Braids & Contact Topology

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Outline

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Intro To Contact Structures

A hyperplane field $\xi^{2n}$ on a manifold $M^{2n+1}$ is called a contact structure if there is (at least locally) a 1-form $\alpha$ such that

$$\xi = \ker \alpha$$

$$\alpha \wedge d\alpha \wedge \ldots \wedge d\alpha \neq 0$$

$n$ times

Example:

On $\mathbb{R}^{2n+1}$ let $\alpha = dz - \sum_{i=1}^{n} y_i \, dx_i$, and $\xi_{\text{std}} = \ker \alpha$

$$= \text{span} \left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} \right\}$$

Darboux:

All contact structures look locally like this one
let \((M,i)\) be a contact manifold

\[ L^n \subset M^{2n+1} \] is a **Legendrian submanifold** if

\[ T_x L \subset T_x M \quad \forall x \in L \]

**Example:** in \((\mathbb{R}^3, i_{\text{std}})\) project \(L\) to the \(xz\)-plane

\[ \gamma = \frac{dz}{dx} \quad \text{on} \quad L \]

this is called the front projection (notice resemblance to wave fronts)

**Thm:** any arc in a contact 3-manifold can be \(C^0\)-approximated by a Legendrian curve (rel end pts)
A natural occurrence of contact structures

consider the configuration space of a skate (or front wheel of a car)

\[(x, y)\] determine the position of the skate in the plane
\[\theta\] determines the angle it forms with the \(x\)-axis

so the configuration space is

\[\mathcal{W} = \mathbb{R}^2 \times S^1\]

**note:**
1) At a fixed point the skate can point in any direction
2) Skate can only move in the direction it is pointing
   (we assume it does not scrape)
So if \( \gamma(t) = (x(t), y(t), \theta(t)) \) is a motion of the skate then
\[
\frac{y'(t)}{x'(t)} = \tan \theta(t)
\]
if we set \( \mathfrak{F} = \ker(\cos \theta dy - \sin \theta dx) \)
then \( \mathfrak{F} \) is a contact structure on \( W \) and
\( \gamma \) is the motion of a skate
\( \iff \)
\( \gamma \) parameterizes a Legendrian curve

Application:
You can always (in theory) parallel park your car.

Legendrian approximation

desired path
Other Occurrences of Contact Structures

- **PDE** (Sophus Lie 1872)
  
  given \( F : \mathbb{R}^{2n+1} \rightarrow \mathbb{R} \)

  finding \( z : \mathbb{R}^n \rightarrow \mathbb{R} \) solving

  \[
  F(x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}, z) = 0
  \]

  is equivalent to finding

  \( u : \mathbb{R}^n \rightarrow \mathbb{R}^{2n+1} \) s.t.

  \[
  F_{ou} = 0
  \]

  \( \text{Image}(u) \) is Legendrian in \( \mathbb{R}^{2n+1} \)

- **Riemannian Geometry**

  \( g \) metric on \( M \)

  \[
  T^*M \cong T^M \quad \text{under } g
  \]

  \[
  S(TM) \quad \text{Geodesic flow}
  \]

  \[
  S(T^*M) \quad \text{Reeb flow}
  \]
• **Optics** via Huygen's Principle

• **Thermodynamics** by work of Gibbs

• **Low-dimensional Topology**

  we will see lots of examples later, but we note a few results here

  recall: **Dehn Surgery** on a knot $K \subset \mathbb{S}^3$
  
  - remove a nbhd $N(K)$ of $K$ from $\mathbb{S}^3$
  
  - glue back $\mathbb{S}^1 \times D^2$ so that
  
  \[ \partial \mathbb{S}^1 \times \partial D^2 \text{ goes to a curve } \gamma \subset \partial (\mathbb{S}^3 \setminus N(K)) \]

  denote the result $\mathbb{S}^3(K)$
Kronheimer-Mrowka proved

Non trivial Knots have property $P$

i.e. $\pi_1(S^3_\lambda(K)) = 1$

$\Rightarrow \lambda = \mu$ or $K = \text{unknot}$

Ozsváth and Szabó proved

If $L$ is $\bigcirc$, $\bigcirc\bigcirc$, or $\bigcirc\bigcirc\bigcirc$

then $S^3_\rho(K) \cong S^3_\rho(L)$ for any $\rho$

implies $K = L$

the $L = O$ was conjectured by Gordon in the 1970's and was originally proven by Kronheimer-Mrowka-Ozsváth-Szabó
Recall $\mathbb{R}^3$ has a contact structure
$$\xi_{std} = \ker (dz + r^2 d\theta)$$

**Question:** Is there another?

Consider $$\xi_{ot} = \ker (\cos r dz + \sin r d\theta)$$

This looks "different" but how do we know there is no diffeomorphism $\Phi: \mathbb{R}^3 \to \mathbb{R}^3$ taking $\xi_{std}$ to $\xi_{ot}$?

