\mathcal{M}^J_1(\gamma_+, \gamma_-)$ to be the moduli space of J-holomorphic cylinders connecting as above, modulo the translation and rotation of the domain $\mathbb{R} \times \mathbb{S}^1$, for a generic choice of J and α, we can see that $\mathcal{M}^J_1(\gamma_+, \gamma_-)$ has a manifold structure near a J-holomorphic curve u in this muduli space, whenever u is *somewhere injective* and $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) = 1$ (for simplicity, we are assuming $2e(\xi) = 0$, which is the case for contact Anosov flows). Finally, we note that $\mathbb{R}$ acts on $\mathcal{M}^J_1(\gamma_+, \gamma_-)$ by vertical translation in $\mathbb{R} \times M$.

Now, we can define the chain complex $CC^{\mathbb{Q}}(M, \alpha, J)$ over $\mathbb{Q}$, generated by the *good* (not necessarily simple) periodic orbits of X_α. That is, the periodic orbits of X_α with either odd μ_{CZ}, or even μ_{CZ} such that they are not a multiple of a periodic orbit with odd μ_{CZ}. We note that that the parity of μ_{CZ} does not depend on the trivialization of ξ (and in fact, does not require the existence of one to be defined).

Then, we can define a differential for this chain complex, by

$$\partial : CC^{\mathbb{Q}}(M, \alpha, J) \to CC^{\mathbb{Q}}(M, \alpha, J),$$

$$\partial a = \sum_{b} \sum_{u \in \mathcal{M}_1^J(a,b)/\mathbb{R}} \epsilon(u) q(a, u)\, b,$$

where $\epsilon(u)$ is an appropriate sign attributed to u as an element of the zero dimensional manifold $\mathcal{M}_1^J(a, b)/\mathbb{R}$ and $q(a, u)$ is a rational number computed based on the multiplicities of a as a periodic orbit and u as a map (we refer the reader to [83, 84] for more details, as this is sufficient for our purpose).

It turns out [83] that in the case when ξ is hypertight (which is the case for contact Anosov flows, or more generally, if ξ is *dynamically convex*), we have $\partial^2 \equiv 0$. Therefore, we can define the homology of the above chain complex and denote it by $CH^{\mathbb{Q}}(M, \alpha, J)$. Furthermore, it can be seen [84] that this definition does not depend of the choice of the almost complex structure J and the contact form α, and we can define the *cylinderical contact homology* $CH(M, \xi) := CH^{\mathbb{Q}}(M, \alpha, J)$ as an invariant of the contact manifold (M, ξ).

Remark 5.2.1. *The remarks below are important for what follows.*

(1) Since in the definition, we only considered the J-holomorphic cylinders connecting two periodic orbits whose Conley-Zehnder indices differ by 1, we have a canonical $\mathbb{Z}_2$-grading by μ_{CZ} mod 2 (in the general case, where we do not necessarily have a global trivialization, such parity is still well defined).

(2) Any J-holomorphic cylinder induces a free homotopy between the two periodic orbits on its ends. Therefore, there is a natural splitting of the above chain complex and homology, based on the free homotopy classes of the periodic orbits.

(3) There is a filtered version of cylinderical contact homology as well [84], denoted by $CH^{<L}(M, \xi)$ with $L > 0$ being a real number, defined by restricting the definition to the periodic orbits of length less than L. This will facilitate the study of the growth rate *of*

these invariants (see [85]).

5.3 Contact dynamics of contact Anosov 3-flows

In what follows, we will see that straightforward computations of the Conley-Zehnder indices associated to the periodic orbits of a contact Anosov flow will help us extract more contact topological information, via the theory of contact dynamics discussed in Section 5.1 and 5.2.

We assume γ is periodic orbit of X, a contact Anosov flow with possibly non-orientable stable and unstable bundles.

By Kobayashi [9] (see Theorem 2 and 5), we have $2e(\xi) = 0 \in H^2(M)$, since ξ admits a line sub bundle (consider the line sub bundle E^{uu}).

Also note that all the periodic orbits of X are non-degenerate, since by Anosovity of the flow, the Poincaré return map along γ hae two distinct eigenspaces with real eigenvalues $|\lambda_1| < 1 < |\lambda_2|$.

The rough idea to compute the Conley-Zehnder indices is that in this case, the Reeb flow does not *twist* with respect to the splitting $\xi = E^{ss} \oplus E^{uu}$. But this splitting does not necessarily induce a trivialization of the contact structure, restricted to the periodic Reeb orbit, since the stable (or unstable) line fields are not necessarily orientable.

To observe this, we consider homologically trivial periodic orbits, where we can compute the Conley-Zehnder indices without any ambiguity.

Lemma 5.3.1. *For any periodic Reeb orbit γ with $[\gamma] = 0 \in H_1(M)$, we have $\mu_{CZ}(\gamma) = 0$.*

Proof. In order to compute the Conley-Zehnder index for γ, we need a symplectic trivialization of $\xi|_\gamma$. Since $[\gamma] = 0 \in H^2(M)$, we can find a Seifert surface $\Sigma_1 \subset M$ for γ (in particular $\partial\Sigma_1 = \gamma$). The splitting $E^{ss} \oplus E^{uu}$ on Σ_1 would induce a trivialization of $\xi|_\gamma$, if $E^{uu}|_{\Sigma_1}$ and $E^{ss}|_{\Sigma_1}$ were orientable. Note that orientability of one will imply orientability of the other one, since ξ is coorientable. However this is not the case in general. Also note that after rescaling of a basis set for ξ, we can assume the trivialization to be symplectic.

Now let $\pi : \Sigma_2 \to \Sigma_1$ to be the orientation double cover for $E^{uu}|_{\Sigma_1}$ and note that $\pi|_{\partial\Sigma_2}$ is a double covering map for $\gamma = \partial\Sigma_1$. This induces a trivialization on γ^2, since the lift of E^{uu} to Σ_2 is orientable.

Since the splitting is preserved by the Reeb flow, the induced path of symplectic maps only includes the ones with positive real eigenvalues and therefore, we have $\mu_{CZ}^{\Sigma_2}(\gamma^2) = 0$ (see Remark 5.1.2).

But since $2e(\xi) = 0$, the Conley-Zehnder index is the same for any other choice of trivialization induced from a 2-chain, in particular from $2\Sigma_1$. i.e.

$$\mu_{CZ}^{2\Sigma_1}(\gamma^2) = \mu_{CZ}^{\Sigma_2}(\gamma^2) = 0.$$

Now noting that γ is a hyperbolic periodic Reeb orbit, by Remark 5.1.4:

$$\mu_{CZ}^{2\Sigma_1}(\gamma^2) = 2 \cdot \mu_{CZ}^{\Sigma_1}(\gamma) = 0$$

and thus

$$\mu_{CZ}(\gamma) = 0.$$

$\square$

In the case where E^{ss} (or E^{uu}) is orientable, we have a global symplectic trivialization of ξ and the same argument as above applies.

Corollary 5.3.2. *[3] If E^{ss} (or E^{uu}) is orientable, then $\mu_{CZ}^{\nu}(\gamma) = 0$ for any periodic Reeb orbit γ, where ν is the trivialization induced from the splitting $E^{ss} \oplus E^{uu}$.*

However, as discussed in Section 5.1 and 5.2, the parity of μ_{CZ} is well defined, even when we do not have any global trivialization, since it only depends on the Poincaré return map along a periodic orbit. As seen above, for a periodic orbit of a contact Anosov flow, the Poincaré return map has two distinct positive (negative) eigenvalues, when either (or both) of the invariant bundles are (non) orientable, when restricted to that periodic orbit.

Corollary 5.3.3. *[3] For any periodic Reeb orbit γ, we have $\mu_{CZ}(\gamma) \equiv 0(mod\, 2)$, when $E^{ss}|_\gamma$ is orientable and $\mu_{CZ}(\gamma) \equiv 1(mod\, 2)$, otherwise.*

Now, we are ready for the implications regarding the cylinderical contact homology in the case of contact Anosov flows. We note that (1) and (2) below are thoroughly discussed in [85].

Theorem 5.3.4. *Let (M, ξ) be an Anosov contact 3-manifold and $CH(M, \xi)$ its associated cylinderical contact homology. Then,*

(1) The differential of $CH(M, \xi)$ is trivial. That is $\partial \equiv 0$;

(2) $CH(M, \xi)$ has infinite rank;

(3) $CH^{<L}(M, \xi)$ has exponential growth as $L \to +\infty$.

Proof. We first note that any periodic orbit of a contact Anosov 3-flow is *good*, unless it is an even multiple of a periodic orbit with odd Conley-Zehnder index (see Section 5.2).

Moreover, Corollary 5.3.3 shows that the parity of μ_{CZ} is an invariant in the free homotopy class of a given periodic orbit in this case. This implies that by Remark 5.2.1 (1) and (2), there is no J-holomorphic cylinder with positive and negative ends γ_+ and γ_-, respectively, such that $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) \equiv 1(mod\, 2)$. Therefore, the differential of the cylinderical contact homology defined in Section 5.2 is trivial.

Moreover, since any contact Anosov flow has (countably) many (good) periodic orbits, (2) would easily follow.

Finally, [86] shows that the conjugacy classes of $\pi_1(M)$ periodic orbits of length $<$ L would increase exponentially as $L \to \infty$ (as a matter of fact they show that the rate exponential growth is equal to the *entropy* of the flow). This implies that the conjugacy classes including good orbits would also increase exponentially, since all the multiples of a simple periodic orbit γ with even $\mu_{CZ}(\gamma)$, or half of its multiples when $\mu_{CZ}(\gamma)$ is odd. $\square$

Since the cylinderical contact homology is invariant of a hypertight contact 3-manifold [84], we can point out the below obstruction for a contact structure to admit an Anosov

Reeb vector field.

Corollary 5.3.5. *Let (M, ξ) be contact 3-manifold whose associated cylinderical contact homology has finite rank. Then, ξ does not admit any Anosov Reeb vector field.*

CHAPTER 6

RIEMANNIAN GEOMETRIC MOTIVATIONS OF CONTACT ANOSOV FLOWS

Contact Anosov 3-flows have been previously studied in the classical literature of of the *Riemannian geometry of contact manifolds*, mainly by David E. Blair and Domenico Perrone. The goal of this chapter is to discuss some of these Riemannian geometric motivations for the study of contact Anosov flows.

