LEGENDRIAN LINKS AND LAGRANGIAN COBORDISMS

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Thomas Rodewald

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Thesis committee:

Dr. John Etnyre
School of Mathematics
Georgia Institute of Technology

Dr. Thang Le
School of Mathematics
Georgia Institute of Technology

Dr. Jennifer Hom
School of Mathematics
Georgia Institute of Technology

Dr. Bulent Tosun
Department of Mathematics
University of Alabama

Dr. Michael Wolf
School of Mathematics
Georgia Institute of Technology

Date approved: March 20, 2026

Oh, come on, man. BELIEVE in something!

Kevin Malone

For my parents Lance and Patricia Rodewald

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SUMMARY

This thesis considers two fundamental objects in contact and symplectic topology: Legendrian links and exact Lagrangian cobordisms. First we study Legendrian realizations of cable links of knot types that are uniformly thick but not Legendrian simple, extending prior work of Dalton, Etnyre, and Traynor. This leads to new phenomena, such as stabilized Legendrian links that are smoothly isotopic and component-wise Legendrian isotopic, but are not Legendrian isotopic. After showing results for three different cases of slopes of cable links, we then consider exact Lagrangian cobordisms between Legendrian knots - in particular the map they induce on the Chekanov-Eliashberg DGAs of their Legendrian ends as defined by Ekholm, Honda, and Kalman. Specifically, we show how to adapt this map to linearizations of the DGA using augmentations. We then show its induced map on linearized Legendrian contact homology and its induced map on higher order product structures are invariant under Lagrangian isotopy under mild hypotheses.

CHAPTER 1

INTRODUCTION

Contact manifolds are smooth, odd dimensional manifolds equipped with a maximally non-integrable hyperplane field. In dimension 3 this means that there is no embedded surface in the manifold whose tangent bundle is a subbundle of the plane field. A plane field can be locally given as the kernel of a 1-form α, and it is a contact structure if it satisfies the *contact condition* $\alpha \wedge d\alpha > 0$. The planes are spinning in a sense, and locally look like the standard contact structure on $\mathbb{R}^3$ (see Fig. 2.1), denoted $(\mathbb{R}^3, ker(dz - ydx))$. A 3-manifold can be equipped with multiple contact structures, and essential to understanding these structures are Legendrian knots. In particular, any contact 3-manifold can be obtained by contact surgery on a Legendrian link in S^3 with the standard tight contact structure [1]. We may also create a symplectic manifold out of a contact manifold via its *symplectization*. Here we take the cross product of the manifold with $\mathbb{R}$, and take the exterior derivative of the product of e^t with α, where t is the variable in $\mathbb{R}$. In particular, $(\mathbb{R} \times \mathbb{R}^3, d(e^t(dz - ydx)))$ is the symplectization of the standard contact structure on $\mathbb{R}^3$. One may then consider Lagrangian cobordisms, which are surfaces on which the symplectic form vanishes, with Legendrian knots as bounds in the symplectization. This thesis is about better understanding Legendrian knots and Lagrangian cobordisms between them.

1.1 Classifying Legendrian cables

Extensive research has been done on Legendrian knots [2, 3, 4, 5, 6, 7, 8]. There has been less research done on Legendrian links. The first classification results for Legendrian links are of two-component links consisting of an unknot and a cable of the unknot that were Legendrian simple, i.e. determined by their classical invariants, due to Ding and Geiges [9]. Since then Dalton, Etnyre, and Traynor classified cable links of knots that are both

Legendrian simple and satisfy the technical condition of uniform thickness. In joint work with Chatterjee, Etnyre, and Min, we expanded these results by removing the Legendrian simple condition. In particular we classified cable links in the positive slope case.

Theorem 1.1.1. *Let K be in the uniformly thick knot type. Let $\frac{q}{p} > \lceil(w(K)\rceil$. The classification of Legendrian representatives of cable links of K of slope $\frac{q}{p}$ is completely determined by the classification of Legendrian representatives of K.*

For a specific statement of the classification result and its proof, see Theorem 3.1.3. The study of cable links of non Legendrian simple knots allows one to consider links with components that have the same classical invariants, but are not Legendrian isotopic to each other. We address this in the following theorem.

Theorem 1.1.2. *The components of a maximum Thurston-Bennequin Legendrian realization of a (np, nq)-cable of a uniform thick knot type K must all be Legendrian isotopic. If $q/p \geq \lceil w(K)\rceil$, then the same result is true without the assumption of uniform thickness of K.*

More generally, for a uniformly thick knot type K, any two components of a Legendrian representative of the (np, nq)-cable of K with the same classical invariants are Legendrian isotopic.

Uniform thickness and $w(K)$ will be defined in Section 2.2. Theorem 1.1.2 follows from Theorem 3.1.3, Theorem 3.2.2, and Theorem 3.3.4. Another corollary of our results established the following new phenomena.

Theorem 1.1.3. *There exists pairs of Legendrian links L and L' that are smoothly isotopic, the components of L and L' have been stabilized many times, and each component of L is Legendrian isotopic to a component of L', but L and L' are not Legendrian isotopic.*

Remark 1.1.4. This has been shown in the case that L and L' are max-tb by Jordan and Traynor in [10]. So the components of L and L' being stabilized are essential to our phenomena being new.

We state further results regarding cables links of negative slope in two cases: integer slope in Section 3.2 and non-integer slope in Section 3.3.

1.2 The cobordism map

Lagrangian cobordisms between Legendrian knots are a central area of study in contact and symplectic geometry and have been intensely studied in the past couple of decades. There are many examples of constructions of such cobordisms [11, 12, 13, 14, 15], as well as obstructions to their existence [16, 12, 17, 18]. In joint work with Knavel, we study Lagrangian cobordisms and, in particular, the Differential Graded Algebra (DGA) map they induce on linearized Legendrian contact homology. The existence of such a map was shown in [19], and a similar version in the language of generating family homology was also shown in [20]. Naturally one might ask how isotoping an exact Lagrangian cobordism through Lagrangians effects this map. After explicitly defining the DGA map on the chain level, we show under mild hypotheses that its induced map on the homology is invariant under Lagrangian isotopy.

To establish this invariance, consider the Chekanov-Eliashberg DGA. Chekanov [21] and Eliashberg [22] independently define a DGA $(\mathcal{A}_\Lambda, \partial)$ associated to a Legendrian Λ in $(\mathbb{R}, \xi_{std})$ to distuinguish Legendrian knots which cannot be distinguished by their classical invariants tb and r [21]. Since this result, the DGA has been an instrumental tool in the study of Legendrian knots [23, 24, 25, 26, 27]. Because the DGA, and by extension its homology, is a non-commutative ring, its computation is difficult. However, in some cases one can distinguish pairs of Legendrian knots by linearizing the DGA using augmentations.

Chekanov [21] defines an augmentation as a chain map

$$\epsilon : (\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}/2\mathbb{Z}, 0)$$

in order to modify the differential of the DGA to a new differential called ∂_ϵ. One can then 'linearize' ∂_ϵ to obtain a differential ∂_ϵ^ℓ on a finite-dimensional vector space F_Λ. The homology of the chain complex yielded by this differential is the *linearized contact homology* of Λ. For a given Legendrian Λ, if an augmentation exists, the linearized contact homology is easier to compute and the set of all linearized contact homologies is invariant up to Legendrian isotopy. The base ring $\mathbb{Z}/2\mathbb{Z}$ has since been generalized to a unital ring [24]; in this thesis we use $\mathbb{Z}/2\mathbb{Z}$.

Let L be an exact Lagrangian cobordism from a Legendrian knot Λ_1 to a Legendrian Λ_2. Eckholm, Honda, and Kalman [13] define the following chain map between their corresponding DGAs.

$$\Phi_L : (\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2}) \to (\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1})$$

Once again, the non-commutativity of this map makes computations difficult, and therefore we aim to linearize Φ_L, show that it induces a homomorphism on the linearized contact homology, and apply it to examples through direct computation. That is, given an augmentation ϵ_1 for Λ_1, we consider the augmentation $\epsilon_1 \circ \Phi_L$ of Λ_2 and then define a map between the complexes F_{Λ_i} (see Section 4.1),

$$(\Phi_L^{\epsilon_1})^\ell : F_{\Lambda_2} \to F_{\Lambda_1}.$$

We show that this gives a chain map, thus inducing a homomorphism on the linearized contact homology.

Proposition 1.2.1. *Let L be an exact Lagrangian cobordism from Λ_1 to Λ_2, and ϵ_1 an augmentation of $(\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1})$ and $\epsilon_2 = \epsilon_1 \circ \Phi_L$. Then*

$$(\Phi_L^{\epsilon_1})^\ell \circ \partial_{\epsilon_2}^\ell = \partial_{\epsilon_1}^\ell \circ (\Phi_L^{\epsilon_1})^\ell.$$

We remind the reader the existence of this map was shown by Pan in [19], and in a different language in [20]. In order to use $(\Phi_L^{\epsilon_1})^\ell$ to study Lagrangian cobordisms, we need to prove some basic facts about the map and its behavior under geometric phenomena, such as Lagrangian isotopy. Essentially, we have a linearized version of [13, Theorems 1.3 part 1) and 1.4].

Theorem 1.2.2. *The DGA map induced by the exact Lagrangian cobordism L from Λ_1 to Λ_2,*

$$(\Phi_L^{\epsilon_1})^\ell : (F_2, \partial^\ell_{\epsilon_2}) \to (F_1, \partial^\ell_{\epsilon_1}),$$

has the following properties:

1) If $L = \Lambda \times \mathbb{R}$, then $(\Phi_L^{\epsilon_1})^\ell = Id_{\mathcal{A}_\Lambda}$.

2) Let L_1 and L_2 be exact Lagrangian cobordisms from Λ_1 and Λ_2 that are isotopic through exact Lagrangian cobordisms. If $\epsilon_1 \circ \Phi_{L_1} = \epsilon_1 \circ \Phi_{L_2}$ and the image of ∂_{ϵ_2} has no quadratic terms with Reeb chords of grading -1, then $(\Phi_{L_1}^{\epsilon_1})^\ell$ and $(\Phi_{L_2}^{\epsilon_1})^\ell$ are chain homotopic.

Remark 1.2.3. It is natural to think that given that the map Φ_L changes by a chain homotopy equivalence when L undergoes an exact Lagrangian isotopy, then the same would be true for the linearized maps, but the above theorem shows that extra information is needed to do this. In particular, we need extra hypotheses on the DGAs involved to guarantee that we can define maps on their linearizations that are invariant under exact Lagrangian isotopy (see Remark 4.1.5). We do not know if these hypotheses are necessary, but it is not clear how one can establish the desired properties of $(\Phi_L^{\epsilon_1})^\ell$ without them.

Given an augmentation ϵ of the DGA for a Legendrian knot Λ, one may define a product structure, in fact an A_∞ structure, on the dualized linear contact homology [28]. It was shown in [28] that the product structure and the entire A_∞ structure is a stronger invariant of a Legendrian knot than just the linearization alone. Pan showed the existence of an A_∞ morphism between the A_∞ structures on their Legendrian ends in [19], and we replicate this result using our argument on the chain level.

Theorem 1.2.4. *Let* $(F_1^*, \{m_k^{\epsilon_1}\}_{k\geq 1})$ *and* $(F_2^*, \{m_k^{\epsilon_2}\}_{k\geq 1})$ *be the* A_∞ *structures on the linearized Legendrian cochain groups. Then the collection of maps* $\{((\Phi_L^{\epsilon_1})^k)^* : (F_1^*)^{\otimes k} \to F_2^*\}_{k\geq 1}$ *is an* A_∞ *morphism.*

An implication of Theorem 1.2.4 is that $((\Phi_L^{\epsilon_1})^\ell)^*$ (which is $((\Phi_L^{\epsilon_1})^1)^*$ in the notation in Theorem 1.2.4 above) preserves the higher order product structures described in Section 2.4.3. As a corollary to Theorem 1.2.2 and Theorem 1.2.4, under certain conditions, we have invariance of our linearized augmented DGA maps respecting these structures under Lagrangian cobordism.

Corollary 1.2.5. *Let* L_1 *be Lagrangian isotopic to* L_2, ϵ_1 *an augmentation for* Λ_1, $\epsilon_1 \circ \Phi_{L_1} = \epsilon_1 \circ \Phi_{L_2}$, *and the quadratics in the image of* ∂_{ϵ_2} *have no degree* -1 *Reeb chords. Then the induced maps of* $((\Phi_{L_1}^{\epsilon_1})^\ell)^*$ *and* $((\Phi_{L_2}^{\epsilon_1})^\ell)^*$ *are equal on the higher order product structures.*

CHAPTER 2

BACKGROUND ON CONTACT AND SYMPLECTIC TOPOLOGY

This chapter covers the relevant definitions and theorems for the results of this thesis (Chapter 3 and Chapter 4). We begin with the basic definitions in contact topology and Legendrian knots in Section 2.1. Next we define convex surfaces, which have proved essential to classifying Legendrian knots, and the relevant theorems about them in Section 2.2, Finally, after briefly defining symplectic manifolds and exact Lagrangian cobordisms in Section 2.3, we cover the Chekanov-Eliashberg DGA and its related material in Section 2.4

2.1 Contact Structures

Let M be a closed, orientable 3-manifold. A *contact structure* ξ is the kernel of a 1-form $\alpha \in \Omega^1(M)$ that satisfies the contact condition: $\alpha \wedge d\alpha \neq 0$. Thus $\xi = \coprod_{p \in M} \xi_p \subseteq TM$, where each $\xi_p = \ker(\alpha_p)$ is a plane. The *standard contact structure* for $\mathbb{R}^3$, denoted $(\mathbb{R}^3, \xi_{std})$ is the kernel of the 1-form $dz - ydx$. See Fig. 2.1 below.

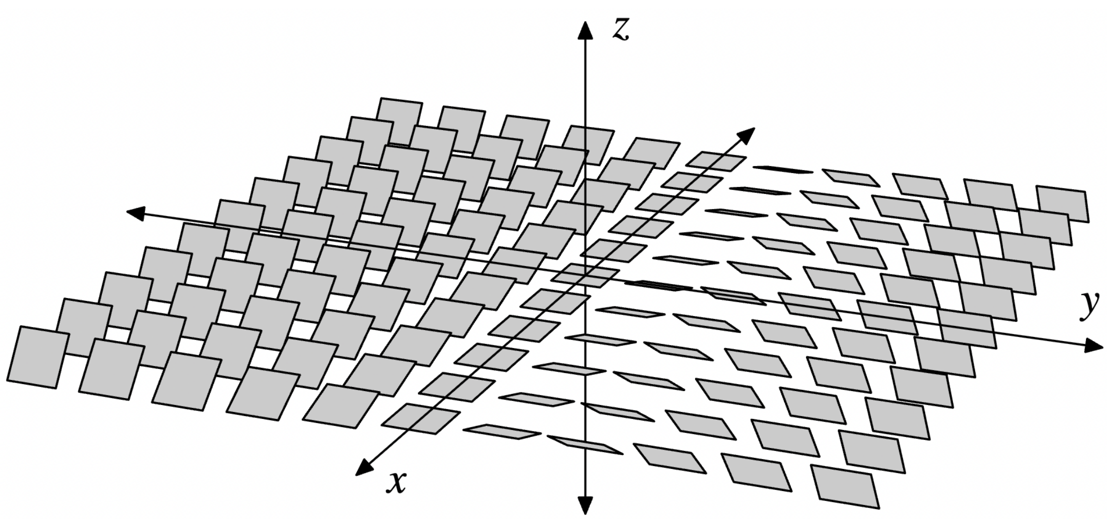


Figure 2.1: The standard contact structure on $\mathbb{R}^3$

2.1.1 Legendrian Knots

A knot Λ is *Legendrian* if it is everywhere tangent to the contact planes, i.e. $T_p\Lambda \subseteq \xi_p$ for all $p \in \Lambda$.

Projections

There are two important projections for Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ used in contact topology: the *front projection*

$$\Pi_F : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, z),$$

and the *Lagrangian projection*

$$\Pi_L : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y).$$

See Fig. 2.2.