We begin this chapter with an overview of the Riemannian geometric theory of contact manifolds. We then discuss two instance when contact Anosov flows naturally appear in this context. That is, the study of *compatible metrics with negative* α*-sectional curvature* and *nowhere Reeb-invariant critical metrics*. In Chapter 7, we will focus our attention on a curvature realization problem in this setting, which goes beyond Anosov dynamical theme of this manuscript. However, we will see that the Anosovity of a Reeb vector field has implications for the Ricci curvature of a *compatible metric*.

6.1 Riemannian geometry of contact 3-manifolds and the local theory of compatibility

The Riemannian geometry of contact manifold has been subject of a thorough study in different contexts, by many including Blair, Hamilton, Chern, etc. and by restricting to certain classes of Riemannian metrics, satisfying natural conditions related to the background contact structure (see [68] for a classical reference). However, we know very little about the global Riemannian geometry of such classes of metrics and therefore their relation to topological aspects of contact structures. A remarkable exception is the analogue of the sphere theorem in the category of contact manifolds [87, 88], when we restrict to a class of Riemannian metrics, namely *compatible metrics*, which seem to be a more natural class of metrics from the topological point of view. It is worth mentioning that the class of compatible metrics is just a slight generalization of the well-studied class of *contact metrics* [68].

On a contact 3-manifold (M, ξ), we can naturally define a Riemannian metric, by choosing a contact form, a complex structure and a positive constant, which measures the *rate of rotation* of ξ.

Definition 6.1.1. *A Riemannian structure g is called* compatible *with (M, ξ) if*

$$g(u, v) = \frac{1}{\theta'} d\alpha(u, Jv) + \alpha(u)\alpha(v)$$

for any $u, v \in TM$, where α is a contact form for ξ, θ' is a positive constant, referred to as instantaneous rotation*, and J is a complex structure on ξ, naturally extended to TM by first projecting along the Reeb vector field associated with α.*

Example 6.1.2. *$(\mathbb{S}^3, \xi_{std})$ and $(\mathbb{T}^3, \xi_n)$ are compatible with round metric on $\mathbb{S}^3$ and flat metric on $\mathbb{T}^3$, respectively.*

Remark 6.1.3. *(1) It can be easily seen that $\theta' = -g([u, v], n) = d\alpha(u, v)$, where (u, v) and (u, v, n) are (locally defined) oriented basis for ξ and TM, respectively and therefore, the positivity of $\theta' > 0$ is equivalent to the (positive) contact condition. In other words θ' measures the rate of rotation ξ with respect to* being integrable. *More precisely, for any point $x \in M$ and basis as above, we can observe that $\theta' = \frac{\partial \theta}{\partial t}\Big|_{t=0}$, where*

$$\theta(t) := \cos^{-1}\left(\frac{g((\phi_{-t})_* v, n)}{||(\phi_{-t})_* v||}\right)$$

and ϕ_t is the flow induced by u. We also observe that the area form of g induced on ξ is $\frac{1}{\theta'} d\alpha$ and similarly for the volume form associated with g,

$$Vol(g) = \frac{1}{\theta'} \alpha \wedge d\alpha.$$

Therefore, such area form and volume form are preserved under X_α by Proposition 1.2.2.

(2) The very well studied class of contact metrics *is the special case of $\theta' = 2$ in the above definition (refer to [68] for the classical literature). However, such restriction is not necessary for our purpose.*

Here, we bring some useful properties of compatible metrics.

Proposition 6.1.4. *For a compatible metric g with associated contact form α and complex structure J, we have*

1) The Reeb vector field X_α is orthonormal to ξ and moreover, is a geodesic field.

2) The Reeb vector field X_α is divergence free with respect to g. Equivalently, for any $e \in \xi$,

$$g(e, \nabla_e X_\alpha) + g(Je, \nabla_{Je} X_\alpha) = 0.$$

By Proposition 6.1.4 1), we have X_α as a geodesic field on M and therefore it is natural to use *Jacobi fields* associated to X_α, measuring the variations of such geodesic field and therefore helping us understand the dynamics and geometry of Reeb vector fields.

More precisely, for a point $p \in M$ and $\gamma : [0, \epsilon] \to M$ being a geodesic flow line of X_α with $\gamma(0) = p$, there exists a map

$$\tilde{\gamma} : [0, \epsilon] \times [0, \epsilon'] \to M,$$

such that

(1) $\frac{\partial \tilde{\gamma}}{\partial t} = X_\alpha$;

(2) $\tilde{\gamma}([0, a] \times \{0\}) = \gamma$;

(3) $v := \frac{\partial \tilde{\gamma}}{\partial s}|_\gamma$ is orthogonal to X_α.

That means that v is a (locally defined) Jacobi field and since for any such map $\tilde{\gamma}$ associated to any geodesic variation, we have $[\frac{\partial \tilde{\gamma}}{\partial t}, \frac{\partial \tilde{\gamma}}{\partial s}] = 0$ (see [89] Lemma 2.2), we can characterize (locally defined) v by

$$X_\alpha^2 \cdot v(t) + \mathcal{R}(v(t), X_\alpha) X_\alpha = 0 \text{ (The Jacobi Identity) };$$

$$X_\alpha \cdot v(t) = \nabla_{v(t)} X_\alpha \ ,$$

where $\mathcal{R}$ is the curvature tensor associated to g and forcing the second condition at an initial point suffices. We refer to such $v(t)$ as an *α-Jacobi field* and note that (locally) $v(t)$ is determined by fixing the initial condition $v(0)$ at p and $v(t)$ is just the push forward of $v(0)$ under X_α. We will exploit such vector fields in the proof of Theorem 6.1.6. With the above remark, it is also useful to compute (see [90]):

Proposition 6.1.5. *For any $e \in \xi$,*

$$\nabla_e X_\alpha = J\left(\frac{\theta'}{2}e - \frac{1}{2}(\mathcal{L}_{X_\alpha} J)(e)\right).$$

Now given a Riemannian manifold (M, g), for any oriented plane field ξ with unit normal n, we can define *the second fundamental form* by:

$$\mathbb{II}(u, v) = g(\nabla_u v, n)$$

for $u, v \in \xi$.

Notice that such bilinear form is symmetric if and only if ξ is integrable. Nevertheless, we can define two geometric invariants of ξ using this second fundamental form, namely the *mean curvature* $H(\xi) := trace(\mathbb{II})$ and the *extrinsic curvature* $G(\xi) := det(\mathbb{II}(\xi))$.

By Proposition 6.1.4, if (M, ξ) is a contact manifold and g a compatible Riemannian metric, we will have:

$$H(\xi) = -div_g(X_\alpha) = 0,$$

while we will show in Theorem 6.1.6 that $G(\xi)$ can be interpreted as (a constant multiplication of) the *Ricci curvature* of X_α.

Theorem 6.1.6. *[4] Let (M, ξ) be a contact 3-manifold, equipped with a compatible metric g. Then for any unit vector $e \in \xi$:*

$$k(e, X_\alpha) = g(Je, \nabla_e X_\alpha)^2 - g(e, \nabla_e X_\alpha)^2 - \frac{\partial}{\partial t} g(e(t), \nabla_{e(t)} X_\alpha)\bigg|_{t=0}$$

where $e(t) := \frac{\tilde{e}(t)}{|\tilde{e}(t)|}$ *and* $\tilde{e}(t)$ *is the unique (locally defined)* α*-Jacobi field with* $\tilde{e}(0) = e$*. Moreover,*

$$Ricci(X_\alpha) := k(e, X_\alpha) + k(Je, X_\alpha) = 2G(\xi).$$

Proof. Since $\nabla_{X_\alpha} X_\alpha = 0$ and $[X_\alpha, \tilde{e}(t)] = 0$,

$$k(X_\alpha, e) = g(R(e, X_\alpha)X_\alpha, e) = -g(\nabla_{X_\alpha} \nabla_{\tilde{e}}(t) X_\alpha, e)\bigg|_{t=0}$$

$$= -\frac{\partial}{\partial t} g(\nabla_{\tilde{e}(t)} X_\alpha, \tilde{e}(t))\bigg|_{t=0} + g(\nabla_e X_\alpha, \nabla_{X_\alpha} \tilde{e}(t))$$

$$= -\frac{\partial}{\partial t} g(\nabla_{\tilde{e}(t)} X_\alpha, \tilde{e}(t))\bigg|_{t=0} + |\nabla_e X_\alpha|^2$$

$$= -\frac{\partial}{\partial t} \left\{ |\tilde{e}(t)|^2 g(\nabla_{e(t)} X_\alpha, e(t)) \right\}\bigg|_{t=0} + |\nabla_e X_\alpha|^2$$

$$= -2g(e, \nabla_e X_\alpha)^2 - \frac{\partial}{\partial t} g(\nabla_{e(t)} X_\alpha, e(t))\bigg|_{t=0} + |\nabla_e X_\alpha|^2$$

$$= g(Je, \nabla_e X_\alpha)^2 - g(e, \nabla_e X_\alpha)^2 - \frac{\partial}{\partial t} g(e(t), \nabla_{e(t)} X_\alpha)\bigg|_{t=0}.$$

Now if we let $e^\perp(t) = \frac{\tilde{e}^\perp(t)}{|\tilde{e}^\perp(t)|}$, where $\tilde{e}^\perp(t)$ is the α-Jacobi field with $\tilde{e}^\perp(0) = Je$,

$$Ricci(X_\alpha) = k(e, X_\alpha) + k(Je, X_\alpha)$$

$$= g(Je, \nabla_e X_\alpha)^2 + g(-e, \nabla_{Je} X_\alpha)^2 - g(e, \nabla_e X_\alpha)^2 - g(Je, \nabla_{Je} X_\alpha)^2 - ...$$

$$... - \frac{\partial}{\partial t} \left\{ g(e(t), \nabla_{e(t)} X_\alpha) + g(e^\perp(t), \nabla_{e^\perp(t)} X_\alpha) \right\}\bigg|_{t=0}$$

$$= (g(Je, \nabla_e X_\alpha) + g(e, \nabla_{Je} X_\alpha))^2 - 2g(Je, \nabla_e X_\alpha) g(e, \nabla_{Je} X_\alpha) - 2g(e, \nabla_e X_\alpha)^2 - ...$$

$$... - \frac{\partial}{\partial t} \left\{ g(e(t), \nabla_{e(t)} X_\alpha) + g(e^\perp(t), \nabla_{e^\perp(t)} X_\alpha) \right\}\bigg|_{t=0}$$

$$= 2G(\xi)+(g(Je,\nabla_e X_\alpha)+g(e,\nabla_{Je}X_\alpha))^2-\frac{\partial}{\partial t}\left\{g(e(t),\nabla_{e(t)}X_\alpha)+g(e^\perp(t),\nabla_{e^\perp(t)}X_\alpha)\right\}\Big|_{t=0}.$$

Therefore, the following lemma will complete the proof:

Lemma 6.1.7. *We have:*

$$(g(Je,\nabla_e X_\alpha)+g(e,\nabla_{Je}X_\alpha))^2=\frac{\partial}{\partial t}\left\{g(e(t),\nabla_{e(t)}X_\alpha)+g(e^\perp(t),\nabla_{e^\perp(t)}X_\alpha)\right\}\Big|_{t=0}.$$

Proof. First compute:

$$g(e(t),\nabla_{e(t)}X_\alpha)+g(e^\perp(t),\nabla_{e^\perp(t)}X_\alpha)=\frac{1}{2}\frac{\partial}{\partial t}\left\{\ln|\tilde{e}(t)|^2+\ln|\tilde{e}^\perp(t)|^2\right\}$$

$$=\frac{1}{2}\frac{\partial}{\partial t}\left\{\ln|\tilde{e}(t)|^2|\tilde{e}^\perp(t)|^2\right\}=-\frac{1}{2}\frac{\partial}{\partial t}\left\{\ln\sin^2\beta(t)\right\}=(-\cot\beta(t))\beta'(t)$$

where $\beta(t)$ is the angle between $\tilde{e}(t)$ and $\tilde{e}^\perp(t)$ and we used the fact that Reeb flow preserves the induced area form of g on ξ and therefore, $\tilde{e}(t)\tilde{e}^\perp(t)\sin\beta(t)=1$ for all t. Now:

$$\frac{\partial}{\partial t}\left\{g(e(t),\nabla_{e(t)}X_\alpha)+g(e^\perp(t),\nabla_{e^\perp(t)}X_\alpha)\right\}\Big|_{t=0}=\left\{-\cot\beta(t).\beta''(t)+\csc^2\beta(t).(\beta'(t))^2\right\}\Big|_{t=0}$$

$$=(\beta'(0))^2.$$

On the other hand:

$$g(Je,\nabla_e X_\alpha)+g(e,\nabla_{Je}X_\alpha)=g(e+Je,\nabla_{e+Je}X_\alpha)=\frac{\partial}{\partial t}\left\{\ln|\tilde{e}(t)+\tilde{e}^\perp(t)|^2\right\}\Big|_{t=0}$$

$$=\frac{\partial}{\partial t}\left\{\ln\left(|\tilde{e}(t)|^2+|\tilde{e}^\perp(t)|^2+2|\tilde{e}(t)||\tilde{e}^\perp(t)|\cos\beta(t)\right)\right\}\Big|_{t=0}$$

$$=\frac{2|\tilde{e}(t)|\frac{\partial|\tilde{e}(t)|}{\partial t}+2|\tilde{e}^\perp(t)|\frac{\partial|\tilde{e}^\perp(t)|}{\partial t}+2|\tilde{e}(t)|\frac{\partial|\tilde{e}^\perp(t)|}{\partial t}\cos\beta(t)+2|\tilde{e}^\perp(t)|\frac{\partial|\tilde{e}(t)|}{\partial t}\cos\beta(t)}{|\tilde{e}(t)+\tilde{e}^\perp(t)|^2}\Big|_{t=0}-...$$

$$..-\frac{2|\tilde{e}(t)||\tilde{e}^\perp(t)|\sin\beta(t).\beta'(t)}{|\tilde{e}(t)+\tilde{e}^\perp(t)|^2}\Big|_{t=0}$$

$$= \left\{ \frac{\partial |\tilde{e}(t)|}{\partial t} + \frac{\partial |\tilde{e}^{\perp}(t)|}{\partial t} \right\} \Big|_{t=0} - \beta'(0) = -\beta'(0).$$

which establishes proof of the lemma. In the last equality, we used the fact that

$$0 = \frac{\partial}{\partial t} \left\{ \tilde{e}(t) \tilde{e}^{\perp}(t) \sin \beta(t) \right\} \Big|_{t=0} = \left\{ \frac{\partial |\tilde{e}(t)|}{\partial t} + \frac{\partial |\tilde{e}^{\perp}(t)|}{\partial t} \right\} \Big|_{t=0}.$$

$\square$

$\square$

Let (e, Je) be any local choice of an orthonormal frame for ξ. Using the above characterization, Remark 6.1.3 and Koszul formula, we can derive the following formula for $Ricci(X_\alpha)$.

Corollary 6.1.8. *For any $x \in M$, we can write $Ricci(X_\alpha)$ as:*

$$Ricci(X_\alpha)(x) = -2P^2(x) + \frac{\theta'^2}{2} - 2Q^2(x)$$

where

$$P(x) = g(e, \nabla_e X) = \frac{1}{\theta'} d\alpha([e, X], Je)$$

and

$$Q(x) = \frac{\theta'}{2} - g(Je, \nabla_e X) = \frac{1}{2\theta'} d\alpha([e, X], e) - \frac{1}{2\theta'} d\alpha([Je, X], Je)$$

for any choice of orthonormal frame (e, Je, X). In particular,

$$Ricci(X_\alpha) \leq \frac{\theta'^2}{2}.$$

It turns out that $Ricci(X_\alpha)$ attaining its maximum has an important geometric meaning.

Proposition 6.1.9. *At any point $x \in M$, the followings are equivalent:*

(1) $Ricci(X_\alpha) = \frac{\theta'^2}{2}$;

(2) $g(e, \nabla_e X_\alpha) = 0$ *for any unit vector* $e \in \xi$*;*

(3) $\mathcal{L}_{X_\alpha} J = 0$;

(4) $\mathcal{L}_{X_\alpha} g = 0$.

Proof.

Claim 6.1.10. *At any point* $x \in M$*,* $g(e, \nabla_e X_\alpha) = 0$ *either for exactly 4 unit vector* e *at* x *or for all unit vectors at* x*.*

Proof. Since $g(e, \nabla_e X_\alpha) + g(Je, \nabla_{Je} X_\alpha) = 0$ for any $e \in \xi$, there exists some $e \in \xi$ such that $g(e, \nabla_e X_\alpha) = 0$. Clearly the same holds for $-e$, Je and $-Je$. Now imagine $g(v, \nabla_v X_\alpha) = 0$ for some other unit vector $v = ae + bJe$ at x (where $ab \neq 0$). Then

$$0 = g(v, \nabla_v X_\alpha) = \frac{1}{2}\frac{\partial}{\partial t} \ln |\tilde{v}(t)|^2 \Big|_{t=0} = \frac{1}{2}\frac{\partial}{\partial t} \ln |a\tilde{e}(t) + b\tilde{e}^{\perp}(t)|^2 \Big|_{t=0}$$

where $\tilde{v}(t)$, $\tilde{e}(t)$ and $\tilde{e}^{\perp}(t)$ are respectively the α-Jacobi field extension of v, e and Je respectively. Letting $\beta(t)$ be the angle between $\tilde{e}(t)$ and $\tilde{e}^{\perp}(t)$, this means

$$0 = \frac{1}{2}\frac{\partial}{\partial t} \ln \left\{a^2|\tilde{e}(t)|^2 + b^2|\tilde{e}^{\perp}(t)|^2 + 2ab|\tilde{e}(t)||\tilde{e}^{\perp}(t)| \cos\beta(t)\right\} \Big|_{t=0} = -ab\beta'(0)$$

So we have $\beta'(0) = 0$. But this computation shows that for any other linear combination $ce + dJe$, we will have $g(ce + dJe, \nabla_{ce+dJe} X_\alpha) = 0$, proving the claim. □

(1) $\Rightarrow$ (2) If $Ricci(X_\alpha) = \frac{\theta'^2}{2}$, then $P(x) = g(e, \nabla_e X) = 0$ for any choice of unit $e \in \xi$.

(2) $\Rightarrow$ (1) In this case, $g(e + Je, \nabla_{e+Je} X_\alpha) = g(e, \nabla_{Je} X_\alpha) + g(Je, \nabla_e X_\alpha) = 0$. Together with Remark 6.1.3, this implies

$$-g(e, \nabla_{Je} X_\alpha) = g(Je, \nabla_e X_\alpha) = \frac{\theta'}{2}$$

which implies $P(x) = Q(x) = 0$.

$(3) \Rightarrow (2)$ By Proposition 6.1.5, for any $e \in \xi$ we have

$$g(e, \nabla_e X_\alpha) = g(e, J\left(\frac{\theta'}{2}e - \frac{1}{2}(\mathcal{L}_{X_\alpha}J)(e)\right)) = g(\frac{\theta'}{2}Je, e) = 0.$$

$(1) \Rightarrow (3)$ In this case, for any $e \in \xi$, we have $g(e, \nabla_e X_\alpha) = 0$ and $g(Je, \nabla_e X_\alpha) = \frac{\theta'}{2}$. Therefore,

$$g((\mathcal{L}_{X_\alpha}J)(e), e) = g((\mathcal{L}_{X_\alpha}J)(e), Je) = 0,$$

which yields $\mathcal{L}_{X_\alpha}J = 0$.

$(3) \iff (4)$ The equivalence follows from the fact that in the definition of a compatible metric, α is invariant under X_α and θ' is constant.