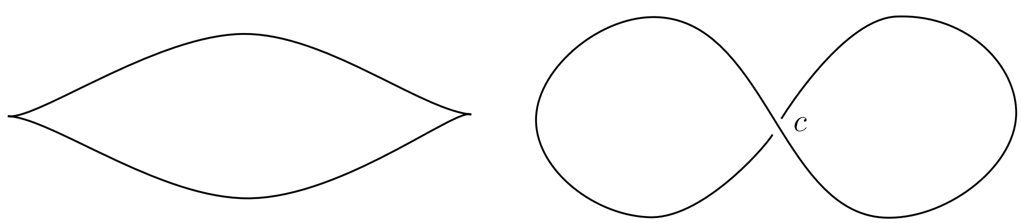


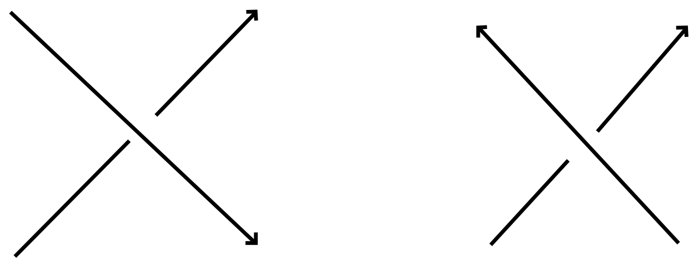
Figure 2.2: The front projection of the unknot, left, and its Lagrangian projection, right, with its Reeb chord labeled c.

The Lagrangian projection results in double points, called *Reeb chords*, which are integral flow lines of the *Reeb vector field* $\frac{\partial}{\partial z}$ that start and end on the Legendrian.

Classical Invariants

An oriented Legendrian knot Λ has two invariants, referred to as the *classical invariants*. They are the *Thurston-Bennequin number*, denoted $tb(\Lambda)$, which measures the twisting of the contact planes along Λ, and the *rotation number*, denoted $r(\Lambda)$.

In $(\mathbb{R}^3, \xi_{std})$, one can easily compute the classical invariants of a given Legendrian Λ from its front projection. Let $Wr(\Lambda)$ denote the *writhe* of the knot, which is a signed count of the crossings in its projection (see Fig. 2.3), and $N(\Lambda)$ denote the number of left cusps resulting from the front projection.



$+1$ $\qquad\qquad$ -1

Figure 2.3: The writhe of a knot adds one for each crossing of the form on the left, and subtracts one for each crossing of the form on the right

After orienting the knot, let $D(\Lambda)$ and $U(\Lambda)$ denote the number of down cusps and up cusps, respectively. Then the formulas are

$$tb(\Lambda) = Wr(\Lambda) - N(\Lambda),$$

and

$$r(\Lambda) = \frac{1}{2}(D(\Lambda) - U(\Lambda)).$$

In the standard contact structure on $\mathbb{R}^3$ any Legendrian knot satisfies the Bennequin bound

$$tb(\Lambda) + |r(\Lambda)| \leq \chi(\Sigma)$$

for any surface Σ with $\partial\Sigma = \Lambda$.

There are two well-defined local operations one can perform on a Legendrian knot Λ. They are both performed by isotoping Λ past a stabilizing disk. They have a convenient description in the front projection. A *positive stablization* of Λ, denoted $S_+(\Lambda)$, is obtained by adding a cusp that is specifically a down cusp. A *negative stabilization*, $S_-(\Lambda)$, is obtained by adding a cusp that is specifically an up cusp. See Fig. 2.4

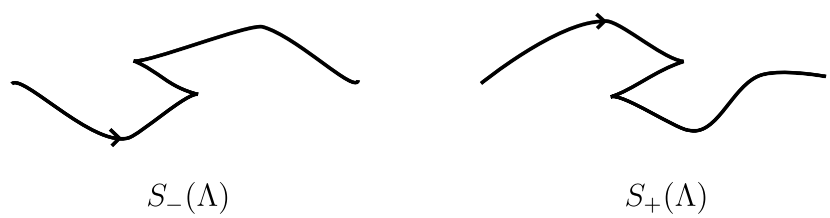


Figure 2.4: The front projections of a negative stabilization, left, and a positive stabilization, right.

Since this is a local operation and no other part of Λ is changed, this yields the following formulas.

$$tb(S_\pm(\Lambda)) = tb(\Lambda) - 1,$$

$$r(S_+(\Lambda)) = r(\Lambda) + 1,$$

and

$$r(S_-(\Lambda)) = r(\Lambda) - 1.$$

Since stabilization of Λ reduces $tb(\Lambda)$ by 1, a given knot type K will have Legendrian respresentatives that yield the maximum tb possible in that knot type. This is referred to as *max-tb*, and denoted $\overline{tb}(K)$.

Classification of Legendrians

A major line of research in contact topology is classification of Legendrian knots by their classical invariants. This is visualized by a knot type's *mountain range*, which is a map

$$M : \mathcal{L}(K) \to \mathbb{Z} \times \mathbb{Z}$$

defined by $M(\Lambda) = (tb(\Lambda), r(\Lambda))$, where $\mathcal{L}(K)$ is the set of Legendrian isotopy classes of Legendrian representatives of K. See Fig. 2.5. In the figure, the knot K_{-5} is a twist knot that is defined in Fig. 3.1, $(K_{-5})_{(2,1)}$ is the cable defined in the next subsection, and the diamond is defined in Chapter 3.

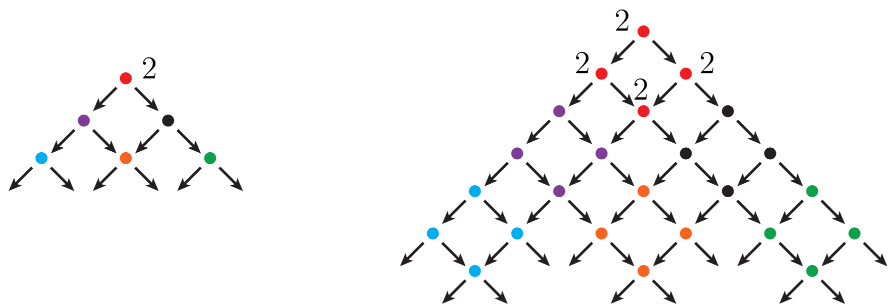


Figure 2.5: The mountain range for K_{-5} on the left. There are two Legendrian representatives when $(\mathrm{rot}, \mathrm{tb}) = (0, -3)$ and one for all other points in the diagram. On the right is the mountain range for $(K_{-5})_{(2,1)}$. The peak of the mountain range is $(0, -5)$. The colors of the dots indicate the diamonds of the Legendrian representatives of K_{-5}.

2.1.2 Cable Links

Let K be an oriented knot type. Let p, $q \in \mathbb{Z}$ be coprime. Then for $n \geq 1$, the *(np,nq)-cable of* K, denoted $K_{(np,nq)}$ or $K_{n(p,q)}$, is the n-component link obtained by taking n (p, q)-curves on the boundary of a tubular neighborhood of a representative of K. By a (p, q)-curve we mean a curve that runs p times longitudinally with respect to the Seifert longitude, and q times meridionally.

2.2 Convex Surfaces

The most powerful tool at our disposal in classifying Legendrian knots has thus far proven to be convex surface theory. A vector field is a *contact vector field* if its flow preserves the contact structure. A surface Σ is *convex* if it is everywhere transverse to a contact vector field. This definition is equivalent to there being an embedding $\phi : \Sigma \times \mathbb{R} \to M$ such that $\phi(\Sigma \times \{0\}) = \Sigma$ and $\phi_*^{-1}(\xi)$ is vertically invariant (in the $\mathbb{R}$ direction).

Given an embedded, oriented surface Σ in a contact manifold, one may consider the intersection its tangent space with the contact planes. Given $x \in \Sigma$, let $l_x = T_x\Sigma \cap \xi_x$. Then l_x is either a line in $T_x\Sigma$ or it is a singular point if $T_x\Sigma = \xi_x$. The *characteristic foliation*, denoted Σ_ξ, is the foliation of the complement of the singularities in Σ by 1-dimensional leaves such that at each point x in a leaf, the leaf is tangent to l_x.

If Σ is convex, then it has a multi-curve Γ, called the *dividing set*, with the following features.

- $\Sigma \backslash \Gamma = \Sigma_+ \coprod \Sigma_-$.
- Γ is transverse to Σ_ξ.
- There is a volume form ω on Σ and a vector field w on Σ such that $\pm L_w \omega > 0$ on $\Sigma_\pm$, w directs Σ_ξ, and w points transversely out of Σ_+ along Γ.

Given a Legendrian curve γ on a surface Σ, we define the *twist of γ relative to Σ*, denoted $tw_\Sigma(\gamma)$, to be the twisting of the contact planes along γ measured with respect to the framing induced by Σ. We have the following theorem due to Kanda which is useful in computing the tb of Legendrian knots [29].

Theorem 2.2.1. *If γ is a Legendrian curve in a surface Σ, then Σ may be isotoped relative to γ to make it convex if and only if $tw_\Sigma(\gamma) \leq 0$. Moreover, if Σ is convex, then*

$$tw_\Sigma(\gamma) = -\frac{1}{2}|\gamma \cap \Gamma|,$$

where Γ is the dividing set of Σ, and $|a \cap b|$ is the intersection number of a and b.

Given a convex torus bounding a solid torus neighborhood of a Legendrian knot, the dividing set will be a set of closed curves on the torus of the same slope. We define the *contact width*, denoted $w(K)$, of a knot type K to be the supremum of the dividing slopes of all convex tori bounding solid tori realizing K.

2.2.1 Giroux flexibility

Let v be a contact vector field and Σ a convex surface that is everywhere transverse to v with dividing set Γ. We say an isotopy $F : \Sigma \times [0,1] \to M$ is *admissible* if $F(\Sigma \times \{t\})$ is transverse to v for all $t \in [0,1]$. We have the following result, referred to as *Giroux flexibility* due to Giroux [30].

Theorem 2.2.2. *Let a convex surface Σ be closed or have Legendrian boundary. Let Γ be its dividing set, and $\mathcal{F}$ another singular foliation on Σ divided by Γ. Then there is an admissible isotopy F of Σ such that $F(\Sigma \times \{0\}) = \Sigma$, $F(\Sigma \times \{1\})_\xi = \mathcal{F}$, and the isotopy is fixed on Γ.*

A useful formulation of Giroux flexibility is the following due to Kanda [29], referred to as the *Legendrian realization principle*.

Theorem 2.2.3. *Let C be a closed curve on a convex surface or a surface with with Legendrian boundary Σ. Assume C is transverse to its dividing set Γ_Σ and every component of $\Sigma \backslash C$ nontrivially intersects Σ. Then there exists an admissible isotopy F_t such that*

- $F_0 = id$
- *$F_t(\Sigma)$ is convex for all $t \in [0,1]$.*
- $F_1(\Gamma_\Sigma) = \Gamma_{F_1(\Sigma)}$
- *$F_1(C)$ is Legendrian.*

2.2.2 Standard neighborhoods

Standard neighborhoods of the Legendrian knots we are cabling are essential to our results. We begin with a local model description. Let N_0 be a neighborhood of the x-axis, denoted Λ_0 in $\mathbb{R}^3/\sim$, where $(x, y, z) \sim (x + 1, y, z)$ and $\mathbb{R}^3$ has the standard contact structure $\xi_{std} = ker(dz - ydx)$. We can assume ∂N_0 is convex and has two dividing curves of slope 0. By Giroux flexibility [30], we can further assume its characteristic foliation is in *standard form*, meaning it has curves of singularities parallel to the dividing curves and the rest of the foliation consists of linear curves of any slope different than that of the dividing curves.

Now given a Legendrian knot Λ, by the Legendrian Neighborhood Theorem there is a *standard neighborhood* N of Λ and a contactomorphism from N_0 to N that sends Λ_0 to Λ and the two dividing curves of slope 0 on ∂N_0 to two dividing curves of slope $tb(\Lambda)$ on N.

We call K *uniformly thick* if any solid torus in the knot type K is contained in a solid torus that is a standard neighborhood of a maximal-tb representative of K. Consequently, the contact width of K is its max-tb: $w(K) = \overline{tb}(K)$.

2.2.3 Bypasses

Let Σ be a convex surface in a contact manifold (M, ξ). A *bypass* for Σ is an embedded disk D such that $\partial D = \alpha_1 \cup \alpha_2$ is Legendrian, ξ is tangent to D at $\alpha_1 \cup \alpha_2$, $D \cap \Sigma = \alpha_1$, α_1 intersects the dividing curves of Σ at three points $\{x_1, x_2, x_3\}$, where x_2 is on the interior of α_1 and $\partial\alpha_2 = \{x_1, x_3\}$, and ξ is not tangent to D on the interior of α_2. Let $\Sigma \times [0, 1]$ be a small $[0, 1]$ invariant neighborhood of Σ such that $\Sigma = \Sigma \times \{0\}$ and part of D intersects the neighborhood. We can push this neighborhood past D so that $\Sigma \times \{1\}$ is convex and the dividing curves differ from that of Σ in a prescribed way. We call this a *bypass attachment* of Σ. In the case of attaching a bypass to a convex torus T, one of 3 things can happen.

1) The number of dividing curves increases by 2.

2) The number of dividing curves decreases by 2.

3) The slope of the dividing curves changes (only in the case that T has exactly 2 dividing curves).

2.2.4 Farey graph

We will use the Farey graph to keep track of curves on a torus. We will use the conventions for the Farey graph shown in Fig. 2.6. Each vertex is labeled with a rational number a/b, and this corresponds to a curve on a torus in the homology class $b\lambda + a\mu$ where λ, μ is a basis for the homology of the torus.

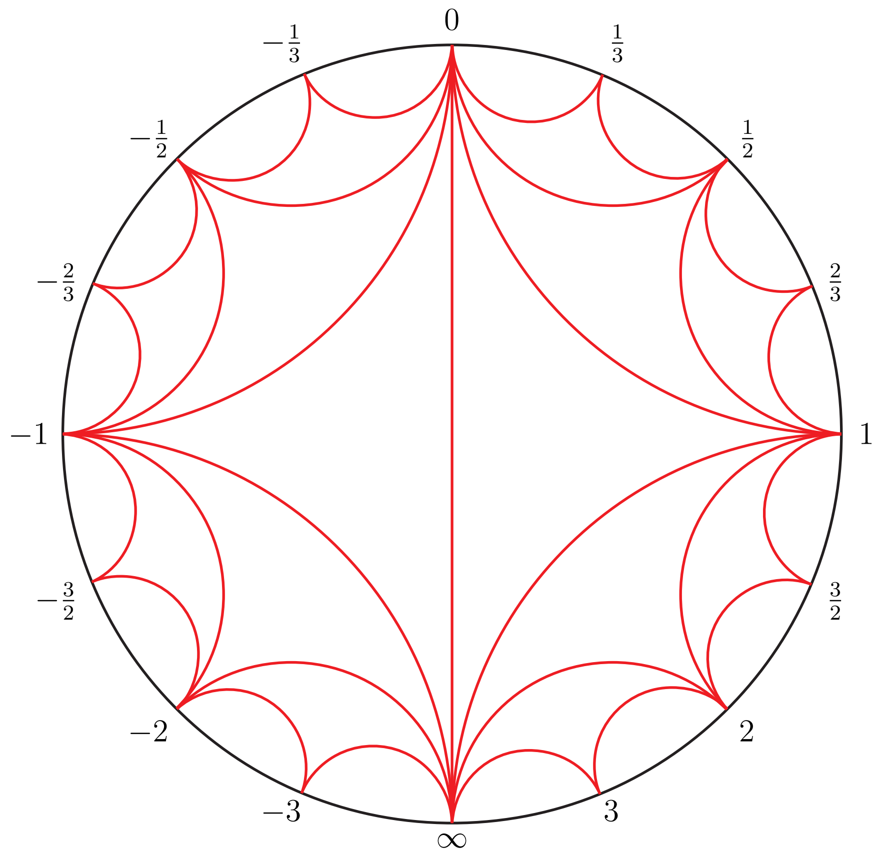


Figure 2.6: The Farey graph

2.2.5 Classification of contact structures $T^2 \times [0, 1]$

Let ξ be a contact structure on $T^2 \times [0, 1]$ with convex boundary such that $T^2 \times \{0\}$ has 2 dividing curves of slope s_0 and $T^2 \times \{1\}$ has 2 dividing curves of slope s_1. We say ξ

is *minimally twisting* if any convex torus in $T^2 \times [0,1]$ parallel to $T^2 \times \{0\}$ has dividing curves of slope in $[s_0, s_1]$. The set of minimally twisting contact structures on $T^2 \times [0,1]$ with the above boundary conditions, denoted $Tight^{min}(T^2 \times [0,1]; s_0, s_1)$, was classified by Giroux in [31] and Honda in [32]. They established the following.