□

Remark 6.1.11. *Note that in the above discussion, since $g|_\xi$ is constantly proportional to $d\alpha|_\xi$ and X_α preserves $d\alpha$, after a (local) trivialization of ξ, we can (locally) describe the action of such Reeb flow on ξ as a path in $Sp(1_\mathbb{C})$, area preserving linear maps of $\mathbb{R}^2$. Now we can decompose any $A \in Sp(1_\mathbb{C})$ as $A = MU$, where $U \in SO(2)$ measures the rotation of the flow with respect to the trivialization and M is a positive definite matrix measuring the hyperbolicity of A. On the other hand, using the above notation, we have $g(e, \nabla_e X_\alpha) = \frac{1}{2}\frac{\partial}{\partial t}\ln|\tilde{e}(t)|^2$ measuring the infinitesimal rate of change of the length of vectors in ξ with respect to g. Therefore by Theorem 6.1.6, the deviation of $Ricci(X_\alpha)$ from its maximum measures the* infinitesimal hyperbolicity *of the flow of X_α with respect to g, leaving the rotation of the flow undetected. In other words, when $Ricci(X_\alpha) = \frac{\theta'^2}{2}$ at a point, the flows acts as pure rotation infinitesimally, while when $Ricci(X_\alpha) < \frac{\theta'^2}{2}$ in a neighborhood, we (locally) have a section e of ξ such that $g(e, \nabla_e X_\alpha) = g(Je, \nabla_{Je} X_\alpha) = 0$ with $g(., \nabla_. X_\alpha)$ having alternating signs in the intermediate regions (see Proposition 6.1.4, Proposition 6.1.9). See Figure 6.1.*

Although in Theorem 7.1.2, we will see that by manipulation of g, we can hide such hyperbolicity locally, the global consequences of such dynamical phenomena can be of

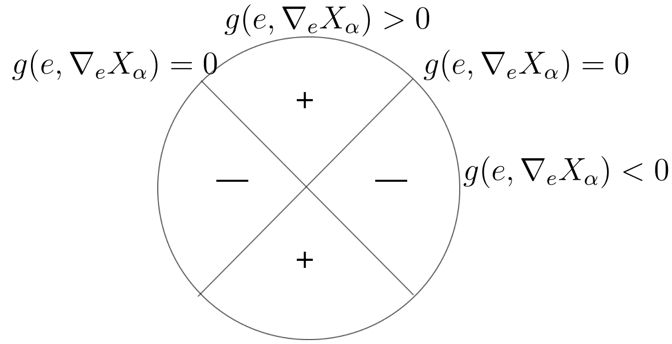


Figure 6.1: Splitting of ξ when $\mathcal{L}_{X_\alpha} g \neq 0$ and regions with alternating signs for $g(e, \nabla_e X_\alpha)$

(contact) topological interest (see Chapter 7).

We note that the case when we have such rigidity everywhere, is studied previously either as K-contact structures *or* geodesible contact structures *[91], defined to be contact manifolds equipped with compatible metrics satisfying $\mathcal{L}_{X_\alpha} J = 0$ everywhere and compatible metrics whose geodesics tangent to ξ at a point remains tangent to ξ (which is equivalent to $g(e, \nabla_e X_\alpha) = 0$ for any $e \in \xi$), respectively.*

On the other hand, the term $\frac{\partial}{\partial t} g(e(t), \nabla_{e(t)} X_\alpha)|_{t=0}$ in the computation of $k(e, X_\alpha)$ can help us measure the infinitesimal rotation of X_α with respect to splitting of TM described above, in the case of $Ricci(X_\alpha) < \frac{\theta'^2}{2}$.

Corollary 6.1.12. *The followings are equivalent.*

1) $Ricci(X_\alpha) = \frac{\theta'^2}{2}$ everywhere;

2) ξ is K-contact;

3) ξ is geodesible.

6.2 Compatible metrics with negative α-sectional curvature

In [32], Blair and Perrone proved that for a contact structure and a compatible metric, if the sectional curvature of the planes including the Reeb direction satisfies a certain upper bound, in particular if it is negative, the projective Anosovity of the underlying contact structure can be concluded. Of course, as we showed in Chapter 3, such flow is Anosov, as is any volume preserving projectively Anosov flow (Corollary 3.2.3. Here for completion, We reprove and improve their result only using the characterization given above, in order to observe the interplay between curvature and Anosovity directly. It is worth mentioning that the compatible Riemannian structure on $(UT\Sigma, \theta)$, when Σ is a hyperbolic surface, satisfies this condition.

Theorem 6.2.1. *[3] Let M^3 be equipped with a contact structure ξ and a compatible metric g, such that for any unit vector $e \in \xi$:*

$$k(e, X_\alpha) < \left[\frac{\theta'}{2} - \sqrt{\frac{\theta'^2}{4} - \frac{1}{2} Ricci(X_\alpha)} \right]^2 .$$

Then X_α is Anosov.

Proof. Recall that the curvature condition implies that the four quadrant as in Remark 6.1.11 exist. Let $e_1, e_2 \in \xi$ be non-parallel unit vectors with $g(e_i, \nabla_{e_i} X_\alpha) = 0$ for $i = 1, 2$ (see Remark 6.1.11). Then by Theorem 6.1.6, we can easily compute for $i = 1, 2$:

$$g(Je_i, \nabla_{e_i} X_\alpha)^2 = \left[\frac{\theta'}{2} \pm \sqrt{\frac{\theta'^2}{4} - \frac{1}{2} Ricci(X_\alpha)} \right]^2$$

and our assumption on sectional curvature will imply:

$$\frac{\partial}{\partial t} g(e_i(t), \nabla_{e_i(t)} X_\alpha) > 0.$$

Then, we know that X_α is projectively Anosov, since $\langle e_1, X_\alpha \rangle$ and $\langle e_2, X_\alpha \rangle$ are positive and negative contact structures. Also notice that this implies that for any $e \in E^u$ ($e \in E^s$) of this projectively Anosov flow, we have $g(e, \nabla_e X_\alpha) > 0$ ($g(e, \nabla_e X_\alpha) < 0$). See Figure 6.2.

Now in order to to prove that X_α is furthermore Anosov, choose $C > 0$ such that for any unit vector $e \in E^u$ at any point in M, $g(e, \nabla_e X_\alpha) > C$ holds (such C exists by the compactness of M). Using the notation of Theorem 6.1.6, we will have:

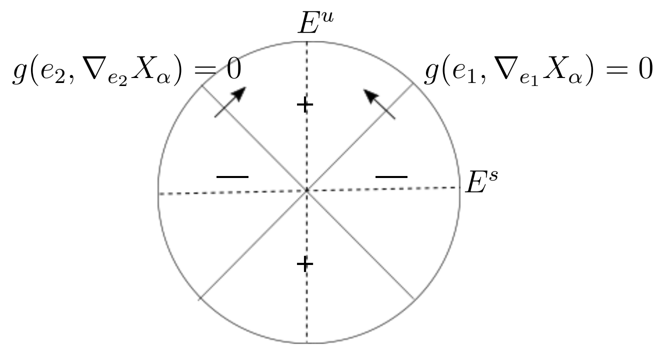


Figure 6.2: Dynamics of contact structures admitting a compatible metric with negative α-sectional curvature

$$g(e(t), \nabla_{e(t)} X_\alpha) = \frac{1}{2} \frac{\partial}{\partial t} \ln g(\tilde{e}(t), \tilde{e}(t)) > C$$

and this implies

$$\ln g(\tilde{e}(t), \tilde{e}(t)) - \ln g(\tilde{e}(0), \tilde{e}(0)) > Ct$$

$$g(\tilde{e}(t), \nabla_{\tilde{e}(t)} X_\alpha) > e^{Ct}.$$

A similar argument for the unit vector $e \in E^s$ yields the Anosovity of X_α. □

6.3 Nowhere Reeb-invariant critical metrics

In [92], Chern and Hamilton initiated the study of a particular class of compatible metrics, namely critical compatible metric, by stating a conjecture that can be generalized to:

Conjecture 6.3.1. *For any closed contact 3-manifold (M, ξ), there exists a compatible metric that realizes the minimum (among compatible metrics) of the energy functional:*

$$E(g) := \int_M |\mathcal{L}_{X_\alpha} g|^2 \quad dVol(g).$$

Motivated by this conjecture, we can study the critical points of this energy functional restricted to the space of compatible metrics. We call such metrics *critical compatible* *metrics*.

This conjecture was proved by Rukimbira [93] for a very specific class of contact manifolds, namely *the generalized Boothby-Wang fibrations*, by characterizing such contact manifolds as the ones admitting a compatible metric with

$$\mathcal{L}_{X_\alpha} g = 0$$

everywhere and therefore satisfying the condition of Chern-Hamilton conjecture.

However, Perrone [33] showed that under the extreme opposite assumption of the compatible metric being *nowhere Reeb-invariant*, i.e. assuming

$$\mathcal{L}_{X_\alpha} g \neq 0$$

everywhere, the existence of such critical compatible metric will imply the projective Anosovity of the underlying contact structure. Again, we now know that we have Anosovity in this case, thanks to Corollary 3.2.3.

Theorem 6.3.2. *[33, 5] If g is a compatible metric which is the critical point of E and we have $\mathcal{L}_{X_\alpha} g \neq 0$ everywhere, then X_α is Anosov with respect to such metric.*

Therefore, Theorem 6.3.2 implies that for a wide range of contact manifolds, critical metrics cannot be nowhere Reeb-invariant.

Corollary 6.3.3. *Let (M^3, ξ) be a contact 3-manifold which is either overtwisted, reducible or admits an exact cobordism to $(\mathbb{S}^3, \xi_{std})$ and g a critical compatible metric. Then, there exists some point at which*

$$\mathcal{L}_{X_\alpha} g = 0,$$

where α is the contact form corresponding to g and X_α is the associated Reeb field.

Remark 6.3.4. *Perrone [33] refers to compatible metrics with $\mathcal{L}_{X_\alpha} g \neq 0$ as* non-Sasakian metrics.

CHAPTER 7

MORE ON THE GLOBAL RIEMANNIAN GEOMETRY OF CONTACT STRUCTURES: RICCI-REEB CURVATURE REALIZATION PROBLEM

In this chapter, we discuss a curvature realization problem in the *compatible Riemannian geometry of contact manifolds*. We will also observe the implications of the Anosovity of Reeb vector fields for this problem.