Theorem 2.2.4. *Each decorated minimal path in the Farey graph from s_0 clockwise to s_1 describes an element of $Tight^{min}(T^2 \times [0,1]; s_0, s_1)$. Two such decorated paths describe the same contact structure if and only if the decorations differ by shuffling in the continued fraction blocks.*

By a *decorated* path we mean an assignment of a $+$ or $-$ to each edge of the path. A continued fraction block is a path in the Farey graph that, after changing basis, is the path $-1, -2, \ldots, -n$, and two decorated paths differ by "shuffling in the continued fraction blocks" if for each continued fraction block in the path the decorated paths have the same number of $+$ signs and the same number of $-$ signs. In the case that there is an edge from s_0 to s_1 in the Farey graph, $Tight^{min}(T^2 \times [0,1]; s_0, s_1)$ has two tight contact structures. These are called *basic slices*, and the correspondence in the theorem can be thought of as stacking basic slices according to the decoration of the path that describes the contact structure. The two different contact structures on a basic slice can be distinguished by their relative Euler class, and we call them *positive* and *negative* basic slices.

2.3 Symplectic Manifolds

A *symplectic manifold* is an even dimensional manifold M^{2n} equipped with a *symplectic form*, i.e. a closed, non-degenerate 2-form. Here non-degenerate means for all $p \in M$, if given some $X \in T_pM$ we have $\omega(X,Y) = 0$ for all $Y \in T_pM$, then $X = 0$. We denote the symplectic manifold (M, ω). A submanifold $L \subseteq M$ of dimension n is *Lagrangian* if the symplectic form vanishes on its tangent space, i.e. $\omega|_{TL} = 0$

Let (M, ξ) be a contact manifold, with $\xi = ker(\alpha)$. Then we call $(\mathbb{R} \times M, d(e^t\alpha))$ its *symplectization*. In this thesis we are interested in the symplectization of $(\mathbb{R}^3, \xi_{std})$.

Our specific objects of interest are exact Lagrangian cobordisms. We say L is an *exact Lagrangian cobordism* from Λ_1 to Λ_2 if it is an embedded, orientable Lagrangian surface in $(\mathbb{R}\times\mathbb{R}^3, d\left(e^t\alpha\right))$ such that for some $T\in\mathbb{R}^+$, the following hold:

- $L\cap([-T,T]\times\mathbb{R}^3)$ is compact
- $L\cap((T,\infty)\times\mathbb{R}^3)=(T,\infty)\times\Lambda_2$
- $L\cap((-\infty,-T)\times\mathbb{R}^3)=(-\infty,-T)\times\Lambda_1$, and
- there exists a function $f:L\to\mathbb{R}$ satisfying $df=e^t\alpha|_{TL}$ which is constant on the cylindrical ends of L.

See Fig. 2.7

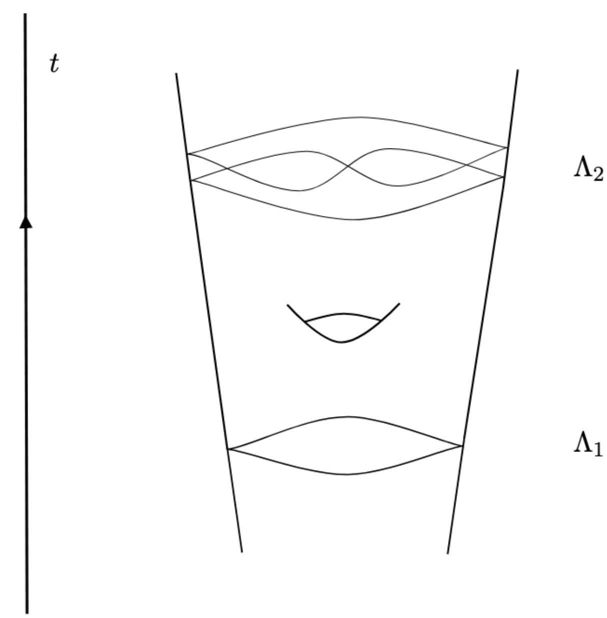


Figure 2.7: A schematic of a Lagrangian cobordism from a Legendrian unknot to a Legendrian trefoil in the symplectization of $(\mathbb{R}^3,\xi_{std})$.

2.4 Chekanov-Eliashberg DGA

Although the classical invariants of Legendrian knots are quite powerful, it was long asked if they were powerful enough to distinguish all Legendrian isotopy classes of a given knot type. This motivated Chekanov and Eliashberg to independently define the *Chekanov-Eliashberg DGA* [22, 21], which Chekanov used to distinguish two Legendrians in the $m(5_2)$ knot type that both have $tb = 1$ and $r = 0$. We define the DGA by its algebra, then the grading, and finally its differential. There are multiple ways to describe the Chekanov-Eliashberg DGA; we will define it in the symplectization $(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$. This will in turn help our description of DGA map induced by a Lagrangian cobordism between two Legendrian knots.

Algebra

Let $\mathcal{C}_\Lambda = \{c_1, \ldots, c_n\}$ be the Reeb chords of the Legendrian knot. The algebra is the associative, noncommutative, unital algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by $c_1, \ldots, c_n, t, t^{-1}$ with relation $tt^{-1} = 1$. This is written as

$$\mathcal{A}_\Lambda = \mathbb{Z}/2\mathbb{Z}\langle c_1, \ldots, c_n, t^{\pm 1}\rangle.$$

It is generated as words in the letters $c_1, \ldots, c_n, t, t^{-1}$. Multiplication in $\mathcal{A}_\Lambda$ is concatenation, and the empty word is the unit 1.

Grading

The grading is defined by associating a degree to each generator, and then the grading of each word is the sum of the gradings of its generators. We define the grading on generators (Reeb chords) as follows. We start by choosing a base point $*$ on Λ disjoint from the end points of all Reeb chords.

Let Λ be a Legendrian knot, c a Reeb chord that has end points a and b on Λ with

$z(a) > z(b)$. Choose a path $\gamma : [0,1] \to \Lambda$ such that $\gamma(0) = a$ and $\gamma(1) = b$ that does not pass through $*$. Then $\Pi_L \circ \gamma$ is a loop in the xy-plane, and $\Gamma(t) = d\Pi_L(T_{\gamma(t)}(\Lambda))$, $0 \leq t \leq 1$ is a path of Lagrangian subspaces in the xy-plane. Since $\Pi_L(c)$ is a transverse double point of $\Pi_L(\Lambda)$, Γ is not a closed loop.

We close Γ by letting $V_0 = \Gamma(0)$ and $V_1 = \Gamma(1)$, and choosing a complex structure I on the xy-plane that is both compatible with $de^t\alpha$ and such that $I(V_1) = V_0$. Then we define the path $\lambda(V_0, V_1)(t) = e^{tI}V_1$, $0 \leq t \leq \frac{\pi}{2}$. The concatenation of Γ and $\lambda(V_0, V_1)$ is a loop in the Grassman manifold of Lagrangian subspaces in $(\mathbb{R}^2, de^t\alpha)$. The fundamental group of Lagrangian lines in $(\mathbb{R}^2, de^t\alpha)$ is $\mathbb{Z}$, so we can associate an integer to this loop, called the *Maslov index* of this loop. We define the *Conley-Zehnder index*, denoted $\nu_\gamma(c)$, to be the Maslov index of this loop. Then the grading of c is defined to be

$$|c| = \nu_\gamma(c) - 1.$$

Differential

To define the boundary map, let J be an almost complex structure that is compatible with $d(e^t\alpha)$. We can assume $J : T(\mathbb{R} \times \mathbb{R}^3) \to T(\mathbb{R} \times \mathbb{R}^3)$ satisfies

$$\begin{aligned}
J(\partial_x) &= \partial_y + y\partial_t \\
J(\partial_y) &= -\partial_x - y\partial_z \\
J(\partial_z) &= -\partial_t \\
J(\partial_t) &= \partial_z
\end{aligned}$$

Let $D^2_n = D^2 \backslash \{x, y_1, \ldots, y_n\}$, where D^2 is the unit disk in $\mathbb{C}$ and $x, y_1, \ldots, y_n$ are points on ∂D^2 arranged in counterclockwise order. Let $u : D^2_n \to \mathbb{R} \times \mathbb{R}^3$ be J-holomorphic, i.e. $J \circ du = du \circ j$, where j is the standard complex structure on $\mathbb{C}$.

We need u to have nice asymptotics near its punctures. So let $u_{\mathbb{R}}$ and $u_{\mathbb{R}^3}$ be its composition with the projection on the the first and second factors of $\mathbb{R} \times \mathbb{R}^3$, respectively. Consider one of the punctures $p \in \{x, y_1, \dots, y_n\}$. Parametrize a neighborhood of p by $(0, \infty) \times [0, 1]$ with coordinates (s, t). Let $a(t)$ be the parametrized Reeb chord a. Then u is *asymptotic to a* at $\pm\infty$ if

$$\lim_{s \to \infty} u_{\mathbb{R}}(s, t) = \pm\infty$$

$$\lim_{s \to \infty} u_{\mathbb{R}^3}(s, t) = a(t)$$

Now let $a, b_1, \dots, b_n$ be points in $\mathcal{C}_\Lambda$ the set of Reeb chords of Λ. Then we define the set

$$\mathcal{M}(a; b_1, \dots b_n) = \{u : (D_n^2, \partial D_n^2) \to (\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda), \text{ satisfying 1) - 4)}\}/\sim,$$

where $\sim$ is holomorphic reparametrization, and

1) u is J-holomorphic.

2) u has finite energy, i.e. $\int_{D_n^2} u^* d\alpha < \infty$.

3) near x, u is asymptotic to a at ∞.

4) near y_i, u is asymptotic to b_i at $-\infty$.

One can show that, for a generic choice of J, $\mathcal{M}(a; b_1, \dots, b_n)$ is a manifold of dimension $|a| - \sum_{i=1}^n |b_i|$, and that given $u \in \mathcal{M}(a; b_1, \dots, b_n)$, adding a constant to $u_{\mathbb{R}}$ gives another element in $\mathcal{M}(a; b_1, \dots, b_n)$. So there is an $\mathbb{R}$ action on $\mathcal{M}(a; b_1, \dots, b_n)$.

Given $u \in \mathcal{M}(a; b_1, \dots, b_n)$, the image of ∂D_n^2 is a union of $n + 1$ paths $\eta_0, \dots \eta_n$ in $\mathbb{R} \times \Lambda$ where η_0 is the path parametrized by the interval starting at x and η_i parametrized by the interval starting at y_i. We define $t(\eta_i)$ to be t^k where k is the number of times η_i crosses $\mathbb{R} \times \{*\}$ counted with sign. The word associated to u is

$$w(u) = t(\eta_0) b_1 t(\eta_1) \cdots b_n t(\eta_n).$$

We can now define the differential of $c \in \mathcal{C}_\Lambda$ to be

$$\partial_\Lambda(c) = \sum_{u \in \mathcal{M}(a;b_1,\ldots,b_n)} w(u)$$

where $n \geq 0$ and $|a| - \sum_{i=1}^{n} |b_i| = 1$. We define $\partial_\Lambda(t) = 0 = \partial_\Lambda(t^{-1})$, and extend ∂_Λ to all of $\mathcal{A}_\Lambda$ via the Leibniz rule, i.e. $\partial_\Lambda(ww') = \partial_\Lambda(w)w' + (-1)^{|w|}w\partial_\Lambda(w')$.

2.4.1 Linearized contact homology

Since the Chekanov-Eliashberg DGA is infinite dimensional, its homology may be too. This hurdle, in addition to the noncommutativity of the algebra, makes computations difficult. To overcome this, Chekanov [21] considered DGAs which have augmentations in order to work with a finite-dimensional linear complex whose homology, called linearized contact homology, is a finite-dimensional vector space.

Like the original DGA, the chain groups are also generated by the Reeb chords with the same grading used in the Chekanov-Eliashberg DGA. Let $\epsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (\mathbb{Z}/2\mathbb{Z}, 0)$ be an augmentation, i.e. a chain map ($\epsilon \circ \partial_\Lambda = 0 \circ \epsilon$) such that the complex $(\mathbb{Z}/2\mathbb{Z}, 0)$ is with both elements of $\mathbb{Z}/2\mathbb{Z}$ graded 0 and 0 denotes the 0 differential. The map $\phi^\epsilon : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, called a *tame isomorphism*, is defined by its action on Reeb chords. Let $c \in \mathcal{C}_\Lambda$. Then

$$\phi^\epsilon(c) = c + \epsilon(c),$$

while fixing constants. Using tame isomorphisms we define the *augmented boundary map* to be

$$\partial_\Lambda^\epsilon := \phi^\epsilon \circ \partial \circ \phi^{-\epsilon}.$$

To define the linearized complex, and the A_∞ structure below, we let F_Λ be the $\mathbb{Z}/2\mathbb{Z}$

vector space generated by the Reeb chords $c_1, \ldots, c_n$. We then define

$$\mathcal{A}_\Lambda^\epsilon = \bigoplus_{n \geq 0} F_\Lambda^{\otimes n},$$

and see that $\partial_\Lambda^\epsilon$ is a boundary map on $\mathcal{A}_\Lambda^\epsilon$. Now define

$$(\mathcal{A}_\Lambda^\epsilon)^k = \bigoplus_{n \geq k} F_\Lambda^{\otimes n},$$

i.e. the words in $\mathcal{A}_\Lambda$ of length k or greater. One can show that the augmented boundary map $\partial_\Lambda^\epsilon$ is a map from $(\mathcal{A}_\Lambda^\epsilon)^k$ to itself (this follows from $\partial_\Lambda^\epsilon(c)$ having no constant terms for any Reeb chord c, and that $\partial_\Lambda^\epsilon$ obeys the Leibniz rule).

Then given a Reeb chord c, the differential for this linearized chain complex, denoted $\partial_\epsilon^\ell(c)$, is defined to be the linear terms of $\partial_\Lambda^\epsilon(c)$, as it is a map from $\frac{(\mathcal{A}_\Lambda^\epsilon)^1}{(\mathcal{A}_\Lambda^\epsilon)^2} \cong F_\Lambda$ to itself satisfying $\partial_\epsilon^\ell \circ \partial_\epsilon^\ell = 0$. Given a Legendrian Λ and an augmentation ϵ, we denote the complex by $(F_\Lambda, \partial_\epsilon^\ell)$. Its homology, which we denote $LCH_*^\epsilon(\Lambda)$, is not itself an invariant. However, Chekanov proved the following theorem.

Theorem 2.4.1. *(Chekanov [21]) The collection* $\{LCH_*^\epsilon(\Lambda) : \epsilon$ *is an augmentation* $(\mathcal{A}_\Lambda, \partial) \to (\mathbb{Z}/2\mathbb{Z}, 0)\}$ *is an invariant of* Λ *up to Legendrian isotopy.*

For more details of the construction of $LCH_*^\epsilon(\Lambda)$ see [33].

2.4.2 The DGA Map

Given an exact Lagrangian cobordism L between two Legendrians, Ekholm, Honda, and Kalman [13] introduced a DGA map on their Chekanov-Eliashberg DGAs. In a similar manner to the boundary map of the Chekanov-Eliashberg DGA ∂_Λ, it is defined by counting rigid punctured holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on L. They further established the following result.

Theorem 2.4.2. *(Ekholm-Honda-Kalman [13, Theorems 1.3 and 1.4]) An exact Lagrangian cobordism L from Λ_1 to Λ_2 induces a DGA map $\Phi_L : (\mathcal{A}_2, \partial_2) \to (\mathcal{A}_1, \partial_1)$ with the following properties:*

1) If $L = \Lambda \times \mathbb{R}$ is a trivial Lagrangian cylinder, then $\Phi_L = Id_{\mathcal{A}_\Lambda}$.

2) If L_1, L_2 have the same ends Λ_1 and Λ_2 and are isotopic through exact Lagrangian cobordisms, then Φ_{L_1} and Φ_{L_2} are chain homotopic.

3) If L_1 and L_2 are exact Lagrangian cobordisms from Λ_0 to Λ_1 and Λ_1 to Λ_2 respectively, then Φ_L and $\Phi_{L_1} \circ \Phi_{L_2}$ are chain homotopic.

2.4.3 The A_∞ structure on Legendrian contact cohomology

Let V be a graded vector space over $\mathbb{Z}/2\mathbb{Z}$. An A_∞ *algebra* on V is an infinite sequence of degree 1 graded maps $\{m_k : V^{\otimes k} \to V\}_{1 \leq k}$ that satisfy:

$$\sum_{i+j+k=l} m_{i+1+k} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = 0$$

for all $1 \leq l \leq n$. The case $l = 1$ yields the equation $m_1 \circ m_1 = 0$, so m_1 is a codifferential on V. The case $l = 2$ gives the equation

$$m_1 \circ m_2(a, b) = m_2(m_1(a), b) + m_2(a, m_1(b)).$$

So m_2 descends to a well-defined product on $H^*(V)$. A full A_∞ structure cannot be defined on $H^*(V)$ for the maps m_k, $k \geq 3$, by simply letting m_k descend to the cohomology. Instead the higher order $(k \geq 3)$ products are inductively defined on quotients of the cohomology. See [28] for details. It was shown in [28] that there is an A_∞ algebra on $(F^*_\Lambda, (\partial^\ell_\epsilon)^*)$.

Theorem 2.4.3. *(Civan-Etnyre-Koprowski-Sabloff-Walker [28]) For each augmentation ϵ, the Legendrian contact homology DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ induces an A_∞ structure on the linearized cochain complex $(F^*_\Lambda, (\partial^\ell_\epsilon)^*)$.*

The cohomology of $(F^*_\Lambda, (\partial^\ell_\epsilon)^*)$, denoted $LCH^*_\epsilon(\Lambda)$, is called the *linearized contact cohomology* of Λ.

Let $(V, \{m_k : V^{\otimes k} \to V\}_{1\leq k})$ and $(W, \{n_k : W^{\otimes k} \to W\}_{1\leq k})$ be A_∞ algebras. An A_∞ *morphism* is an infinite collection of degree 0 maps $\{\phi_n : V^{\otimes n} \to W\}_{n\geq 1}$ that satisfy

$$\sum_{i+j+k=n} \phi_{i+1+k} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = \sum_{r=1}^{n} \sum_{i_1+\ldots+i_r} n_r \circ (\phi_{i_1} \otimes \cdots \otimes \phi_{i_r})$$

for all $n \geq 1$. A consequence of $\{\phi_n : V^{\otimes n} \to W\}_{n\geq 1}$ being an A_∞ morphism is that ϕ_1 preserves the product structures, i.e.

$$\phi_1 \circ m_k([c_{i_1}], \ldots, [c_{i_k}]) = n_k(\phi_1[c_{i_1}], \ldots, \phi_1[c_{i_k}])$$

on the level of cohomology.

CHAPTER 3

CLASSIFICATION OF LEGENDRIAN CABLES OF UNIFORMLY THICK KNOTS

The results in this chapter are joint with Chatterjee, Etnyre, and Min [34]. Recall $w(K)$ denotes the contact width of the knot type K. We classify Legendrian cable links whose underlying knot is the uniformly thick. Recall the $n(p,q)$-cable link of K from Section 2.1.2 is denoted $K_{n(p,q)}$. The classification is broken up into three cases: Section 3.1 covers the *positive slope* case ($\frac{q}{p} > \lceil w(\Lambda)\rceil$), Section 3.2 covers the *negative integer slope* case ($\frac{q}{p} \leq \lceil w(\Lambda)\rceil$, $p = 1$), and Section 3.3 covers the *negative noninteger slope* case ($\frac{q}{p} < \lceil w(\Lambda)\rceil$, $p \neq 1$). We provide the proofs used in the positive slope case, and omit them for the negative slope cases. They can be found in [34].

3.1 Positive slope cables

The mountain range of positive slope cables can be visualized by its diamond expansion. Given a point (a,b) in the integer lattice $\mathbb{Z}^2$, the (p,q) *diamond of (a,b)* is the set of points

$$\begin{aligned} D_{p,q)}(a,b) = \{(r,t) : \; & t + |r - pa| \leq pq - |pb - q|, \\ & t + |r - pa| \geq pq - |pb - q| - 2p + 2, \\ & t + r \text{ odd}\}. \end{aligned}$$

See [35] for more details.

For classification of positive slope cable links, we define the *cone* of a link Λ to be the set

$$C(\Lambda) = \{S_-^m \circ S_+^n(\Lambda) | m, n \geq 0\},$$

i.e. the set of all stabilizations of Λ. Next we define our max-tb positive slope cable links in the following way. Let the coordinates on ∂N be given by the Seifert longitude. Let N be a standard neighborhood of Λ whose ruling curves have slope $\frac{q}{p}$. Then the *standard (np,nq)-cable* of Λ is the union of n such ruling curves. Now we can describe an important property of cabling.

Lemma 3.1.1. *Let Λ be in the knot type K. Then If $\Lambda' \in C(\Lambda)$, then*

1. $C(\Lambda'_{n(p,q)}) \subseteq C(\Lambda_{n(p,q)})$.

2. *If $\Lambda' \in C(\Lambda_{n(p,q)})$, then Λ' sits on a standard neighborhood of Λ.*

Proof. Let $\Lambda' = S_+^k \circ S_-^l(\Lambda)$. We show that $\Lambda'_{n(p,q)}$ is obtained by stabilizing each component of $\Lambda_{n(p,q)}$ pk times positively and pl times negatively. Then item $1)$ will follow.

Let N be a standard neighborhood of Λ. Then we can take the standard neighborhood N' of Λ' to be inside of N. Since $\Lambda_{n(p,q)}$ is n ruling curves on N and $\Lambda'_{n(p,q)}$ is n ruling curves on N', we have a union of convex annuli $\bigcup_{i=1}^{n} A_i$ each with one boundary component a component of $\Lambda_{n(p,q)}$ and the other a component of $\Lambda'_{n(p,q)}$. We can assume the dividing set of these annuli intersect $\Lambda'_{n(p,q)}$ exactly $|p(tb(\Lambda) - k - l) - q|$ times and $\Lambda_{n(p,q)}$ exactly $|ptb(\Lambda) - q|$ times. Therefore we have $p(k + l)$ boundary parallel dividing curves on each A_i giving bypasses for each component of $\Lambda'_{n(p,q)}$. Considering the rotation numbers of the components of $\Lambda'_{n(p,q)}$ and $\Lambda_{n(p,q)}$, we see there are pk positive bypasses and pl negative bypasses. Thus $\Lambda'_{n(p,q)}$ destabilizes to $\Lambda_{n(p,q)}$.

For item $2)$, since $\Lambda_{n(p,q)}$ lies on ∂N, then any stabilization of $\Lambda_{n(p,q)}$ can be obtained by perturbing $\Lambda_{n(p,q)}$ on ∂N so that it intersects the dividing set of ∂N a certain number of times. □

In order to classify positive slope cables, we need the following definitions and lemma. Given a Legendrian link $\Lambda \in \mathcal{L}(K_{n(p,q)})$, we say that $L \in \mathcal{L}(K)$ is an *underlying Legendrian of* Λ if Λ is Legendrian isotopic to an n copy of (p, q) curves on the boundary of a neighborhood of L. We define a *minimal underlying Legendrian* L_{min} to be an underlying

Legendrian of Λ such that neither $S_+(L_{min})$ nor $S_-(L_{min})$ is an underlying Legendrian of Λ.

Lemma 3.1.2. *Let Λ be a Legendrian link in $\mathcal{L}(K_{n(p,q)})$. Then there exists a minimal underlying Legendrian knot $L_{min} \in \mathcal{L}(K)$ and if L'_{min} is another minimal underlying Legednrian of Λ, then $tb(L_{min}) = tb(L'_{min})$ and $r(L_{min}) = r(L'_{min})$.*

Proof. The existence of L_{min} is clear. By the definition, we know $S_\pm(L_{min})$ are not underlying Legendrians of Λ. Therefore either at least one component of Λ is $(L_{min})_{(p,q)}$, or at least two components of Λ are $S^i_+((L_{min})_{(p,q)})$ and $S^j_-((L_{min})_{(p,q)})$ for $i, j > 0$, respectively. Thus one component of Λ is at the top of the cone of $(L_{min})_{(p,q)}$, or there are two components of Λ, one of which is on the left boundary edge of the cone $(L_{min})_{(p,q)}$ and the other is on the right boundary edge.

Now assume that there exists another minimal underlying knot L'_{min} for Λ. Suppose that $tb(L'_{min}) > tb(L_{min})$. Also, without loss of generality, we may assume $r(L_{min}) \leq r(L'_{min})$. Then we do not have that at least one component of Λ is $(L_{min})_{(p,q)}$, for otherwise there exists a component of Λ with $tb = tb((L'_{min})_{(p,q)}) > tb((L_{min})_{(p,q)}$, contradicting that L_{min} is an underlying Legendrian of Λ. Thus only the condition that at least two components of Λ are $S^i_+((L_{min})_{(p,q)}$ and $S^j_-((L_{min})_{(p,q)}$ for $i, j > 0$, respectively holds for $L_{min'}$, which implies that there exists a component of Λ that lies on the right boundary edge of the cone $C((L'_{min})_{(p,q)})$. However, since $tb(L_{min}) < tb(L'_{min})$ and $r(L_{min}) \leq r(L'_{min})$, the cone $C((L_{min})_{(p,q)}$ does not contain the right boundary edge of the cone $C((L'_{min})_{(p,q)}$, which again contradicts that L_{min} is an underlying Legendrian of Λ.

Now consider the case that $tb(L_{min}) = tb(L'_{min})$. Again, without loss of generality, we may assume $r(L_{min}) < r(L'_{min})$. In this case $C((L_{min})_{(p,q)})$ cannot contain the right boundary edge of $C((L'_{min})_{(p,q)}$. That means neither the condition that at least one component of Λ is $(L_{min})_{(p,q)}$ nor the condition that at least two components of Λ are $S^i_+((L_{min})_{(p,q)})$ and $S^j_-((L_{min})_{(p,q)})$ for $i, j > 0$, respectively, can hold, which is a contradiction. □

Now we state the classification of positive slope cables.

Theorem 3.1.3. *Let $L^1, \ldots, L^n$ be the nondestabalizable Legendrians in a knot type K. For relatively prime integers p and q with $\frac{q}{p} > \lceil w(K) \rceil$ and $n > 1$, we have*

1. $$\mathcal{L}(K_{n(p,q)}) = \bigcup_{i=1}^{n} C((L^i)_{n(p,q)})$$

2. *Two legendrian links in $C((L^i)_{n(p,q)})$ are isotopic if and only if they have the same classical invariants.*

3. *Λ and Λ' in $\mathcal{L}(K_{n(p,q)})$ are isotopic if and only if there is some $L \in \mathcal{L}(K)$ such that Λ and Λ' are both in $C(L_{n(p,q)})$ and the components of Λ and Λ' have the same classical invariants.*

4. *For $L, L' \in \mathcal{L}(K)$, $\Lambda \in C(L_{n(p,q)}) \cap C(L'_{n(p,q)})$ if and only if there is some $L'' \in C(L) \cap C(L')$ and $\Lambda \in C(L''_{n(p,q)})$*

Moreover, given any $\Lambda \in \mathcal{L}(K_{n(p,q)})$, any permutation of the components of Λ preserving the classical invariants can be achieved by a Legendrian isotopy.

Proof. Since we are taking $L^1, \ldots, L^n$ to be the nondestabilizable Legendrian knots in $\mathcal{L}(K)$ and relatively prime integers in p and q such that $\frac{q}{p} > \lceil w(K) \rceil$,

Item 1. follows from Proposition 7.7 in [36].

Item 2. follows as they are all stabilizations of a fixed Legendrian link and we know any permutations of the components of the link that respect the classical invariants can be realized by Legendrian isotopy.

For item 3., the reverse implication follows from item 2. Now let us assume Λ and Λ' are Legendrian isotopic. Then they must have the same classical invariants. From item 1 of the theorem there is some L^i such that $\Lambda \in C((L^i)_{n(p,q)})$, and thus Λ' must also be in $C((L^i)_{n(p,q)})$

Now, for item 4., the reverse implication follows from item 1. of Lemma 3.1.1. Now suppose that $\Lambda \in C(L_{n(p,q)}) \cap C(L'_{n(p,q)})$. By Lemma 3.1.2, there exist minimal underlying Legendrian knots $L_{min} \in C(L)$ and $L'_{min} \in C(L')$ such that L_{min} and L'_{min} have the same classical invariants. Then by item 2. of Lemma 3.1.1, Λ lies both on T, the boundary of a standard neighborhood of L_{min}, and T', the boundary of a standard neighborhood of L'_{min}. So the intersection of T and T' contains Λ. Thus there is a smooth isotopy from T to T' fixing Λ, and by discretization of isotopy there is a sequence of convex tori $T = T_1, \dots, T_n = T'$ such that T_{i+1} can be obtained by a bypass attachment to T_i where the bypass attachment does not intersect Λ.

Since L_{min} is minimal, it is clear that the dividing slopes of T_i are in $(tb(L_{min})-1, \infty)$. We show that the dividing slopes are not in $(tb(L_{min}) - 1, tb(L_{min}))$, and thus must be in $[tb(L_{min}), \infty)$. Assume Λ sits on a torus T'' with dividing slope in $(tb(L_{min})-1, tb(L_{min}))$. Then T'' bounds a solid torus which contains another solid torus T''' which has dividing slope $tb(L_{min})-1$, and the ruling curves of slope $\frac{q}{p}$ on T'' are stabilizations of those on T''' (this can be seen by taking annuli between the ruling curves). Therefore Λ also sits on T''', but this contradicts Lemma 3.1.2 since T''' is the boundary of a standard neighborhood of a Legendrian knots with tb different from L_{min}.

Now we inductively claim that each T_i bounds a solid torus that contains a standard neighborhood of L_{min}. For $i = 1$ the claim is true. Now assume true for i. We can obtain T_{i+1} by attaching a bypass to T_i. There are three non-trivial bypasses: one that changes the number of dividing curves, one that increases the slope of the dividing curves, and one that decreases the slope of the dividing curves.

In the first case, the bypass can be attached from the outside or the inside of the solid torus that T_i bounds. If it is attached from the outside, then T_{i+1} bounds a solid torus that contains the solid torus bounded by T_i and hence also contains a neighborhood of L_{min}. If it is attached from the inside, then T_{i+1} bounds a solid torus that contains a torus T'_{i+1} isotopic to it such that T_i and T'_{i+1} cobound $T^2 \times I$ with an I-invariant contact structure.

Thus T_{i+1} bounds a solid torus containing a neighborhood of L_{min} since T_i does and T_{i+1} bounds a solid torus containing the solid torus that T'_{i+1} bounds, completing the induction in this case.

If the bypass increases the dividing slope, then it must be attached from the outside of T_i, so T_{i+1} bounds a solid torus containing T_i, and we are done as above.

Lastly, consider the case where the bypass decreases the dividing slope. That means the bypass is attached from the inside of T_i, so the solid torus S_i that T_i bounds contains T_{i+1}. We consider two cases. First, suppose that s_i is not an integer. According to [37], there is a unique solid torus S'_i in S_i up to contact isotopy with convex boundary having two diving curves of slope $\lceil s_i \rceil$. By the induction hypothesis, there is a standard neighborhood N_i of L_{min} in S_i, and since $tb(L_{min}) \leq \lceil s_i \rceil$, we know that N_i is inside of S'_i. Now since T_{i+1} is obtained from ∂S_i by a bypass attachment, its slope must be greater than or equal to $\lceil s_i \rceil$ and hence S_{i+1} will also contain N_i, so we can let $N_{i+1} = N_i$.