In Riemannian geometry, it is well known that local restrictions on a Riemannian metric, in particular its curvature tensor, can result in topological consequences. A classical example is the celebrated sphere theorem, introduced by Berger [94] and Klingenberg [95] in early 1960s:

Theorem 7.0.1. *[94, 95] Let (M, g) be a Riemannian manifold of arbitrary dimension n with $\frac{1}{4}$-pinched sectional curvature. i.e. if there exists some positive constant K, for which $\frac{1}{4}K < Sec(g) \leq K$. Then the universal cover of M is homeomorphic to $\mathbb{S}^n$.*

In dimension 3, this was generalized extensively by Hamilton and his theory of Ricci flow [96] in 1982.

Theorem 7.0.2. *[96] Let (M, g) be a Riemannian 3-manifold such that $Ricci(g) > 0$. Then the universal cover of M is diffeomorphic to $\mathbb{S}^3$.*

Beside the above *rigidity theorems*, we also have *flexibility theorems*, showing the lack of relation to topology. For instance, in 1994 Lohkamp [97] showed:

Theorem 7.0.3. *[97] Let M be a smooth manifold of arbitrary dimension. Then it admits a Riemannian metric g with $Ricci(g) < 0$.*

which means negative Ricci curvature does not yield any information about the topology of the underlying manifold.

It is natural to ask whether results similar to above theorems hold in other categories of 3-manifolds, since after the proof of geometrization conjecture by Perelman, we can expect to be able to relate topological theories of 3-manifolds to their underlying Riemannian geometry. On the other hand, we have learned that contact structures, which are known to have subtle and rich relation to the topology of 3-manifolds.

As discussed in Chapter 6, although the Riemannian geometry of contact manifolds has been thoroughly studied by restricting to certain classes of Riemannian metrics, satisfying natural conditions related to the background contact structure (see [68] for an overview), the global aspects of this theory is left mostly unexplored. An important exception is the analogue of sphere theorem for Riemannian metrics compatible with a contact 3-manifold [87, 88].

Theorem 7.0.4. *[87, 88] Let (M, ξ) be a contact 3-manifold, admitting a compatible metric g with $\frac{1}{4}$-pinched sectional curvature. Then the universal cover of (M, ξ) is contactomorphic to $(\mathbb{S}^3, \xi_{std})$.*

Note that by Eliashberg's classification of contact structures [11, 12], we have a $\mathbb{Z}$-family of distinct contact structures on $\mathbb{S}^3$. Therefore in the above theorem, the universal cover of M being $\mathbb{S}^3$ is concluded from the classical sphere theorem and specifying the contact structure as the standard contact structure on $\mathbb{S}^3$ is the consequence of the compatibility condition. A natural generalization would be

Conjecture 7.0.5. *Let (M, ξ) be a contact 3-manifold, equipped with a compatible metric g, such that $Ricci(g) > 0$. Then the universal cover of (M, ξ) is contactomorphic to $(\mathbb{S}^3, \xi_{std})$.*

which is still not known to be true.

For more global results, regarding curvature realization of such metrics see [98], about contact topology of compatible metrics with negative *α-sectional curvatures* Chapter 6, and regarding the more restricted class of *Sasakian metrics*, positive curvature and contact topology in higher dimensions, see [99].

Motivated by the above discussion, it is natural to study Ricci curvature realization problems in the category of contact 3-manifolds. In this paper, we study the Ricci curvature of Reeb vector fields (also known as characteristic vector fields) associated to a contact manifold. Reeb vector fields have played a central role in contact geometry, going back to its classical development, comparable to Hamiltonian vector fields in symplectic geometry. Moreover, since the early 1990s, we have learned that they can be used to extract contact topological information about the underlying contact manifold as well and by now, we have useful invariants of contact manifold, based on understanding of such dynamics (see [78] for early developments). Therefore, it is natural to investigate if Ricci curvature of such vector fields contain any contact topological informations and what functions can be realized as such Ricci curvature of a given contact manifolds.

Question 7.0.6 (Ricci-Reeb Realization Problem)**.** *Given a contact manifold (M,ξ), what functions can be realized as Ricci curvature of the Reeb vector field associated to a compatible metric?*

First, we will see that the subtlety of such realization is of global nature, since any function can be realized locally.

Theorem 7.0.7. *[4][Local realization] Let (M,ξ) be a contact 3-manifold equipped with a compatible metric g and $x \in M$ an arbitrary point, and a given function $f : M \to \mathbb{R}$. Then there exists a neighborhood U containing x and a compatible metric g_* such that:*

1) $Ricci_*(X_\alpha)(x) = f(x)$ on U;

2) $g = g_*$ at x,

where $Ricci_*(X_\alpha)$ is the Ricci curvature of the Reeb vector field associated with g_*.

In an attempt to extend such solution to a global one, we will use the topological tool of *open book decompositions*, which has been widely used in contact topology since the establishment of *Giroux's correspondence* between such structures and contact structures in 2000 [100]. This method will yield an *almost global* realization, reducing the pursuit of a global solution to resolving a codimension one embedded submanifold of singularities.

Theorem 7.0.8. *[4][Almost global realization] Let (M, ξ) be a closed oriented contact 3-manifold, $f(x) : M \to \mathbb{R}$ a function on M and V a positive real number. Then there exists a singular metric g_∞ and an embedded compact surface with boundary $F \subset M$ such that:*

1) g_∞ is a compatible metric on $M \backslash F$;

2) $Ricci(X_\alpha)(x) = f(x)$ on $M \backslash F$, where X_α is the Reeb vector field associated with g_∞;

3) $Vol(g_\infty) = V$;

4) g_∞ can be realized as an element of the completion of the space of compatible Riemannian metrics $\overline{\mathcal{M}_\xi} \subset \overline{\mathcal{M}}$. More precisely, given any $\epsilon > 0$, $[g_\infty]$ is the limit of a L^2-Cauchy sequence of compatible metrics $\{g_n\} \to [g_\infty] \in \overline{\mathcal{M}_\xi} \subset \overline{\mathcal{M}} \simeq \mathcal{M}_f / \sim$, such that g_n realizes the given function $f(x)$ as $Ricci(X_\alpha)$, outside a $\frac{\epsilon}{2^n}$-neighborhood of F.

As we saw in Chapter 6 that for any compatible metric with instantaneous rotation θ' (see Remark 6.1.3), we have $Ricci(X_\alpha) \leq \frac{\theta'^2}{2}$ (see Corollary 6.1.8). Therefore in the above theorems, we need to choose the constant θ' such that $f(x) \leq \frac{\theta'^2}{2}$ (note that M is compact). On the other hand, for a fixed θ', these theorems hold for any function, respecting such upper bound.

As we will learn about the geometric meaning of such Ricci curvature attaining its maximum (see Proposition 6.1.9), we recognize that the dichotomy of achieving such maximum or not seems to be of central importance for complete understanding of the Ricci-Reeb realization problem. In particular, when considered globally, the dichotomy will result in topological obstructions to realization of a function as $Ricci(X_\alpha)$, showing that the resolution of the singularity set in Theorem 7.0.8 depends on topological data.

Using the previous works of [101, 93, 91, 102], we will see that forcing $Ricci(X_\alpha) = \frac{\theta'^2}{2}$ everywhere has strong rigidity consequences for the underlying contact manifold.

Theorem 7.0.9. *[4] Let (M, ξ) be a closed contact 3-manifold and g a compatible Riemannian metric with $Ricci(X_\alpha) = \frac{\theta'^2}{2}$ everywhere, where θ' is the instantaneous rotation of g. Then (M, ξ) is finitely covered by a Boothby-Wang fibration with ξ being a tight symplectically fillable contact structure. Moreover, if all the periodic Reeb orbits associated with g are non-degenerate, then (M, ξ) is finitely covered by 3-sphere with the standard tight contact structure.*

On the other hand, we can easily find topological obstructions for the extreme opposite case of nowhere attaining such maximum, i.e. admitting a *nowhere Reeb-invariant* compatible metric, strengthening a theorem of Krouglov [98].

Theorem 7.0.10. *[4] Let (M, ξ) be any contact 3-manifold with $2e(\xi) \in H^2(M) \neq 0$. Then for any compatible metric g with instantaneous rotation θ', there exists some point $x \in M$ at which $Ricci(X_\alpha)(x) = \frac{\theta'^2}{2}$, where X_α is the Reeb vector field associated with g.*

Note that this also means that the analogue of Lohkamp's flexibility theorem, Theorem 7.0.3, does not hold in this category.

We will also observe that as long as (M, ξ) admits a compatible metric satisfying $Ricci(X_\alpha)(x) < \frac{\theta'^2}{2}$, we can find a compatible metric for which $Ricci(X_\alpha)$ is arbitrary far from the maximum, confirming the observation that the described dichotomy is of primary importance, compared to other natural dichotomies like $Ricci(X_\alpha)$ being positive versus negative (however, for a survey on the known results concerning the sign of curvature and contact metric geometry see [103]):

Theorem 7.0.11. *[4] Assume (M, ξ) admits some compatible metric with instantaneous rotation θ' and $Ricci(X_\alpha) < \frac{\theta'^2}{2}$ everywhere. Then for any $c \leq \frac{\theta'^2}{2}$, there exists some compatible metric with instantaneous rotation θ' and $Ricci(X_\alpha) < c$.*

It is worth mentioning that we can establish existence of such metric, based on the dynamical assumption of *Anosovity* of a contact manifold, i.e. when (M, ξ) admits an Anosov Reeb vector field. Such a class of flows were introduced by Eliashberg and Thurston [2] and Mitsumatsu [1] in mid 1990s and has showed up naturally in the study of Riemannian geometry of contact manifolds by Blair and Perrone [32, 33]. We have studied such flows in the category of three dimensional contact topology in Chapter 6.

Theorem 7.0.12. *[4] Let (M, ξ) be an Anosov contact 3-manifold. Then ξ admits a Reeb vector field and a complex structure J, satisfying*

$$\mathcal{L}_{X_\alpha} J \neq 0$$

everywhere, or equivalently, (M, ξ) admits a compatible metric with instantaneous rotation θ' and

$$Ricci(X_\alpha) < \frac{\theta'^2}{2}$$

everywhere.

However, it is interesting to know whether there are *contact topological* obstructions to global realization of a given function, or equivalently resolving the codimension one singularity set described in Theorem 7.0.8. Based on our study of Ricci-Reeb realization problem and our other results in Chapter 6, we conjecture.