We are left to consider the case when s_i is an integer. If $s_i = tb(L_{min})$, then the slope of T_{i+1} is also s_i and the bypass attachment was trivial and did not decrease the dividing slope. So we can assume $s_i > tb(L_{min})$. The solid torus S_i is a standard neighborhood of a Legendrian knot L'', and since S_i contains a standard neighborhood of L_{min}, we know that L_{min} is a stabilization of L''. If we have to stabilize L'' both positively and negatively to obtain L_{min}, then one may easily check that no matter what T_{i+1} is, the solid torus S_{i+1} that it bounds will contain a neighborhood of L_{min} (this is because it will contain a neighborhood of $S_+(L'')$ or $S_-(L)$ which both contain a neighborhood of L_{min} in this case). However, if L_{min} is obtained by stabilizing L'' with only one sign, say positively, then it is possible that attaching a bypass from the inside of S_i will result in a torus that does not bound a solid torus containing a neighborhood of L_{min}. But if this is the case then S_{i+1} does not contain a neighborhood of $S_-(L'')$ and as argued above (in the first paragraph when discussing Item (4)) we see that Λ will sit on the boundary of the standard neighborhood of $S_-(L'')$. Thus Λ is in $C((S_-(L''))_{n(p,q)})$. However, if one considers how

$C((S_-(L''))_{n(p,q)})$ and $C((L_{min})_{n(p,q)})$ intersect, we see that Λ cannot satisfy Conditions $1)$ and $2)$ in the proof of Lemma 3.1.2 and thus L_{min} is not a minimal underlying Legendrian knot for Λ. This contradiction implies that the S_{i+1} does contain a neighborhood of L_{min} as desired.

We now notice that a standard neighborhood of L'_{min} (bounded by T') contains a standard neighborhood of L_{min} (by our inductive hypothesis), so L'_{min} must be isotopic to L_{min}. Thus L_{min} is in the cone of L and L' finishing the proof. $\square$

Proof of Theorem 1.1.3. Consider the negative twist knot $(K_{-5})_{2(2,1)}$. Here we have a two component link. Note that if the components have the same tb and r, then they must be Legendrian isotopic. This is because they are stabilizations of max-tb components. As max-tb components, they are sitting on the boundary of a standard neighborhood of a max-tb Legendrian respresentative of K_{-5} and intersecting the dividing set minimally. Therefore by Giroux flexibility the boundary of the standard neighborhood can be isotoped such that it has ruling curves of slope $\frac{q}{p}$, providing a Legendrian isotopy between the two components. So if $\Lambda_1 \cup \Lambda_2 \in \mathcal{L}((K_{-5})_{2(2,1)})$ and Λ_1 and Λ_2 have the same classical invariants, they must be Legendrian isotopic because they are either max-tb or stabilizations of a max-tb respresentative of $\mathcal{L}((K_{-5})_{2(2,1)})$.

Let Λ^1 and Λ^2 be the two max-tb representatives of $\mathcal{L}((K_{-5})_{2(2,1)})$, and denote $\Lambda^i_{m,n,k,l}$ to be Λ^i after stabilizing the first component of Λ^i positively m times and negatively n times, and stabilizing the second component k times positively and l times negatively.

If $m, l \leq 1$ and $n, k \geq 2$, then $\Lambda^1_{m,n,k,l}$ sits in $C(\Lambda^1)$ and $\Lambda^2_{m,n,k,l}$ sits in $C(\Lambda^2)$. Since $m, l \leq 1$, it is clear that both $\Lambda^1_{m,n,k,l}$ and $\Lambda^2_{m,n,k,l}$ do not sit in $C(L'_{2(2,1)})$ where L' is either $S_+(L^1) = S_+(L^2)$ or $S_-(L^1) = S_-(L^2)$. That is, there is no $L \in C(L^1) \cap C(L^2)$ such that $\Lambda^i_{m,n,k,l} \in C(L_{2(2,1)})$ for $i = 1, 2$. By part $4)$ of Theorem 3.1.3, $\Lambda^i_{m,n,k,l} \notin C(\Lambda^1) \cap C(\Lambda^2)$, and thus by part $3)$ of Theorem 3.1.3, $\Lambda^1_{m,n,k,l}$ is not Legendrian isotopic to $\Lambda^2_{m,n,k,l}$. But clearly each component of $\Lambda^1_{m,n,k,l}$ is Legendrian isotopic to each component of $\Lambda^2_{m,n,k,l}$. $\square$

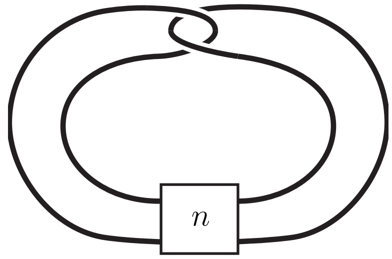


Figure 3.1: The twist knot T_n where the box contains n positive half=twists if $n > 0$ and $|n|$ negative half twists if $n < 0$.

3.2 Integer negative slope cables

We omit the proofs in this section. They can be found in [34]. For the classification of integer negative sloped cales, we begin by defeining the max-tb cables and the non-destabilizable non max-tb cables. Let N be a standard neighborhood of a Legendrian knot L. Then in the standard model, there is an annulus A that contains L and has a linear characteristic foliation in which each leaf is Legendrian isotopic to L. The *n-copy* of L is the link nL consisting of n leaves in the characteristic foliation of A. Another way of defining the n-copy of L in $(\mathbb{R}^3, \xi_{std})$ is taking the front projection of L and shifting L slightly in the z-direction $n-1$ times. Then their union is nL. Note that nL is an $(n, ntb(L))-$ cable of L. See Fig. 3.2 below.

For the non-destabilizable cables, we define the *t-twisted n-copy of* L, denoted $T^t(nL)$, to be the link consisting of L and $(n-1)$-ruling curves of slope $tb(L) - t$ on ∂N. So then $T^t(nL)$ is an $(n, n(t + tb(L))$-cable of L. Note that the t-twisted 2-copy is also max-tb. If $n > 2$, then the t-twisted n-copy of L is non max-tb. An alternative description of the t-twisted n-copy in $(\mathbb{R}^3, \xi_{std})$ is to take the front projection of the n-copy of L and then replace a region in the front projection as shown in Fig. 3.2.

Figure 3.2: The n-copy of a Legendrian knot L on the left. To create the t-twisted n-copy of L, replace t regions of the n-copy as shown at the top of the right-hand diagram with the region shown on the bottom.

Given an n-component Legnedrian link λ, we label the components $\Lambda_1 \cup \ldots \cup \Lambda_n$. We denote the result of $\pm$-stabilizing the i^{th} component of Λ by $S_{i,\pm}(\Lambda)$. Given the t-twisted n-copy of L, we say the component isotopic to L is the first component and then label the others, cyclically, by 2 through n. (Cyclically here means we order them as they occur on ∂N. This leads us to the following lemma regarding t-twisted n-copies.

Lemma 3.2.1. *We have the following relations between the stabilizations of the twisted n-copies:*

$$S_{1,\pm}(T^t(nL)) = S_{2,\mp} \circ \cdots \circ S_{n,\mp}(T^{t-1}(nS_\pm(L))).$$

Additionally, we have

$$S_{1,\pm}(T^1(nL)) = S_{2,\mp} \circ \cdots \circ S_{n,\mp}(nS_\pm(L)).$$

Proof. Recall that $T^t(nL)$ is formed by taking L and the union of $n-1$ ruling curves $L_2, \ldots, L_n$ of slope $tb(L) - t$ on the boundary of a standard neighborhood $N(L)$ of L. Now let $N(S_\pm(L))$ be a standard neighborhood of $S_\pm(L)$ inside of $N(L)$. Note $S_\pm(L)$ together with $n-1$ curves $L'_2, \ldots, L'_n$ of slope $tb(L) - t$, which is also equal to the slope $tb(S_\pm(L)) - (t-1)$, on $\partial N(S_\pm(L))$ is the $(t-1)$-twisted n-copy of $S_\pm(L)$. Moreover, we can take annuli between L_i and L'_i to see a bypass for L'_i. Thus the L'_i are stabilizations of the L_i. As we know that rotation numbers of the L'_i match that of $S_\pm(L)$ and those of L_i

match that of L, we see the signs of the stabilization are as claimed.

The second relation between the 1-twisted n-copy and the n-copy of a stabilization follows similarly, except to see the destabilization of the we need to use [38] $\square$

Now we state our main results for Legendrian links in integer negative sloped cabled of uniformly thick knots.

Theorem 3.2.2. *Let K be a uniform thick knot type and $q \leq \overline{\mathrm{tb}}(K)$ an integer. Let $L_1, \ldots, L_k$ be the distinct Legendrian representatives of K with $\mathrm{tb} \geq q$. Then the non-destabilized Legendrian representatives of $K_{n(1,q)}$ are*

$$\{T^t(nL_i) : t = \mathrm{tb}(L_i) - q \text{ and } i = 1, \ldots, k\}.$$

Any other Legendrian representative of $K_{n(1,q)}$ destabilizes to one of these.

Let Λ^1 and Λ^2 be two Legendrian representatives of $K_{n(1,q)}$ that are not stabilizations of n-copies of a Legendrian L_i with $\mathrm{tb} = q$. Let Λ^i_1 be the component of Λ^i with the largest Thurston-Bennequin invariant (if there is more than one such component, then choose one), then Λ^1 and Λ^2 are Legendrian isotopic if and only if Λ^1_1 and Λ^2_1 are Legendrian isotopic and the other components of Λ^1 and Λ^2 can be paired to have the same Thurston-Bennequin invariant and rotation number.

Let Λ^1 and Λ^2 be two Legendrian representatives of $K_{n(1,q)}$ such that Λ^1 is obtained from a n-copy of L_i with $\mathrm{tb}(L_i) = q$ by stabilizing at least one of the components positively some number of times (but not negatively) and at least one of the components negatively some number of times (but not positively) and similarly Λ^2 is obtained from an n-copy of L_j with $\mathrm{tb}(L_j) = q$ by such stabilizations. If $i \neq j$, then Λ^1 and Λ^2 are not Legendrian isotopic.

Let Λ^1 and Λ^2 be two Legendrian representatives of $K_{n(1,q)}$ such that Λ^1 is obtained from a n-copy of L_i with $\mathrm{tb}(L_i) = q$ by stabilizing all the components positively some number of times (but not negatively) and Λ^2 is obtained by the same stabilizations of an

n-copy of L_j with $\operatorname{tb}(L_j) = q$. Then Λ^1 and Λ^2 are Legendrian isotopic if and only if the components with the largest Thurston-Bennequin invariant are Legendrian isotopic. The same statement is true when all components are stabilized negatively.

Now we consider the possible permutations of the components of integer lesser-sloped cables that are realizable by Legendrian isotopy.

Theorem 3.2.3 (Ordered Classification)**.** *Let K be a uniform thick knot type and $q \leq \overline{\operatorname{tb}}(K)$ an integer. Given an element $\Lambda \in \mathcal{L}(K_{n(1,q)})$ then:*

1. *Any permutation of the components of Λ are can be realized by a Legendrian isotopy unless $\Lambda = nL$ or nL with some, but not all, components stabilized.*

2. *If $\Lambda = nL$, possibly with some components stabilized, and $q = \overline{\operatorname{tb}}(K)$, then no permutations of the maximal Thurston-Bennequin invariant components can be realized by a Legendrian isotopy, but non-maximal Thurston-Bennequin invariant components can be permuted if they have the same classical invariants.*

3. *If $\Lambda = nL$, possibly with some components stabilized, and $q < \overline{\operatorname{tb}}(K)$, then Λ sits on a convex torus bounding a solid torus in the knot type of K and the components of Λ have a cyclic ordering given by this torus. Only cyclic permutations of the components of Λ with maximal Thurston-Bennequin invariant can be realized by Legendrian isotopy, but non-maximal Thurston-Bennequin invariant components can be permuted if they have the same classical invariants.*

3.3 Non-integer negative slope cables

We omit the proofs in this section. They can be found in [34]. We first construct max-tb cables for the non-integer negative slope case. Consider a standard neighborhood N of a Legendrian knot L. Inside of N we can consider a standard neighborhood $N_\pm$ of a stabilization $S_\pm(L)$ of L, with $N_\pm \subseteq N$. For any $\frac{q}{p} \in (tb(L) - 1, tb(L))$ we can find a unique, up to contact isotopy, torus $T_\pm$ in $N\backslash N_\pm$ isotopic to ∂N whose characteristic foliation is linear of slope $\frac{q}{p}$. We define the $\pm$ *standard (np,nq)-cable of L* to be n leaves in the characteristic foliation of $T_\pm$ and we denote this link by $L^\pm_{n(p,q)}$.

Before stating our results on cable links, we discuss cable knots. Negative slope cable knots were first considered in [35]. The following theorem, except for Item 3., was proven in that paper. In [34], we proved the full version of the following theorem that allows us to understand cables of twist knots (among others).

Theorem 3.3.1. *Let K be a uniformly thick knot type and p and q relatively prime with $q/p < \overline{\mathrm{tb}}(K)$. Let $L_1, \ldots, L_k$ be the distinct Legendrian knots in $\mathcal{L}(K)$ with $\mathrm{tb} = \lceil q/p \rceil$. Every element in $\mathcal{L}(K_{(p,q)})$ destabilizes to $(L_i)^\pm_{(p,q)}$ for some i and $\pm$.*

We distinguish the $(L_i)^\pm_{(p,q)}$ as follows.

1. *If $\mathrm{rot}(L_i) \neq \mathrm{rot}(L_j)$, then $(L_i)^\pm_{(p,q)}$ and $(L_j)^\pm_{(p,q)}$ are distinct.*

2. *If $\mathrm{rot}(L_i) = \mathrm{rot}(L_j)$ but $S_\pm(L_i) \neq S_\pm(L_j)$, then $(L_i)^\pm_{(p,q)}$ and $(L_j)^\pm_{(p,q)}$ are distinct.*

3. *If $\mathrm{rot}(L_i) = \mathrm{rot}(L_j)$ but Legendrian surgery in L_i and L_j yield distinct contact manifolds, then $(L_i)^\pm_{(p,q)}$ and $(L_j)^\pm_{(p,q)}$ are distinct.*

4. *Otherwise, it is not clear if $(L_i)^\pm_{(p,q)}$ and $(L_j)^\pm_{(p,q)}$ are distinct.*

We distinguish stabilizations of the $(L_i)^\pm_{(p,q)}$ as follows.

1. *Stabilizations of $(L_i)^+_{(p,q)}$ and $(L_i)^-_{(p,q)}$ remain distinct until the first knot has been $-$-stabilized $|p\,\mathrm{tb}(L) - q|$ or more times, the second knot has been $+$-stabilized $|p\,\mathrm{tb}(L) - q|$ or more times, and the classical invariants match. In this case they are Legendrian isotopic.*

2. *If $S_+(L_i) = S_-(L_j)$, then $(L_i)^+_{(p,q)}$ and $(L_j)^-_{(p,q)}$ will have distinct rotation numbers until the first has been $+$-stabilized $|p(\mathrm{tb}(L) - 1) - q|$ times and the second has been $-$-stabilized $|p(\mathrm{tb}(L) - 1) - q|$ times. After each has been stabilized as indicated, they become Legendrian isotopic.*

3. *If $S_{\pm}(L_i) = S_{\pm}(L_j)$, but Legendrian surgery on L_i and L_j yield distinct contact manifolds, then stabilizations of $(L_i)^{\pm}_{(p,q)}$ and $(L_j)^{\pm}_{(p,q)}$ remain distinct until each has been $\pm$-stabilized $|p(\mathrm{tb}(L) - 1) - q|$ or more times, and the classical invariants are the same. In this case they are Legendrian isotopic.*

4. *If $S^l_{\pm}(L_i)$ is not Legendrian isotopic to $S^l_{\pm}(L_j)$ for any l, then any number of $\pm$-stabilizations of $(L_i)^{\pm}_{(p,q)}$ and $(L_j)^{\pm}_{(p,q)}$ remain distinct.*

Although the statements in the theorem do not necessarily classify all Legendrian cable knots of uniformly thick knot types, the theorem and its proof are sufficient in many cases, such as negative twist knots. In particular, we have the following result.

Theorem 3.3.2. *Let K be the $-2n$-twist knot, $-q/p \in (-m-1, -m)$ for $m \geq 0$ an integer, and $k = \lceil \frac{n}{2} \rceil$. Any (r,t) in the mountain range of $\mathcal{L}(K_{(p,-q)})$ with $t \leq pq$ and $r \in [p - q + (pq + t), -q + p + (t - pq)]$ is realized by a unique Legendrian knot and any other pair (r', t') in the mountain range is realized by k distinct Legendrian knots.*

More specifically, there are $2m + 4k$ Legendrian knots in $\mathcal{L}(K_{(p,q)})$ with $\mathrm{tb} = pq$. Of those, $2m$ are determined by their rotation numbers, which are

$$\pm(p - q + 2pl) \quad \textit{for } l = 0, \dots m-1,$$

and for each rotation number

$$\mathrm{rot} = \pm(p+q) \textit{ and } \pm((2m+1)p - q).$$

there are k Legendrian knots. All other Legendrian knots are stabilizations of these. The mountain range for $K_{(p,q)}$ is given in Fig. 3.3.

The only range of cabling slopes in which we cannot classify cables for $-2n$ twist knots is $(0,1)$, but under one assumption, we can understand this range as well.

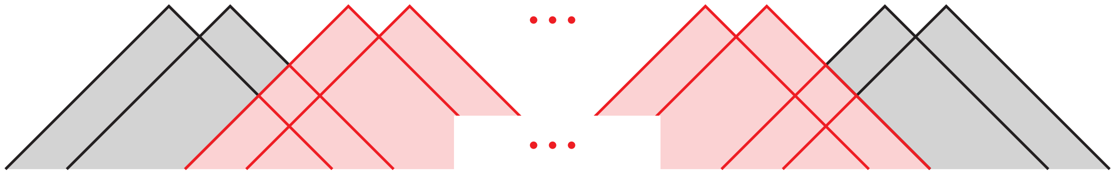


Figure 3.3: The mountain range for $K_{(p,q)}$ where K is a $-2n$-twist knots and $q/p \in (-m-1,-m)$. Any point (r,t) with $r+t$ odd in the red region is realized by a unique Legendrian knot, while in the black region is realized by k Legendrian knots, where k is as in Theorem 3.3.2. The peaks are at $\mathrm{tb} = pq$ and the values of the rotation numbers at the peaks are given in Theorem 3.3.2.

Theorem 3.3.3. *Let K be the $-2n$-twist knot, $q/p \in (0,1)$, $k = \lceil \frac{n}{2} \rceil$ and $l = \lceil \frac{n^2}{2} \rceil$. Suppose that Legendrian surgery on the maximal Thurston-Bennequin invariant Legendrian representatives of K yields distinct contact manifolds. Then any (r,t) in the mountain range of $\mathcal{L}(K_{(p,q)})$ with $r \in [-(pq-p-q-t), pq-p-q-t]$ is realized by a unique Legendrian knot, any such pair with $r \in [\pm p-(pq-q-t), \pm p+(pq-q-t)]$ that is not in the first region has exactly k Legendrian representatives, and any other pair of (r,t) in the mountain range is realized by l distinct Legendrian knots.*

More specifically, there are $2l$ distinct Legendrian knots $L_i^\pm, i = 1,\ldots,l$ in $\mathcal{L}(K_{(p,q)})$ with $\mathrm{tb} = pq$ and $\mathrm{rot}(L_i^\pm) = \pm(p-q)$. All other Legendrian knots in $\mathcal{L}(K_{(p,q)})$ destabilize to one of these. In addition, $S_-^{p-q}(L_i^+) = S_+^{p-q}(L_i^-)$, the collection $\{L_i^+\}_{i=1}^l$ after $+$-stabilizing q times has k distinct members and similarly for $\{L_i^-\}_{i=1}^l$ after $-$-stabilizing q times. After the $L_i^\pm$ have been stabilized into the first regions mentioned above, then they are determined by their classical invariants. The mountain range $K_{(p,q)}$ is given in Fig. 3.4.

Now we consider negative slope, non-integral cable links.

Theorem 3.3.4. *Let K be a uniformly thick knot type and p and q relatively prime with $q/p < \overline{\mathrm{tb}}(K)$. Let $L_1,\ldots,L_k$ be the distinct Legendrian knots in $\mathcal{L}(K)$ with $\mathrm{tb} = \lceil q/p \rceil$. Every element in $\mathcal{L}(K_{n(p,q)})$ destabilizes to $(L_i)^\pm_{n(p,q)}$ for some i and $\pm$.*

We distinguish the $(L_i)^\pm_{n(p,q)}$ as follows.

1. *If $\mathrm{rot}(L_i) \neq \mathrm{rot}(L_j)$, then $(L_i)^\pm_{n(p,q)}$ and $(L_j)^\pm_{n(p,q)}$ are distinct.*

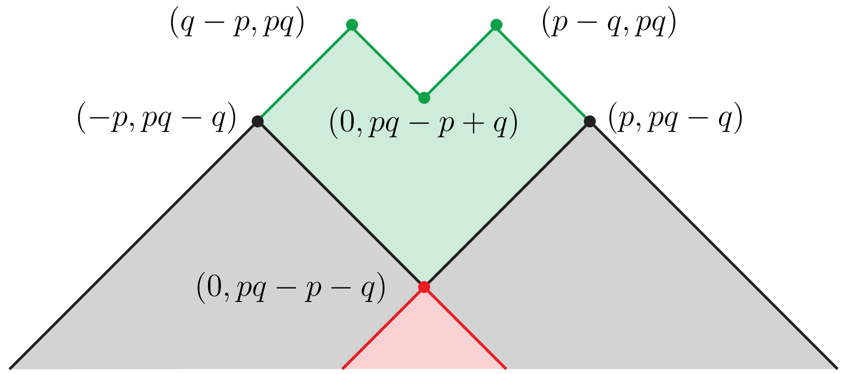


Figure 3.4: The (p,q)-cables of K_{-2n} where the (r,t) with $r+t$ odd in the upper region have $l = \lceil \frac{n^2}{2} \rceil$ Legendrian representatives, such pairs in the regions on the right and left have $k = \lceil \frac{n}{2} \rceil$ Legendrian representatives, and such pairs in the bottom region have a unique Legendrian representative.

2. *If* $\mathrm{rot}(L_i) = \mathrm{rot}(L_j)$ *but* $S_\pm(L_i) \neq S_\pm(L_j)$, *then* $(L_i)^\pm_{n(p,q)}$ *and* $(L_j)^\pm_{n(p,q)}$ *are distinct.*

3. *If* $\mathrm{rot}(L_i) = \mathrm{rot}(L_j)$ *but Legendrian surgery in* L_i *and* L_j *yield distinct contact manifolds, then* $(L_i)^\pm_{n(p,q)}$ *and* $(L_j)^\pm_{n(p,q)}$ *are distinct.*

4. *Otherwise, it is not clear if* $(L_i)^\pm_{n(p,q)}$ *and* $(L_j)^\pm_{n(p,q)}$ *are distinct.*

We distinguish stabilizations of the $(L_i)^\pm_{n(p,q)}$ *as follows.*

1. *Stabilizations of* $(L_i)^+_{n(p,q)}$ *and* $(L_i)^-_{n(p,q)}$ *remain distinct until each component of the first link has been* $-$*-stabilized* $|p\,\mathrm{tb}(L) - q|$ *or more times, each component of the second link has been* $+$ *-stabilized* $|p\,\mathrm{tb}(L) - q|$ *or more times, and the classical invariants of the components of one link match those of the other, in this case they are Legendrian isotopic.*

2. *If* $S_+(L_i) = S_-(L_j)$, *then* $(L_i)^+_{n(p,q)}$ *and* $(L_j)^-_{n(p,q)}$ *will have distinct rotation numbers until each component of the first has been* $+$*-stabilized* $|p(\mathrm{tb}(L)-1) - q|$ *times and each component of the second has been* $-$*-stabilized* $|p(\mathrm{tb}(L)-1) - q|$ *times. After each has been stabilized as indicated, they become Legendrian isotopic.*

3. *If* $S_\pm(L_i) = S_\pm(L_j)$, *but Legendrian surgery on* L_i *and* L_j *yield distinct contact manifolds, then stabilizations of* $(L_i)^\pm_{n(p,q)}$ *and* $(L_j)^\pm_{n(p,q)}$ *remain distinct until each*

component of the of both links have been $\pm$-stabilized $|p(\mathrm{tb}(L)-1)-q|$ or more times, and the classical invariants of the components of one link match those of the other, in this case they are Legendrian isotopic.

4. *If $S^l_\pm(L_i)$ is not Legendrian isotopic to $S^l_\pm(L_j)$ for any l, then any number of $\pm$-stabilizations of $(L_i)^\pm_{n(p,q)}$ and $(L_j)^\pm_{n(p,q)}$ remain distinct.*

Like in the case of cable knots, the statements in the theorem do not necessarily classify all Legendrian cable links of uniformly thick knot types, but the theorem and its proof are sufficient in many cases. In particular, we have the following result.

Theorem 3.3.5. *Let K be the $-2n$-twist knot, p and q relatively prime with $-q/p < 1$ and $n > 1$ an integer. Let $L_1, \ldots, L_k$ be the distinct Legendrian knots in $\mathcal{L}(K)$ with $\mathrm{tb} = \lceil -q/p \rceil$. Every element in $\mathcal{L}(K_{n(p,-q)})$ destabilizes to $(L_i)^\pm_{n(p,-q)}$ for some i and $\pm$. Two Legendrian links in $\mathcal{L}(K_{n(p,-q)})$ are Legendrian isotopic if and only if they are component-wise isotopic. The classification of Legendrian realizations of $K_{n(p,q)}$ now follows from Theorem 3.3.2 when $q/p < 0$ and from Theorem 3.3.3 if Legendrian surgery on the maximal Thurston-Bennequin invariant representatives of K are distinct.*

CHAPTER 4
THE COBORDISM MAP

Let L be an exact Lagrangian cobordism from Λ_1 to Λ_2. The map $\Phi_L : (\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2}) \to (\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1})$ defined by Ekholm, Honda, and Kalman [13] has proven useful in distinguishing exact Lagrangian isotopy classes of Lagrangian cobordisms and fillings. Due to the difficult nature of computing Legendrian contact homology, we would like to adapt Φ_L to a map defined on its easier-to-compute linearized version $LCH_*^{\epsilon}(\Lambda)$. In Section 4.1 we do just that, and show its invariance under exact Lagrangian isotopy. Then in Section 4.2 we show that L also induces an A_∞ morphism on the linearized cohomology, $LCH^*_{\epsilon}(\Lambda)$, of its Legendrian ends that is also invariant under exact Lagrangian isotopy.

4.1 Linearizing the cobordism map

In this section, we show how to obtain a map on the linearized contact homology of Legendrian knots from an exact Lagrangian cobordism. We then explore the properties of this map when it is a trivial cylinder or altered through exact Lagrangian cobordisms by an isotopy. All of the results are in joint work with Knavel [39]

Let L be an exact Lagrangian cobordism from Λ_1 to Λ_2, Φ_L its induced DGA map. Let $\epsilon_1 : (\mathcal{A}_{\Lambda_1}, \partial_1) \to (\mathbb{Z}, 0)$ be an augmentation and $\epsilon_2 = \epsilon_1 \circ \Phi_L$. We define the *augmented DGA map*, $\Phi_L^{\epsilon_1} : (\mathcal{A}_{\Lambda_2}, \partial_{\epsilon_2}) \to (\mathcal{A}_{\Lambda_1}, \partial_{\epsilon_1})$, to be $\Phi_L^{\epsilon_1} = \phi^{\epsilon_1} \circ \Phi_L \circ \phi^{-\epsilon_2}$. Recall the map $\phi^{\epsilon_i} : \mathcal{A}_{\Lambda_i} \to \mathcal{A}_{\Lambda_i}$ was defined in Section 2.4.1. We first show that $\Phi_L^{\epsilon_1}$ is a chain map from $(\mathcal{A}_{\Lambda_2}, \partial_{\epsilon_2})$ to $(\mathcal{A}_{\Lambda_1}, \partial_{\epsilon_1})$, where $\partial_{\epsilon_i} = \phi^{\epsilon_i} \circ \partial \circ \phi^{-\epsilon_i}, i = 1, 2$, which we refer to as the *augmented differentials*.

Lemma 4.1.1. *Let L be an exact Lagrangian cobordism from Λ_1 to Λ_2. Then $\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_2} = \partial_{\epsilon_1} \circ \Phi_L^{\epsilon_1}$.*

Proof. A computation gives

$$\begin{aligned}
\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_2} &= \phi^{\epsilon_1} \circ \Phi_L \circ \phi^{-\epsilon_2} \circ \phi^{\epsilon_2} \circ \partial_2 \circ \phi^{-\epsilon_2} \\
&= \phi^{\epsilon_1} \circ \Phi_L \circ \partial_2 \circ \phi^{-\epsilon_2} \\
&= \phi^{\epsilon_1} \circ \partial_1 \circ \Phi_L \circ \phi^{-\epsilon_2} \\
&= \phi^{\epsilon_1} \circ \partial_1 \circ \phi^{-\epsilon_1} \circ \phi^{\epsilon_1} \circ \Phi_L \circ \phi^{-\epsilon_2} \\
&= \partial_{\epsilon_1} \circ \Phi_L^{\epsilon_1},
\end{aligned}$$

where the third equality follows from Theorem 2.4.2 Thus we see that $\Phi_L^{\epsilon_1}$ commutes with the augmented differentials. $\square$

In order to 'linearize' the equation in Lemma 4.1.1, we will need the following two lemmas. The first shows that $\Phi_L^{\epsilon_1}$ has no constant terms.

Lemma 4.1.2. *Let c be a Reeb chord. Then $\Phi_L^{\epsilon_1}(c)$ has no constant terms.*

Proof. Let $\mathcal{C}_{\Lambda_1} = \{c_1, \dots, c_m\}$ be the Reeb chords of Λ_1, c a Reeb chord of Λ_2. We start by showing the constant terms of $\phi^{\epsilon_1} \circ \Phi_L(c)$ are $\epsilon_1 \circ \Phi_L(c)$. There are constants $a_0, a_{i_1,\dots,i_j} \in \mathbb{Z}$ for $j \geq 1$ and $i_k \in \{1, \dots, n\}$, with only finitely many being nonzero such that

$$\Phi_L(c) = a_0 + \sum_{j=1}^{\infty} \left(\sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} c_{i_1} \cdots c_{i_j} \right).$$

Then

$$\phi^{\epsilon_1} \circ \Phi_L(c) = a_0 + \sum_{j=1}^{\infty} \left(\sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} (c_{i_1} + \epsilon(c_{i_1})) \cdots (c_{i_j} + \epsilon_1(c_{i_j})) \right).$$

The constant terms of this sum are

$$\phi^{\epsilon_1} \circ \Phi_L(c) = a_0 + \sum_{j=1}^{\infty} \left(\sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} \epsilon(c_{i_1}) \cdots \epsilon_1(c_{i_j}) \right) = \epsilon_1 \circ \Phi_L(c).$$

Now we show $\Phi_L^{\epsilon_1}(c)$ has no constant terms. We have

$$\begin{aligned}\Phi_L^{\epsilon_1}(c) &= \phi^{\epsilon_1} \circ \Phi_L \circ \phi^{-\epsilon_2}(c) \\ &= \phi^{\epsilon_1} \circ \Phi_L(c - \epsilon_2(c)) \\ &= \phi^{\epsilon_1}(\Phi_L(c) - \Phi_L(\epsilon_2(c))) \\ &= \phi^{\epsilon_1}(\Phi_L(c) - (\epsilon_2(c))) \\ &= \phi^{\epsilon_1}(\Phi_L(c) - (\epsilon_1 \circ \Phi_L(c))) \\ &= \phi^{\epsilon_1} \circ \Phi_L(c) - \epsilon_1 \circ \Phi_L(c).\end{aligned}$$

As shown in the previous paragraph, the constant terms of $\phi^{\epsilon_1} \circ \Phi_L(c)$ are exactly $\epsilon_1 \circ \Phi_L(c)$.

□

It is well-known that the augmented differentials satisfy the Leibniz rule. We provide a proof of this here as we will need it below, and it does not appear in the literature.