Conjecture 7.0.13. *If (M, ξ) admits a Reeb vector field X_α and a complex structure J, satisfying*

$$\mathcal{L}_{X_\alpha} J \neq 0$$

everywhere, or equivalently if (M, ξ) admits a compatible metric with instantaneous rotation θ' and

$$Ricci(X_\alpha) < \frac{\theta'^2}{2}$$

everywhere, then it is tight.

7.1 Deformation and local realization

In order to prescribe a function for $Ricci(X_\alpha)$, we want to understand the effect of perturbing J on $Ricci(X_\alpha)$.

Lemma 7.1.1 (Perturbation of complex structure)**.** *Let (M,ξ) be equipped with a compatible metric $g(.,.) = \frac{1}{\theta'}d\alpha(.,J.) + \alpha(.)\alpha(.)$ and assume that there exist a line sub bundle of ξ (equivalently $2e(\xi)=0$). Define a new complex structure by*

$$J_*:\ e \mapsto \eta^2 Je + \lambda e$$

where e is any vector on the above line section, λ is any function on M and η is a positive function on M. The Ricci curvature for the new compatible metric $g_*(.,.) = \frac{1}{\theta'}d\alpha(.,J_*.) + \alpha(.)\alpha(.)$ is given by

$$Ricci_*(X_\alpha)(x) = -2\left(P_*(x) + \frac{X_\alpha\eta}{\eta}\right)^2 + \frac{\theta'^2}{2} - 2\left(Q_*(x) - \frac{\lambda}{2\eta}X_\alpha\eta + \frac{\eta}{2}X_\alpha(\frac{\lambda}{\eta})\right)^2$$

where

$$P_*(x) = \frac{1}{\theta'}d\alpha([e,X_\alpha],Je) + \frac{1}{\theta'}\frac{\lambda}{\eta^2}d\alpha([e,X_\alpha],e)$$

and

$$Q_*(x) = \frac{1}{2\theta'}\frac{1}{\eta^2}d\alpha([e,X_\alpha],e) - \frac{\eta^2}{2\theta'}d\alpha([Je,X_\alpha],Je) - \frac{\lambda}{\theta'}d\alpha([e,X_\alpha],Je) - \dots$$

$$\dots - \frac{1}{2\theta'}\frac{\lambda^2}{\eta^2}d\alpha([e,X_\alpha],e).$$

Proof. Let $\overset{*}{\nabla}$ be the Levi-Civita connection associated to g_*. Note that under the above perturbation the length of e will become η. So $\frac{e}{\eta}$ will be the unit vector in the direction of e. Applying Koszul formula as in Corollary 6.1.8 we have

$$g_*(\frac{e}{\eta}, \overset{*}{\nabla}_{\frac{e}{\eta}} X_\alpha) = \frac{1}{\theta'} d\alpha([\frac{e}{\eta}, X_\alpha], J_* \frac{e}{\eta}) = \frac{1}{\theta'} d\alpha(\frac{1}{\eta}[e, X_\alpha] - X_\alpha(\frac{1}{\eta})e, \eta Je + \frac{\lambda}{\eta} e)$$

$$= \frac{1}{\theta'} d\alpha([e, X_\alpha], Je) + \frac{1}{\theta'} \frac{\lambda}{\eta^2} d\alpha([e, X_\alpha], e) + \frac{X_\alpha \eta}{\eta}$$

We will also have

$$g_*(J_* \frac{e}{\eta}, \overset{*}{\nabla}_{\frac{e}{\eta}} X_\alpha) = \frac{\theta'}{2} - \frac{1}{2\theta'} d\alpha([\frac{e}{\eta}, X_\alpha], \frac{e}{\eta}) + \frac{1}{2\theta'} d\alpha([J_* \frac{e}{\eta}, X_\alpha], J_* \frac{e}{\eta})$$

$$= \frac{\theta'}{2} - \frac{1}{2\theta'} d\alpha(\frac{1}{\eta}[e, X_\alpha] + \frac{X_\alpha \eta}{\eta^2} e, \frac{e}{\eta}) + \frac{1}{2\theta'} d\alpha(\eta[Je, X_\alpha] - (X_\alpha \eta) Je + \frac{\lambda}{\eta}[e, X_\alpha] - (X_\alpha \frac{\lambda}{\eta}) e, \eta Je + \frac{\lambda}{\eta} e)$$

$$= \frac{\theta'}{2} - \frac{1}{2\theta'} \frac{1}{\eta^2} d\alpha([e, X_\alpha], e) + \frac{\eta^2}{2\theta'} d\alpha([Je, X_\alpha], Je) + \frac{\lambda}{2\theta'} d\alpha([Je, X_\alpha], e) + \frac{\lambda}{2\theta'} d\alpha([e, X_\alpha], Je) + \dots$$

$$\dots + \frac{1}{2\theta'} \frac{\lambda^2}{\eta^2} d\alpha([e, X_\alpha], e) + \frac{\lambda}{2\eta} X_\alpha \eta - \frac{\eta}{2} X_\alpha \frac{\lambda}{\eta}.$$

□

As a result, starting from any compatible metric, it is enough to perturb the associated complex structure to realize any function as $Ricci(X_\alpha)$ locally, assuming it respects the upper bound on Ricci curvature.

Theorem 7.1.2. *[4][Local realization] Let (M, ξ) be a contact 3-manifold equipped with a compatible metric g and $x \in M$ an arbitrary point, and $f : M \to \mathbb{R}$ a function such that $f(x) \leq \frac{\theta'^2}{2}$. Then there exists the neighborhood U containing x and a compatible metric g_* with instantaneous rotation θ' such that*

1) $Ricci_*(X_\alpha)(x) = f(x)$ on U;

2) $g = g_*$ at x.

Proof. Let $\mu = \frac{\lambda}{\eta^2}$. After choosing local trivialization e, we can rewrite the equations of Lemma 7.1.1 for the corresponding perturbation of the almost complex structure J:

$$Ricci_*(X_\alpha)(x) = -2\left(\hat{P}_*(x) + X_\alpha(\ln\eta)\right)^2 + \frac{\theta'^2}{2} - 2\eta^4\left(\hat{Q}_*(x) + \frac{1}{2\theta'}\frac{1}{\eta^4}d\alpha([e, X_\alpha], e) + \frac{1}{2}X_\alpha\mu\right)^2$$

where

$$\hat{P}_*(x) = \frac{1}{\theta'}d\alpha([e, X_\alpha], Je) + \frac{1}{\theta'}\mu d\alpha([e, X_\alpha], e)$$

and

$$\hat{Q}_*(x) = -\frac{1}{2\theta'}d\alpha([Je, X_\alpha], Je) - \frac{\mu}{\theta'}d\alpha([e, X_\alpha], Je) - \frac{1}{2\theta'}\mu^2 d\alpha([e, X_\alpha], e)$$

Now in order to solve the PDE $Ricci_*(X_\alpha)(x) = f(x)$ locally, let U be an open neighborhood around x such that $x \in \Sigma_0 \subset U$, where Σ_0 is a (local) smooth surface transverse to X_α including x and X_α gives the neighborhood U a smooth product structure $U \simeq \Sigma_0 \times (-\epsilon, \epsilon)$. Now, we can solve our PDE on U, by solving the following two PDEs.

$$(1)\begin{cases} \hat{P}_*(x) - X_\alpha(\ln\eta) = 0 \\ \eta|_{\Sigma_0} = 1 \end{cases}$$

$$(2)\begin{cases} \frac{\theta'^2}{4} - \eta^4(\hat{Q}_*(x) + \frac{1}{2\theta'}\frac{1}{\eta^4}d\alpha([e, X_\alpha], e) + \frac{1}{2}X_\alpha\mu)^2 = \frac{f(x)}{2} \\ \mu|_{\Sigma_0} = 0 \end{cases}$$

But exploiting the (local) product structure above, we can translate these two PDEs into two ODEs on Σ_0.

$$(1)\begin{cases} \frac{\partial}{\partial t}\ln\eta = -\frac{1}{\theta'}d\alpha([e, X_\alpha], Je) - \frac{1}{\theta'}\mu d\alpha([e, X_\alpha], e) \\ \eta(0) = 1 \end{cases}$$

$$(2)\begin{cases} \frac{1}{2}\frac{\partial}{\partial t}\mu = \frac{1}{\eta^2}\sqrt{\frac{\theta'^2}{4} - \frac{f(x(t))}{2}} - \frac{1}{2\theta'}\frac{1}{\eta^4}d\alpha([e, X_\alpha], e) + \frac{1}{2\theta'}d\alpha([Je, X_\alpha], Je) + ... \\ \qquad\qquad ... + \frac{\mu}{\theta'}d\alpha([e, X_\alpha], Je) + \frac{1}{2\theta'}\mu^2 d\alpha([e, X_\alpha], e) \\ \mu(0) = 0 \end{cases}$$

Now because of existence and uniqueness of the solution of ODEs, we can solve these two equations in the following way. First solve (1) for η in terms of μ. More explicitly,

$$\eta(x(t)) = e^{\int_0^t \hat{P}_*(x(s))ds}$$

which depends on the unknown $\mu(x(t))$ (note that η stays positive). But replacing this solution (in terms of μ) into (2), we will have another ODE

$$(2)\begin{cases} \frac{\partial}{\partial t}\mu(x(t)) = F(x(t), \mu) \\ \mu(0) = 0 \end{cases}$$

for the appropriate function F. Now we can locally solve this ODE to find μ. Replacing this into the solution for η which was in terms of μ, we find η. Hence, we also have found $\lambda = \mu\eta^2$. The complex structure defined by these two parameters will define the desired Riemannian metric g_*. Notice that $g = g_*|_{\Sigma_0}$ by our initial conditions. □

7.2 Open book decompositions and Giroux correspondence

In order to study the discussed curvature realization problem globally, we use a standard tool from contact topology, named *open book decompostions*. In this section, we review some basic fact on such topological structures.

Open book decompositions have become one of the main topological tools in contact topology, thanks to the celebrated *Giroux correspondence*, established by Emmanuel Giroux in 2000 [100], which was built upon the previous work of Thurston and Wilkelnkemper [104] and gives a purely topological description of contact structures.

Theorem 7.2.1. *[4][Giroux Correspondence] On a given 3-manifold M, contact structures up to isotopy are in 1-to-1 correspondence with* open book decompositions up to positive stabilization.

In this paper, we only use the fact that for any contact structure on a given manifold, there exists an open book decomposition *adapted* to it. Therefore, we only include the necessary elements (and exclude describing notions like *stabilization of open books*).

Definition 7.2.2. *An open book decomposition of a 3-manifold M is a pair (B, π) such that B is an oriented link in M, referred to as the* binding *of the open book, and $\pi : M \backslash B \to \mathbb{S}^1$ is a fibration. For any $\tau \in \mathbb{S}^1$, $\pi^{-1}(\tau)$ is the interior of a compact surface Σ_τ with $\partial \Sigma_\tau = B$. We refer to the surfaces Σ_τ as the* pages *of the open book.*

Example 7.2.3. *1) Considering $\mathbb{S}^3$ as compactified $\mathbb{R}^3$, the z-axis can be thought of as the binding of an open book decomposition of $\mathbb{S}^3$, with pages being diffeomorphic to disks.*

2) Considering $\mathbb{S}^3 \subset \mathbb{C}^2$ as the unit sphere, the set $B := \{(z_1, z_2) \in \mathbb{S}^3 | z_1 z_2 = 0\}$ is the Hopf link and together with the projection $\pi : \mathbb{S}^3 \backslash B \to \mathbb{S}^1 : (z_1, z_2) \to \frac{z_1 z_2}{|z_1 z_2|}$ forms an open book decomposition of $\mathbb{S}^3$.

While Alexander has proved the existence of such structures on any 3-manifold [105], in proof of Theorem 7.2.1 Giroux showed that we can construct an open book decomposition *adapted* to a given contact manifold, in the following sense:

Definition 7.2.4. *We say the open book decomposition (B, π) on M is adapted to the contact structure ξ if there exists some Reeb vector field X, such that it is (positively) tangent to B and is (positively) transverse to the pages of π.*

We note that both open book decompositions in the above example can be isotoped to be adapted to the standard contact structure on $\mathbb{S}^3$.

7.3 Towards an almost global realization: Topological obstructions

In order to move towards the almost global realization of Theorem 7.0.8, we will discuss some (contact) topological obstructions for a global realization. We will then recall some results from metric geometry of the space of (compatible) Riemannian metrics in Section 7.4, mostly due to Brian Clarke [106, 107], which will prepare us for the proof for the almost realization theorem in Section 7.5.

First we note that forcing $Ricci(X_\alpha)$ to obtain its maximum everywhere restricts the contact topology significantly, since this is equivalent to $\mathcal{L}_{X_\alpha} J = 0$ everywhere. Putting the previous works of previous works of [101, 93, 91, 102] together, we have

Theorem 7.3.1. *[4] Let (M, ξ) be a closed contact 3-manifold and g a compatible Riemannian metric with instantaneous rotation θ', such that $Ricci(X_\alpha) = \frac{\theta'^2}{2}$ everywhere. Then (M, ξ) is finitely covered by a Boothby-Wang fibration with ξ being a tight symplectically fillable contact structure. Moreover, if all the periodic Reeb orbits associated with g are* non-degenerate*, i.e. their Poincare return map does not have 1 as eigenvlaue, then (M, ξ) is finitely covered by $(\mathbb{S}^3, \xi_{std})$.*

Proof. The implication follows from classification of K-contact structures by Rukimbira [93] (see Corollary 6.1.12). In fact, after an arbitrary small perturbation (M, ξ, g) can be approximated by arbitrary close *almost regular* K-contact structure. i.e. X_α induces a S^1 action as Killing vector field. It turns out [101] that this induces a Seifert fibration structure on (M, ξ), whose fibers are tangent to a vector field which keeps ξ invariant. This is called a *generalized Boothby-Wang fibration* and is finitely covered by a Boothby-Wang fibration. Furthermore, [108] shows that these contact structures are symplectically fillable and tight.

Moreover, since in this case X_α preserves the length of any vector $e \in \xi$, all periodic orbits will be *elliptic*. i.e. have (complex) Poincare return map with unit length eigenvalues. If furthermore, all periodic orbits are non-degenerate as well, [102] shows that (M, ξ) is either $(\mathbb{S}^3, \xi_{std})$ or a Lens space with ξ being universally tight. $\square$

Now it is also interesting to understand the extreme opposite of the above situation. That is when (M,ξ) admits a nowhere Reeb-invariant compatible metric. i.e. a metric for which $\mathcal{L}_{X_\alpha}g \neq 0$ everywhere. First, we easily observe that there are algebraic obstructions for the existence of such metrics, improving [98].

Theorem 7.3.2. *[4] Let (M,ξ) be any contact 3-manifold with $2e(\xi) \in H^2(M) \neq 0$. Then for any compatible metric g with instantaneous rotation θ', there exists some point $x \in M$ at which $Ricci(X_\alpha)(x) = \frac{\theta'^2}{2}$, where X_α is the Reeb vector field corresponding to g.*

Proof. The proof immediately follows from the fact that if we have $Ricci(X_\alpha) < \frac{\theta'}{2}$ everywhere (see Remark 6.1.11), there exists a (unique up to homotopy) line field $\langle e\rangle \subset \xi$ with $g(e,\nabla_e X_\alpha) > 0$, and therefore ξ admits a globally defined line field. By [9], this is equivalent to $2e(\xi) \in H^2(M) = 0$. $\square$

However, we still do not know whether this is the only obstruction or if there are others of contact topological nature. In fact, we can conjectured the following statement in support of the latter viewpoint, which can be seen to partly generalize Corollary 6.3.3.

Conjecture 7.3.3. *If ξ admits a Reeb vector field and a complex structure J, satisfying*

$$\mathcal{L}_{X_\alpha}J \neq 0$$

everywhere, or equivalently if (M,ξ) admits a compatible metric with instantaneous rotation θ' and

$$Ricci(X_\alpha) < \frac{\theta'^2}{2}$$

everywhere, then it is tight.

It is worth mentioning that using our computation, we can see that when it does admit such compatible metric, then we can make $Ricci(X_\alpha)$ arbitrary far from the upper bound, confirming the significance of the dichotomy discussed above.

Theorem 7.3.4. *[4] Assume (M,ξ) admits some compatible metric with instantaneous rotation θ' and $Ricci(X_\alpha) < \frac{\theta'^2}{2}$ everywhere, in particular if (M,ξ) is Anosov. Then for any $c \leq \frac{\theta'^2}{2}$, there exists some compatible metric with instantaneous rotation θ' and $Ricci(X_\alpha) < c$.*

Proof. Since (M,ξ) admits a metric with $Ricci(X_\alpha) < \frac{\theta'^2}{2}$, we have $2e(\xi) = 0 \in H^2(M)$. Then there exists a line sub bundle $\langle e\rangle \subset \xi$. Choose some contact from α and complex structure J. For some constant λ define a perturbation of complex structure $J_\lambda : \langle e\rangle \to J\langle e\rangle + \lambda\langle e\rangle$. Letting $\eta = 1$ and $X\lambda = 0$ in Lemma 7.1.1, we have

$$Ricci_\lambda(X_\alpha)(x) = -2\left(P_\lambda(x)\right)^2 + \frac{\theta'^2}{2} - 2\left(Q_\lambda(x)\right)^2$$

where

$$P_\lambda(x) = -\frac{1}{\theta'}d\alpha([e,X],Je) - \frac{1}{\theta'}\lambda d\alpha([e,X],e)$$

and

$$Q_\lambda(x) = \frac{1}{2\theta'}d\alpha([e,X],e) - \frac{1}{2\theta'}d\alpha([Je,X],Je) - \frac{\lambda}{\theta'}d\alpha([e,X],Je) - \dots$$

$$\dots - \frac{1}{2\theta'}\lambda^2 d\alpha([e,X],e)$$

So $Ricci_\lambda(X_\alpha)(x)$ is a non-constant (since we start with $Ricci(X_\alpha) < \frac{\theta'^2}{2}$) polynomial with even degree in terms of λ and function coefficients. At each point, we can choose λ such that we have $Ricci_\lambda(X_\alpha)(x) < c$ at that point. Since M is compact, we can choose such λ globally. □

Finally, we note that the existence of a nowhere-Reeb invariant metric can be concluded, under the dynamical assumption of Anosovity on (M,ξ). An Anosov contact manifold is a contact manifolds (M,ξ) admitting an Anosov Reeb vector field. i.e. some X_α and the

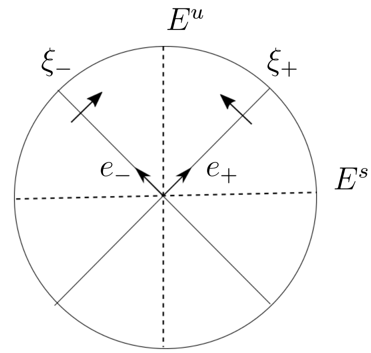


Figure 7.1: Anosovity of contact structures

continuous X_α-invariant splitting $\xi \simeq E^s \oplus E^u$, such that for any $u \in E^s$ and $v \in E^u$,

$$||\phi_*^t(v)||/||\phi_*^t(u)|| \geq Ae^{Ct}||v||/||u||;$$

where ϕ^t is the flow of X_α and $A, C > 0$ are positive constants.

As discussed in Chapter 2, the projective Anosovity of X_α is equivalent to $\langle X_\alpha \rangle = \xi_+ \cap \xi_-$, where ξ_+ and ξ_- are transverse positive and negative contact structures on M, which in this case implies Anosovity, since Reeb vector fields are volume preserving. Now if we (locally) consider sections $e_+ \in \xi \cap \xi_+$ and $e_- \in \xi \cap \xi_-$ such that (e_+, e_-) form an oriented basis for ξ, positivity of ξ_+ and negatively of ξ_- will imply $g([e_+, X_\alpha], e_-) > 0$ and $g([e_-, X_\alpha], e_+) > 0$, respectively. Therefore, the dynamics of X_α cannot be *purely rotational* (see Figure 7.1) and by discussion in Remark 6.1.11, we have

Theorem 7.3.5. *[4] Let (M, ξ) be a Anosov contact 3-manifold. Then ξ admits a Reeb vector field and a complex structure J, satisfying*

$$\mathcal{L}_{X_\alpha} J \neq 0$$

everywhere, or equivalently (M, ξ) admits a compatible metric with

$$Ricci(X_\alpha) < \frac{\theta'^2}{2}$$

everywhere.