Lemma 4.1.3. *The augmented boundary map ∂_ϵ satisfies $\partial_\epsilon(c_1 \cdots c_m) = \partial_\epsilon(c_1)c_2 \cdots c_m + (-1)^{|c_1|}c_1\partial_\epsilon(c_2)c_3 \cdots c_m + \ldots + (-1)^{|c_1 \cdots c_{m-1}|}c_1 \cdots c_{m-1}\partial_\epsilon(c_m)$.*

Proof. We first show

$$\partial(\phi^{-\epsilon}(c_1 \cdots c_m)) = \sum_{i=1}^{m}(-1)^{|c_1 \cdots c_{i-1}|}\phi^{-\epsilon}(c_1 \cdots c_{i-1})\partial(\phi^{-\epsilon}(c_i))\phi^{-\epsilon}(c_{i+1} \cdots c_m)$$

by induction. Assume true for $m-1$. Then for a word of length m we have

$$\begin{aligned}\partial(\phi^{-\epsilon}(c_1 \cdots c_m)) &= \partial((c_1 - \epsilon_1(c_1))\phi^{-\epsilon}(c_2 \cdots c_m)) \\ &= \partial(c_1\phi^{-\epsilon}(c_2 \cdots c_m)) - \epsilon(c_1)\partial(\phi^{-\epsilon}(c_2 \cdots c_m)).\end{aligned}$$

Let $\sum_{i=1}^{n} w_i$ be the sum of words of length at most $m-1$ after expanding $\phi^{-\epsilon}(c_2 \cdots c_m)$.

Then

$$\begin{aligned}
\partial(c_1\phi^{-\epsilon}(c_2\cdots c_m)) &= \partial(c_1\sum_{i=1}^{n} w_i) \\
&= \sum_{i=1}^{n} \partial(c_1 w_i) \\
&= \sum_{i=1}^{n} \partial(c_1)w_i + (-1)^{|c_1|}c_1\partial(w_i) \\
&= \partial(c_1)\sum_{i=1}^{n} w_i + (-1)^{|c_1|}c_1\partial(\sum_{i=1}^{n} w_i).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\partial(c_1\phi^{-\epsilon}(c_2\cdots c_m)) - \epsilon(c_1)\partial(\phi^{-\epsilon}(c_2\cdots c_m)) \\
&= \partial(c_1)\phi^{-\epsilon}(c_2\cdots c_m) + (-1)^{|c_1|}c_1\partial(\phi^{-\epsilon}(c_2\cdots c_m)) - \epsilon(c_1)\partial(\phi^{-\epsilon}(c_2\cdots c_m)) \\
&= \partial(\phi^{-\epsilon}(c_1))\phi^{-\epsilon}(c_2\cdots c_m) + (-1)^{|c_1|}\phi^{-\epsilon}(c_1)\partial(\phi^{-\epsilon}(c_2\cdots c_m)),
\end{aligned}$$

where $(-1)^{|c_1|}c_1 - \epsilon(c_1) = (-1)^{|c_1|}\phi^{-\epsilon}(c_1)$ because ϵ vanishes on nonzero graded Reeb chords. Since we assumed true for $m-1$, the last term becomes

$$\sum_{i=2}^{m}(-1)^{|c_1\cdots c_{i-1}|}\phi^{-\epsilon}(c_1\cdots c_{i-1})\partial(\phi^{-\epsilon}(c_i))\phi^{-\epsilon}(c_{i+1}\cdots c_m),$$

and so we get the desired equality.

Now we prove the lemma. We compute that

$$\begin{aligned}
\partial_\epsilon(c_1...c_m) &= \phi^\epsilon\circ\partial\circ\phi^{-\epsilon}(c_1...c_m) \\
&= \phi^\epsilon\circ(\sum_{i=1}^{m}(-1)^{|c_1\cdots c_{i-1}|}\phi^{-\epsilon}(c_1\cdots c_{i-1})\partial(\phi^{-\epsilon}(c_i))\phi^{-\epsilon}(c_{i+1}\cdots c_m) \\
&= \sum_{i=1}^{m}(-1)^{|c_1\cdots c_{i-1}|}\phi^\epsilon\circ\phi^{-\epsilon}(c_1\cdots c_{i-1})\partial_\epsilon(c_i)\phi^\epsilon\circ\phi^{-\epsilon}(c_{i+1}\cdots c_m) \\
&= \partial^\epsilon(c_1)c_2\cdots c_m + \ldots + (-1)^{|c_1\cdots c_{m-1}|}c_1\cdots c_{m-1}\partial_\epsilon(c_m)
\end{aligned}$$

$\square$

We now consider a linearized version of $\Phi_L^{\epsilon_1}$. Let $(F_i, \partial_{\epsilon_i}^\ell)$ denote the linearized chain groups as defined in Section 2.4.1. We define the *linearized augmented DGA map* to be

$$(\Phi_L^{\epsilon_1})^\ell : (F_2, \partial_{\epsilon_2}^\ell) \to (F_1, \partial_{\epsilon_1}^\ell),$$

where $(\Phi_L^{\epsilon_1})^\ell(c)$ for a generator c of F_2 is the linear terms in $\Phi_L^{\epsilon_1}(c)$. Now we show that $(\Phi_L^{\epsilon_1})^\ell$ is a DGA map.

Proof of Proposition 1.2.1. Let c be a Reeb chord in Λ_2. We begin by observing that the linear part of $\Phi_L^{\epsilon_1}(\partial_{\epsilon_2}(c))$ is the linear part of the image of the linear part of $\partial_{\epsilon_2}(c)$ under $\Phi_L^{\epsilon_1}$. Let $\mathcal{C}_{\Lambda_1} = \{c_1, \ldots, c_m\}$ be the Reeb chords of Λ_1, and $a_{i_1,\ldots,i_j} \in \mathbb{Z}$. Recall that $\partial_{\epsilon_1}(c)$ has no constant terms. If

$$\partial_{\epsilon_2}(c) = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a_{i_1,\ldots,i_j} c_{i_1} \cdots c_{i_j} \right),$$

with only finitely many i_k nonzero, then

$$\Phi_L^{\epsilon_1}(\partial_{\epsilon_2}(c)) = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a_{i_1,\ldots,i_j} \Phi_L^{\epsilon_1}(c_{i_1}) \cdots \Phi_L^{\epsilon_1}(c_{i_j}) \right).$$

We know that for $j > 1$, $\Phi_L^{\epsilon_1}(c_{i_1}) \cdots \Phi_L^{\epsilon_1}(c_{i_j})$ must be a sum of words all of length greater than 1 due to Lemma 4.1.2. So all of the linear parts of $\Phi_L^{\epsilon_1}(\partial_{\epsilon_2}(c))$ must come from terms when $j = 1$, i.e. the linear part of $\partial_{\epsilon_2}(c)$. We have just shown that

$$(\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_1})^\ell(c) = (\Phi_L^{\epsilon_2})^\ell \circ \partial_{\epsilon_2}^\ell(c).$$

Now let

$$\Phi_L^{\epsilon_1}(c) = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a'_{i_1,\ldots,i_j} c'_{i_1} \cdots c'_{i_j} \right),$$

with only finitely many i_k nonzero, where $c'_{i_1}, \ldots, c'_{i_j} \in \mathcal{C}_{\Lambda_1}$ and $a'_{i_1,\ldots,i_j} \in \mathbb{Z}$. Then taking the augmented differential ∂_{ϵ_1} of both sides gives

$$\partial_{\epsilon_1}(\Phi_L^{\epsilon_1}(c)) = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a'_{i_1,\ldots,i_j} \partial_{\epsilon_1}(c'_{i_1} \cdots c'_{i_j}) \right).$$

By Lemma 4.1.3 any word of length $j \geq 2$ will be mapped to a sum of words each of which has $j-1$ Reeb chords multiplied by some $\partial_{\epsilon_1}(c_i)$. Since $\partial_{\epsilon_1}(c_i)$ has no constant terms, these words are length at least 2. Thus the linear part of $\partial_{\epsilon_1}(\Phi_L^{\epsilon_1}(c))$ is the linear part of the image of the linear part of $\Phi_L^{\epsilon_1}(c)$ under ∂_{ϵ_1}, and we conclude that

$$(\partial_{\epsilon_1} \circ \Phi_L^{\epsilon_1})^{\ell}(c) = \partial_{\epsilon_1}^{\ell} \circ (\Phi_L^{\epsilon_1})^{\ell}(c).$$

Using the facts proven in the previous paragraphs and Lemma 4.1.1, we have

$$\begin{aligned} \partial_{\epsilon_1}^{\ell} \circ (\Phi_L^{\epsilon_1})^{\ell}(c) &= (\partial_{\epsilon_1} \circ \Phi_L^{\epsilon_1})^{\ell}(c) \\ &= (\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_2})^{\ell}(c) \\ &= (\Phi_L^{\epsilon_1})^{\ell} \circ \partial_{\epsilon_2}^{\ell}(c). \end{aligned}$$

Thus $(\Phi_L^{\epsilon_1})^{\ell}$ is a chain map. □

Now that we have shown the linearized augmented DGA map $(\Phi_L^{\epsilon_1})^{\ell}$ is indeed a well-defined DGA map, we would like to consider how different geometric phenomena, such as Lagrangian isotopy, affect $(\Phi_L^{\epsilon_1})^{\ell}$. We do so by proving a linearized version of Theorem 2.4.2 parts 1) and 2). Let $\mathcal{C}_{\Lambda_2}$ be the set of Reeb chords of Λ_2. In [13], the chain homotopy in part 2) of Theorem 2.4.2 is defined on words of Reeb chords as the $\mathbb{Z}/2\mathbb{Z}$-

linear map

$$\Omega(c_1...c_m) = \sum_{j=1}^{m} \Phi_{L_1}(c_1...c_{j-1})K(c_j)\Phi_{L_2}(c_{j+1}...c_m)$$

where $c_i \in \mathcal{C}_{\Lambda_2}$ for $1 \leq i \leq m$ and K is a degree 1 map that maps c_j to a finite sum of words in Reeb chords of degree $|c_j| + 1$. For our Theorem 1.2.2, we define a linearized version of Ω to use as our chain homotopy in the following lemma. In order to do this we also define a linearized version of K by $K^{\epsilon_1} = \phi^{\epsilon_1} \circ K \circ \phi^{-\epsilon_2}$.

Lemma 4.1.4. *Let $c_1 \cdots c_m$ be a word in $(\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2})$, and define Ω^{ϵ_1} by*

$$\Omega^{\epsilon_1}(c_1 \cdots c_m) = \sum_{j=1}^{m} \Phi^{\epsilon_1}_{L_1}(c_1 \cdots c_{j-1})K^{\epsilon_1}(c_j)\Phi^{\epsilon_1}_{L_2}(c_{j+1} \cdots c_m).$$

Then

$$\Omega^{\epsilon_1}(c_1 \cdots c_m) = \phi^{\epsilon_1} \circ \Omega \circ \phi^{-\epsilon_2}(c_1 \cdots c_m).$$

Proof. Let $c_i \in \mathcal{C}_{\Lambda_2}$ for $1 \leq i \leq m$. First note that $\Omega^{\epsilon_1}(c_1 \cdots c_m)$

$$\begin{aligned}
&= \sum_{j=1}^{m} (\phi^{\epsilon_1} \circ \Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1}))(\phi^{\epsilon_1} \circ K \circ \phi^{-\epsilon_2}(c_j))(\phi^{\epsilon_1} \circ \Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m)) \\
&= \sum_{j=1}^{m} \phi^{\epsilon_1} \circ (\Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1})K \circ \phi^{-\epsilon_2}(c_j)\Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m)) \\
&= \phi^{\epsilon_1} \circ (\sum_{j=1}^{m} \Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1})K \circ \phi^{-\epsilon_2}(c_j)\Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m)).
\end{aligned}$$

So to prove the lemma we only need to show

$$\Omega \circ \phi^{-\epsilon_2}(c_1 \cdots c_m) = \sum_{j=1}^{m} \Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1})K \circ \phi^{-\epsilon_2}(c_j)\Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m).$$

We do so by showing any word on the left exists uniquely in the sum on the right, and vice versa.

Let

$$\phi^{-\epsilon_2}(c_1\cdots c_m) = c_1\cdots c_m + \sum_{i=1}^{m} c_1\cdots \hat{c}_i \cdots c_m + \sum_{i=1}^{m-1}\sum_{k>i}^{m} c_1\cdots \hat{c}_i\cdots \hat{c}_k\cdots c_m + \ldots + \hat{c}_1\cdots \hat{c}_m$$

where $\hat{c}_i = -\epsilon_2(c_i)$. This is a sum of words of length at most m. Let w_n be a word in this sum with $n \leq m$ hatted Reeb chords. Then

$$\Omega(w_n) = (-1)^n \epsilon_2(c_{i_1}\cdots c_{i_n}) \sum_{j=1}^{m-n} \Phi_{L_1}(c_{i_{n+1}}\cdots c_{i_{n+j-1}})K(c_{i_{n+j}})\Phi_{L_2}(c_{i_{n+j+1}}\cdots c_{i_m}).$$

Now fix j. Let $i_r < i_{n+j}$ be the largest index such that it's Reeb chord is hatted, and i_s, $i_{n+j} < i_s$, be the smallest such that its Reeb chord is hatted as well, with $r + s = n$. Then $(-1)^r\epsilon_2(c_{i_1}\cdots c_{i_r})\Phi_{L_1}(c_{i_{n+1}}\cdots c_{i_{n+j-1}})$ is a unique term in $\Phi_{L_1}\circ\phi^{-\epsilon_2}(c_1\cdots c_{i_{n+j}-1})$, while $K(c_{i_{n+j}}) = K\circ\phi^{-\epsilon_2}(c_{i_{n+j}})$, and $(-1)^s\epsilon_2(c_{i_s}\cdots c_m)\Phi_{L_2}(c_{i_{n+j+1}}\cdots c_m)$ is a unique term in $\Phi_{L_2}\circ\phi^{-\epsilon_2}(c_{i_{n+j}+1}\cdots c_m)$. Therefore their concatenation is a unique word in

$$\Phi_{L_1}\circ\phi^{-\epsilon_2}(c_1\cdots c_{i_{n+j}-1})K\circ\phi^{-\epsilon_2}(c_{i_{n+j}})\Phi_{L_2}\circ\phi^{-\epsilon_2}(c_{i_{n+j}+1}\cdots c_m).$$

Thus any word in $\Omega\circ\phi^{-\epsilon_2}(c_1\cdots c_m)$ is a unique word in

$$\sum_{j=1}^{m}\Phi_{L_1}\circ\phi^{-\epsilon_2}(c_1\cdots c_{j-1})K\circ\phi^{-\epsilon_2}(c_j)\Phi_{L_2}\circ\phi^{-\epsilon_2}(c_{j+1}\cdots c_m).$$

Now consider a word in the sum

$$\sum_{j=1}^{m}\Phi_{L_1}\circ\phi^{-\epsilon_2}(c_1\cdots c_{j-1})K\circ\phi^{-\epsilon_2}(c_j)\Phi_{L_2}\circ\phi^{-\epsilon_2}(c_{j+1}\cdots c_m).$$

For a fixed j, we have

$$\begin{aligned}
&\Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1}) K \circ \phi^{-\epsilon_2}(c_j) \Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m) \\
&= (-1)^r \epsilon_2(c_{i_1} \cdots c_{i_r}) \Phi_{L_1}(c_1 \cdots c_{j-1}) K(c_j) (-1)^{s-r} \epsilon_2(c_{i_{r+1}} \cdots c_s) \Phi_{L_2}(c_{j+1} \cdots c_m) \\
&= (-1)^s \epsilon_2(c_{i_1} \cdots c_{i_s}) \Phi_{L_1}(c_1 \cdots c_{j-1}) K(c_j) \Phi_{L_2}(c_{j+1} \cdots c_m),
\end{aligned}$$

which is a unique word in the sum $(-1)^s \epsilon_2(c_{i_1} \cdots c_{i_s}) \Omega(w_s)$, where w_s is the word of unhatted Reeb chords in some word in $\phi^{-\epsilon_2}(c_1 \cdots c_m)$ whose hatted Reeb chords are $c_{i_1}, \ldots, c_{i_s}$. Thus any word in $\sum_{j=1}^m \Phi_{L_1} \circ \phi^{-\epsilon_2}(c_1 \cdots c_{j-1}) K \circ \phi^{-\epsilon_2}(c_j) \Phi_{L_2} \circ \phi^{-\epsilon_2}(c_{j+1} \cdots c_m)$ is a unique word in $\Omega \circ \phi^{-\epsilon_2}(c_1 \cdots c_m)$, and so the lemma follows. $\square$

Now that we have shown that $\Omega^{\epsilon_1} = \phi^{\epsilon_1} \circ \Omega \circ \phi^{-\epsilon_2}$, we can prove the linearized version of parts 1) and 2) of Theorem 2.4.2. We remind the reader that the exponent ℓ means taking the linear terms of the expression.