7.4 Completion of the space of compatible metrics

In Section 7.3, we observed the contact topological subtlety of finding global solutions for the Ricci-Reeb realization problem and we can ask what is the best we can do to realize a function as $Ricci(X_\alpha)$. In order to establish almost global solutions to the Ricci-Reeb realization problem, we need some elements from the geometry of the space of Riemannian metrics on M, denoted by $\mathcal{M}$. Although the Riemannian geometry of $\mathcal{M}$, like geodesics, sectional curvature, etc. is studied in the classical literature, its metric geometry and in particular, its completion, was not understood well, until relatively recently, in the works of Brian Clarke [106, 107].

It can be seen that $\mathcal{M}$ admits a natural Riemannian metric, often called L^2-metric, denoted by $(.,.)$ and induced from its inclusion into S^2T^*M, the space of symmetric (0,2)-tensor fields on M. Let $g \in \mathcal{M}$ and $h, k \in T_g\mathcal{M}$:

$$(h,k) := \int_M trace(g^{-1}hg^{-1}k)dVol(g).$$

One can easily confirm that $(.,.)$ is a metric on $\mathcal{M}$ (note that the trace of a symmetric matrix is the sum of its eigenvalues and using a partition of unity, it suffices to define the trace in local coordinates). This is in fact the generalization of Weil-Peterson metric in Teichmuller theory. This inner product naturally defines a distance function d on $\mathcal{M}$, which satisfies the following interesting and useful property, letting us control the distance between two metrics by controlling the volume of the set they differ on.

Proposition 7.4.1. *Let $g_0, g_1 \in \mathcal{M}$ and $E := \{x \in M | g_0(x) = g_1(x)\}$. Then*

$$d(g_0, g_1) \leq C\left(\sqrt{Vol(E, g_1)} + \sqrt{Vol(E, g_0)}\right),$$

where C is a constant only depending on the dimension of M and $Vol(E, g_i)$ is the volume *of E measured by g_i for $i \in \{0, 1\}$.*

Brian Clark characterized the completion of $\mathcal{M}$ as follows. Let $\overline{\mathcal{M}}$ be such completion and $\mathcal{M}_f$ be the space of measurable, symmetric, finite volume semi-metrics on M.

Theorem 7.4.2. *[106, 107] Using the above notations, we have the natural identification*

$$\overline{\mathcal{M}} \simeq \mathcal{M}_f / \sim,$$

where for $g_0, g_1 \in \mathcal{M}_f$, we have $g_0 \sim g_1$ if and only if for almost any $x \in M$, $g_0(x) = g_1(x)$ when at least one of them is non-degenerate. Such identification can be improved to an isometry.

Moreover, in order to understand L^2-limit of metrics, we need to control how metrics *degenerate* on measurable subsets of M.

Definition 7.4.3. *Let $\tilde{g} \in \mathcal{M}_f$. We define*

$$X_{\tilde{g}} := \{x \in M | \tilde{g}(x) \textit{ is degenerate}\} \subset M,$$

which we call the deflated *set of $\tilde{g}$.*

Definition 7.4.4. *Let $\{g_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ be any sequence. We define the set*

$$D_{\{g_k\}_{k \in \mathbb{N}}} := \{x \in M | \forall \delta > 0, \exists k \in \mathbb{N} \textit{ s.t. } \det G_k(x) < \delta\},$$

where G_k is g-dual of g_k for some fixed $g \in \mathcal{M}$. We call $D_{\{g_k\}_{k \in \mathbb{N}}}$ the deflated *set of $\{g_k\}$. This definition does not depend on the choice of g.*

Although the conditions of convergence in the following theorem can be relaxed extensively, in order to avoid introducing further notions, we give the following theorem which suffices for our purpose (see [106] Definition 4.4, Theorem 4.3 and Theorem 5.19).

Theorem 7.4.5. *[106, 107] Using the above characterization of $\overline{\mathcal{M}}$, we have*

$$\{g_k\} \to [g_\infty],$$

if $\{g_k\}$ is d-Cauchy and

1) $\Sigma_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty$;

2) X_{g_∞} and $D_{\{g_k\}}$ differ at most by a null-set;

3) $g_k(x) \to g_\infty(x)$ for almost every $x \in M \backslash D_{\{g_k\}}$.

7.5 The proof of almost global realization theorem

Theorem 7.5.1. *[4] Let (M, ξ) be a closed oriented contact 3-manifold, $\frac{\theta'^2}{2} \geq f(x) : M \to \mathbb{R}$ a function on M and V a positive real number. Then there exists a singular metric g_∞ with instantaneous rotation θ' and an embedded compact surface with boundary $F \subset M$ such that*

1) g_∞ is a compatible metric on $M \backslash F$,

2) $Ricci(X_\alpha)(x) = f(x)$ on $M \backslash F$, where X_α is the Reeb vector field associated with g_∞,

3) $Vol(g_\infty) = V$,

4) g_∞ can be realized as an element of the completion of the space of compatible Riemannian metrics $\overline{\mathcal{M}_\xi} \subset \overline{\mathcal{M}}$. More precisely, given any $\epsilon > 0$, $[g_\infty]$ is the limit of a L^2-Cauchy sequence of compatible metrics $\{g_n\} \to [g_\infty] \in \overline{\mathcal{M}_\xi} \subset \overline{\mathcal{M}} \simeq \mathcal{M}_f / \sim$, such that g_n realizes the given function as $Ricci(X_\alpha)$, outside a $\frac{\epsilon}{2^n}$-neighborhood of F.

Proof. Given $f : M \to \mathbb{R}$, fix the real number $\theta' > 0$ such that $f(x) \leq \frac{\theta'^2}{2}$. Let (B, π) be an open book decomposition adapted to ξ and α a contact form for ξ satisfying the condition of Definition 7.2.4. After multiplying α by a constant, we can assume $Vol(g) = \frac{1}{\theta'} \alpha \wedge d\alpha = V$.

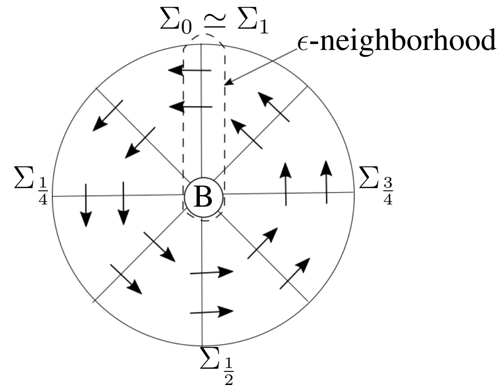


Figure 7.2: Using open book decomposition and the flow of X_α to establish almost global realization

Choose an arbitrary complex structure J on ξ, inducing the compatible metric g. Parametrizing $\mathbb{S}^1 \simeq [0,1]/0 \sim 1$, consider $J|_{\Sigma_0 \backslash B}$ to be initial condition for the PDE described in Lemma 7.1.1 and since the interior of pages of (B,π) are transverse to X_α, we can solve such PDE (as in local realization theorem) and extend the solution of realization problem over $\Sigma_\tau \backslash B$ for $0 < \tau < 1$, i.e. $M \backslash \Sigma_0$. The achieved complex structure $J_{(\lambda,\eta)}$ on $M \backslash \Sigma_0$ yields a singular (measurable) compatible metric g_∞, satisfying 1)-3) with $F := \Sigma_0$ being the singular set. Also note that the volume form of g_∞ is the same as g, since F is measure zero. See Figure 7.2.

Now we can realize the measurable semi-metric g_∞ as the limit of a L^2-Cauchy sequence of compatible metric, using Theorem 7.4.2 and Theorem 7.4.5 in the following way. For any fixed $\epsilon > 0$, choose small enough $\delta > 0$, such that

$$Vol(E := \bigcup_{1-\delta \leq \tau \leq 1} (\Sigma_\tau \backslash B), g) < \frac{\epsilon}{2}.$$

Now since X_α induces a product structure on E, we can use a smooth interpolation function $h_\delta : [0,1] \to [0,1]$ with $h_\delta(\tau) = 0$ for $0 \leq \tau \leq 1 - \delta$ and $h_\delta(1) = 1$ and see that the the complex structure

$$\tilde{J}_\epsilon|_{\Sigma_\tau} := (1 - h_\delta(\tau))J_{(\lambda,\eta)} + h_\delta(\tau)J$$

for $\tau \in \mathbb{S}^1$ can be extended over $\Sigma_0 \backslash B \simeq \Sigma_1 \backslash B$, yielding a singular compatible metric, which is singular on B and has $Ricci(X_\alpha) = f(x)$ outside of a $\frac{\epsilon}{2}$-neighborhood of Σ_0. Similarly, with a smooth radial interpolation between $\tilde{J}_\epsilon$ and J in a product $\frac{\epsilon}{2}$-neighborhood of B, we can define J_ϵ and consequently the compatible g_ϵ on all of M, such that $Ricci(X_\alpha) = f(x)$ outside of a ϵ-neighborhood of Σ_0. We claim that repeating this procedure for $\epsilon_n := \frac{\epsilon}{2^n}$ gives the sequence described in 4).

First, notice that for all the metrics above, we only perturbed the complex structure, leaving the volume form unchanged. Therefore by Proposition 7.4.1,

$$d(g_{\epsilon_n}, g_{\epsilon_m}) \leq 2C\sqrt{\frac{\epsilon}{2^{min\{m,n\}}}}$$

and g_{ϵ_n} is a Cauchy sequence and moreover satisfies condition 1) of Theorem 7.4.5. Now, note that for any $x \in M \backslash \Sigma_0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $g_{\epsilon_n} = g_\infty$ and hence, we have condition 3) of Theorem 7.4.5. That also means that $D_{\{g_{\epsilon_n}\}}$ is included in the measure zero set $X_{g_\infty} = F = \Sigma_0$. Therefore by Theorem 7.4.5, $\{g_{\epsilon_n}\}_{n \in \mathbb{N}}$ L^2-converges to $[g_\infty] \in \overline{\mathcal{M}_\xi}$. $\square$

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