Proof of Theorem 1.2.2. For Item 1) in the theorem, let c be a Reeb chord of Λ. Then

$$\begin{aligned}
(\Phi_L^{\epsilon_1})^\ell(c) &= (\phi^{\epsilon_1} \circ \Phi_L \circ \phi^{-\epsilon_2})^\ell(c) \\
&= (\phi^{\epsilon_1} \circ \Phi_L)^\ell(c - \epsilon_2(c)) \\
&= (\phi^{\epsilon_1})^\ell(c - \epsilon_2(c)) \\
&= (c + \epsilon_1(c) - \epsilon_2(c))^\ell \\
&= c.
\end{aligned}$$

To prove Statement 2), we start with the homotopy equation from [13].

$$\Phi_{L_1} - \Phi_{L_2} = \Omega \circ \partial_2 + \partial_1 \circ \Omega,$$

which is equivalent to

$$\Phi_{L_1}^{\epsilon_1} - \Phi_{L_2}^{\epsilon_1} = \phi^{\epsilon_1} \circ \Omega \circ \phi^{-\epsilon_2} \circ \phi^{\epsilon_2} \circ \partial_2 \circ \phi^{-\epsilon_2} + \phi^{\epsilon_1} \circ \partial_1 \circ \phi^{-\epsilon_1} \circ \phi^{\epsilon_1} \circ \Omega \circ \phi^{-\epsilon_2}.$$

By Lemma 4.1.4, this is equivalent to

$$\Phi_{L_1}^{\epsilon_1} - \Phi_{L_2}^{\epsilon_1} = \Omega^{\epsilon_1} \circ \partial_{\epsilon_2} + \partial_{\epsilon_1} \circ \Omega^{\epsilon_1}.$$

Linearizing both sides yields

$$(\Phi_{L_1}^{\epsilon_1})^{\ell} - (\Phi_{L_2}^{\epsilon_1})^{\ell} = (\Omega^{\epsilon_1} \circ \partial_{\epsilon_2})^{\ell} + (\partial_{\epsilon_1} \circ \Omega^{\epsilon_1})^{\ell}.$$

We need to show that the linear terms of the compositions $\Omega^{\epsilon_1} \circ \partial_{\epsilon_2}$ and $\partial_{\epsilon_1} \circ \Omega^{\epsilon_1}$ equal the composition of their linearizations, respectively. Let $a_{i_1,\ldots,i_j} \in \mathbb{Z}$ and $\mathcal{C}_{\Lambda_2} = \{c_1, \ldots, c_n\}$ be Reeb chords in Λ_2 such that

$$w = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a_{i_1,\ldots,i_j} c_{i_1} \cdots c_{i_j} \right) \in \mathrm{Im}\partial_{\epsilon_2}.$$

We must show words of length greater than 1 are mapped to words also of length greater than 1. We have that

$$\Omega^{\epsilon_1}(w) = \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j} a_{i_1,\ldots,i_j} \Omega^{\epsilon_1}(c_{i_1} \cdots c_{i_j}) \right).$$

The image of words of Reeb chords of length greater than 2 under Ω^{ϵ_1} will be a sum of words that each have $\Phi_{L_i}^{\epsilon_1}$ applied to at least two of the letters, which have no constant terms by Lemma 4.1.2. Thus those words will be length greater than 1. Now consider the images of the quadratic terms $a_{i_1,i_2}\Omega^{\epsilon_1}(c_{i_1}c_{i_2}) = a_{i_1,i_2}(\Phi_{L_1}^{\epsilon_1}(c_{i_1})K^{\epsilon_1}(c_{i_2}) + K^{\epsilon_1}(c_{i_1})\Phi_{L_2}^{\epsilon_1}(c_{i_2}))$. Since none of the Reeb chords in the image of ∂_{ϵ_2} have grading -1, $K \circ \phi^{-\epsilon_2}(c_{i_1})$ and $K \circ \phi^{-\epsilon_2}(c_{i_2})$ are both sums of words that have nonzero grading. Therefore all of those words in their sums must have a nonzero graded Reeb chord, and since ϕ^{ϵ_1} is the identity on generators with nonzero grading, $\phi^{\epsilon_1} \circ K \circ \phi^{-\epsilon_2}(c_{i_1})$ and $\phi^{\epsilon_1} \circ K \circ \phi^{-\epsilon_2}(c_{i_2})$ have no constant terms.

We have just shown that

$$(\Omega^{\epsilon_1} \circ \partial_{\epsilon_2})^\ell = (\Omega^{\epsilon_1})^\ell \circ \partial_{\epsilon_2}^\ell.$$

By the same argument used in the proof of Proposition 1.2.1, $(\partial_{\epsilon_1} \circ \Omega^{\epsilon_1})^\ell = \partial_{\epsilon_1}^\ell \circ (\Omega^{\epsilon_1})^\ell$, and so we have the chain homotopy equation:

$$(\Phi_{L_1}^{\epsilon_1})^\ell - (\Phi_{L_2}^{\epsilon_1})^\ell = (\Omega^{\epsilon_1})^\ell \circ \partial_{\epsilon_2}^\ell + \partial_{\epsilon_1}^\ell \circ (\Omega^{\epsilon_1})^\ell.$$

$\square$

Remark 4.1.5. In the proof of statement 2), we impose the condition that the quadratics $c_i c_j$ in the image of ∂_{ϵ_2} not have any Reeb chords of grading -1 because $K(c_j)$ could have a word of Reeb chords in its sum that all have grading zero and are mapped to a nonzero constant by ϵ_1. This in turn would result in a contant term in the sum $K^{\epsilon_1}(c_j)$, and thus $\Phi_{L_1}^{\epsilon_1}(c_i)K^{\epsilon_1}(c_j)$ could have a linear term, obstucting the desired equality $(\Omega^{\epsilon_1} \circ \partial_{\epsilon_2})^\ell = (\Omega^{\epsilon_1})^\ell \circ (\partial_{\epsilon_2})^\ell$.

4.2 Preserving the A_∞ structure

Let $((\Phi_L^{\epsilon_1})^k)^*$ be the adjoint of the map $(\Phi_L^{\epsilon_1})^k : F_{\Lambda_2} \to F_{\Lambda_1}^{\otimes k}$ that sends Reeb chords c to the length k words in the sum $\Phi_L^{\epsilon_1}(c)$. We will show the collection $\{((\Phi_L^{\epsilon_1})^k)^*\}_{k\geq 1}$ is an A_∞ morphism. This was shown by Pan in [19], but we will show it using an argument on the chain level similar to how we did in Section 4.1. We first prove the following two lemmas. We remind the reader that $\partial_{\epsilon_i}^k(c)$ denotes the length k terms in the sum $\partial_{\epsilon_i}(c)$, $i = 1, 2$. For this section we are working in $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Lemma 4.2.1. *Let L be an exact Lagrangian cobordisms from Λ_1 to Λ_2, and assume $j < k$. Then*

$$(\partial_{\epsilon_1} \circ (\Phi_L^{\epsilon_1})^j)^k(c) = \sum_{i+n+1=j} (1^{\otimes i} \otimes \partial_{\epsilon_1}^{k-j+1} \otimes 1^{\otimes n}) \circ (\Phi_L^{\epsilon_1})^j(c)$$

Proof. Let

$$(\Phi_L^{\epsilon_1})^j(c) = \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} c_{i_1} \cdots c_{i_j}.$$

Then

$$\begin{aligned}\partial_{\epsilon_1} \circ (\Phi_L^{\epsilon_1})^j(c) &= \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} \partial_{\epsilon_1}(c_{i_1} \cdots c_{i_j}) \\ &= \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} (\partial_{\epsilon_1}(c_{i_1}) c_{i_2} \cdots c_{i_j} + \ldots + c_{i_1} c_{i_2} \cdots \partial_{\epsilon_1}(c_{i_j})).\end{aligned}$$

The length k words of this sum will be exactly

$$\sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} (\partial_{\epsilon_1}^{k-j+1}(c_{i_1}) c_{i_2} \cdots c_{i_j} + \ldots + c_{i_1} c_{i_2} \cdots \partial_{\epsilon_1}^{k-j+1}(c_{i_j})),$$

or

$$\sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} (\partial_{\epsilon_1}^{k-j+1} \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes \partial_{\epsilon_1}^{k-j+1})(c_{i_1} \cdots c_{i_j}),$$

which can be rewritten as

$$\sum_{i+1+n=j} \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} (1^{\otimes i} \otimes \partial_{\epsilon_1}^{k-j+1} \otimes 1^{\otimes n})(c_{i_1} \cdots c_{i_j}),$$

and the lemma follows. □

Lemma 4.2.2. *Let L be an exact Lagrangian cobordisms from Λ_1 to Λ_2, and assume $j < k$. Then*

$$((\Phi_L^{\epsilon_1}) \circ \partial_{\epsilon_2}^j)^k(c) = \sum_{k_1+\ldots+k_j=k} ((\Phi_L^{\epsilon_1})^{k_1} \otimes \ldots \otimes (\Phi_L^{\epsilon_1})^{k_j}) \circ \partial_{\epsilon_2}^j(c)$$

.

Proof. Let

$$\partial_{\epsilon_2}^j(c) = \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} c_{i_1} c_{i_2} \cdots c_{i_j}.$$

Then

$$\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_2}^j(c) = \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} \Phi_L^{\epsilon_1}(c_{i_1} c_{i_2} \cdots c_{i_j}) = \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_j} \Phi_L^{\epsilon_1}(c_{i_1}) \cdots \Phi_L^{\epsilon_1}(c_{i_j}).$$

Then length k words coming from this product will be exactly those resulting from

$$(\Phi_L^{\epsilon_1})^{k_1}(c_{i_1})(\Phi_L^{\epsilon_1})^{k_2}(c_{i_2}) \cdots (\Phi_L^{\epsilon_1})^{k_j}(c_{i_j}),$$

where $k_1 + \ldots + k_j = k$. Summing over all such $k_1, \ldots, k_j$ proves the lemma. $\square$

We now show $\{((\Phi_L^{\epsilon_1})^k)^*\}_{k \geq 1}$ is an A_∞ morphism. We denote the adjoint of $\partial_{\epsilon_i}^k$ by $m_k^{\epsilon_i}$, $i = 1, 2$.

Proof of Theorem 1.2.4. We first consider the chain groups and then dualize. Consider the equation

$$(\partial_{\epsilon_1} \circ \Phi_L^{\epsilon_1})^k(c) = (\Phi_L^{\epsilon_1} \circ \partial_{\epsilon_2})^k(c).$$

This can be rewritten

$$\sum_{j=1}^{k} (\partial_{\epsilon_1} \circ (\Phi_L^{\epsilon_1})^j)^k(c) = \sum_{j=1}^{k} ((\Phi_L^{\epsilon_1}) \circ \partial_{\epsilon_2}^j)^k(c)$$

because if $j > k$, then the Leibniz rule for ∂_{ϵ_1} and $\Phi_L^{\epsilon_1}$ being a homomorphism combined with there being no constants in their images imply that a word of length $j > k$ would be sent to a sum of words of length greater than k. By the lemmas above, this can be rewritten

$$\sum_{j=1}^{k} \sum_{i+n+1=j} (1^{\otimes i} \otimes \partial_{\epsilon_1}^{k-j+1} \otimes 1^{\otimes n}) \circ (\Phi_L^{\epsilon_1})^j(c) = \sum_{j=1}^{k} \sum_{k_1+\ldots+k_j=k} ((\Phi_L^{\epsilon_1})^{k_1} \otimes \cdots \otimes (\Phi_L^{\epsilon_1})^{k_j}) \circ \partial_{\epsilon_2}^j(c).$$

Dualizing yields the equation

$$\sum_{j=1}^{k} \sum_{i+n+1=j} ((\Phi_L^{\epsilon_1})^j)^* \circ (1^{\otimes i} \otimes m_{k-j+1}^{\epsilon_1} \otimes 1^{\otimes n}) = \sum_{j=1}^{k} \sum_{k_1+\ldots+k_j=k} m_j^{\epsilon_2} \circ (((\Phi_L^{\epsilon_1})^{k_1})^* \otimes \cdots \otimes ((\Phi_L^{\epsilon_1})^{k_j})^*).$$

Reindexing the left side gives the equation

$$\sum_{i+p+n=k}((\Phi_L^{\epsilon_1})^{i+n+1})^*\circ(1^{\otimes i}\otimes m_p^{\epsilon_1}\otimes 1^{\otimes n})=\sum_{j=1}^{k}\sum_{k_1+\ldots+k_j=k} m_j^{\epsilon_2}\circ(((\Phi_L^{\epsilon_1})^{k_1})^*\otimes\cdots\otimes((\Phi_L^{\epsilon_1})^{k_j})^*).$$

Hence $\{((\Phi_L^{\epsilon_1})^k)^* : (F_1^*)^{\otimes k}\to F_2^*\}_{k\geq 1}$ is an A_∞ morphism. □

Now that we have shown an A_∞ morphism between $(F_1^*, \{m_k^{\epsilon_1}\}_{k\geq 1})$ and $(F_2^*, \{m_k^{\epsilon_2}\}_{k\geq 1})$, let us consider the induced maps on their higher order product structures by two Lagrangian isotopic cobordisms. If two exact Lagrangian cobordisms from Λ_1 to Λ_2 are Lagrangian isotopic (and satisfy the conditions in Theorem 1.2.2), then we can dualize the chain homotopy equation from Theorem 1.2.2 to give us a chain homotopy equation between $((\Phi_{L_1}^{\epsilon_1})^\ell)^*$ and $((\Phi_{L_2}^{\epsilon_2})^\ell)^*$. Since the product structures on $(F_1^*, \{m_k^{\epsilon_1}\}_{k\geq 1})$ and $(F_2^*, \{m_k^{\epsilon_2}\}_{k\geq 1})$ are defined on quotients of the cohomologies of $(F_1^*, (\partial_{\epsilon_1}^\ell)^*)$ and $(F_2^*, (\partial_{\epsilon_2}^\ell)^*)$, the chain homotopy equation between $((\Phi_{L_1}^{\epsilon_1})^\ell)^*$ and $((\Phi_{L_2}^{\epsilon_2})^\ell)^*$ implies their induced maps on these product structures are equal. Thus we have the following corollary.

Corollary 4.2.3. *Let L_1 be Lagrangian isotopic to L_2, ϵ_1 an augmentation for Λ_1, $\epsilon_1\circ\Phi_{L_1}=\epsilon_1\circ\Phi_{L_2}$, and the quadratics in the image of ∂_{ϵ_2} have no degree -1 Reeb chords. Then the induced maps of $((\Phi_{L_1}^{\epsilon_1})^\ell)^*$ and $((\Phi_{L_2}^{\epsilon_1})^\ell)^*$ are equal on the higher order product structures.*

4.3 Forward directions

Sabloff and Ma have shown the A_∞ structures of its Legendrian ends can be equipped with a *weak relative Calabi-Yau* structure [40], and Ng has shown the existence of an L_∞ structure on Legendrian contact homology [41]. We hope to show in future work that Φ_L can be modified to preserve these structures as well, and that it is also invariant under Lagrangian isotopy.

Another useful algebraic object extracted from the DGA of a Legendrian knot is the *characteristic algebra* defined by Ng in [26]. There it is shown that the characteristic algebra is a Legendrian isotopy class invariant. This invariant is used to distinguish Legendrian representatives of the same smooth knot type. It is easy to show that Φ_L also induces a well-defined map on the characteristic algebra. In future work, we hope to show that this induced map is also invariant under Lagrangian isotopy.

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