**Bennequin's Answer:** Use braid theory!

(and in the process give birth to contact topology and initiate tools in Braid theory independently developed by Birman-Menasco) more fully!)
Outline Of Proof:

A knot \( K \) in a contact manifold \((M,\xi)\) is called **transverse** if

\[
T_xK \not\subset \xi_x \quad \forall x \in K
\]

**Step 1:** such a \( K \) in \((\mathbb{R}^3, \xi_{\text{std}})\) can be braided

**Step 2:** define the **self-linking number** of \( K \): \( sl(K) \)

compute \( sl(K) \) using

- Seifert surface \( \Sigma \) for \( K \)
- braid representation of \( K \)

**Step 3:** analyze the Birman–Menasco braid foliations on \( \Sigma \) to prove

\[
sl(K) \leq -\chi(\Sigma)
\]

**Step 4:** Notice in \((\mathbb{R}^3, \xi_{\text{tot}})\) there is a transverse unknot with \( sl = 1 \)

\[\text{Bennecquin inequality}\]
Some Details of Proof

Step 1: notice that for large $r$ the contact planes
$$\{s_{\theta} = \text{span} \{ \frac{\partial}{\partial r}, \frac{r^2 \partial}{\partial \theta} - \frac{\partial}{\partial \theta} \}$$
are almost tangent to the constant $\theta$ half-planes $H_{\theta_0} = \{(r, \theta, z) : \theta = \theta_0\}$

Since braids are $\mathcal{A}$ to the $H_{\theta_0}$'s

we see (closed) braids naturally give $\mathcal{A}$ knots

Thm (Bennequin '82)

Any transverse knot in $(\mathbb{R}^3, \tau_{std})$ can be isotoped through $\mathcal{A}$ knots to a closed braid.
one can prove this by observing that the proof that any knot can be braided can easily be adapted to this situation.

**Fact (Orevkov–Shevchishin, Wrinkle’03):**

To closed braids representing in $(\mathbb{R}^3, 3_{std})$ are isotopic as transverse knots $\iff$ they are related by

1) conjugation in braid group (i.e. braid isotopy)

2) positive Markov moves

Recall:

So classifying transverse knots in $(\mathbb{R}^3, 3_{std})$ can be done purely in terms of braids!
Step 2: The **Self-linking number**

**Note:** there is a never zero vector field $\mathbf{v}$ in $\mathbb{R}^2$. To see this notice $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ gives $d\pi|_p : T_p \mathbb{R}^2 = \mathbb{R}^2 \\ \pi(p)$ on isomorphism for all $p$ so $\exists!$ vector field $\mathbf{v}$ in $\mathbb{R}^2$ such that $d\pi(\mathbf{v}) = \frac{\partial}{\partial x}$ in $\mathbb{R}^2$

Now if $K$ is a transverse knot let $K' = K + \varepsilon \mathbf{v}$ for some small $\varepsilon$

The self-linking number of $K$ is

$$SL(K) = \text{linking } (K, K')$$

**Exercise:** $SL(K)$ is independent of $\varepsilon$
Recall: any $n$-strand braid can be represented by a word in $\sigma_1 \ldots \sigma_{n-1}$ and $\sigma_1^{-1} \ldots \sigma_{n-1}^{-1}$ where...

\[ \{ \text{strands} \} \]

Lemma: If the transverse knot $K$ is given as the closure of the $n$-braid $B = \sigma_1^{\varepsilon_1} \ldots \sigma_k^{\varepsilon_k}$ where $\varepsilon_i = \pm 1$ Then $\ell(K) = a(B) - n(B)$

algebraic length \quad braid index

\[ \sum \varepsilon_i \]

Proof

- get $-1$ for each strand
- get writhes from $B$
let $\Sigma$ be an oriented surface such that $\partial \Sigma = \mathcal{K}$

for each $x \in \Sigma$ let $l_x = T_x \Sigma \cup T_x \mathcal{K}$

at most points $l_x$ is a line but at some points $l_x = T_x \Sigma = T_x \mathcal{K}$

these are called singular points

exercise: There is a singular 1-dimensional foliation $\Sigma_1$ whose tangents are given by $l_x$. We call $\Sigma_1$ the characteristic foliation

note that since $\mathcal{K} = \partial \Sigma$ is $\mathcal{A}$ to $\mathcal{K}$ we see
**Exercise:** Can perturb $\Sigma$, rel $\partial \Sigma$, so that all singular points "look like"

- elliptic
- hyperbolic

$\Omega$ is oriented (by $2\pi dr d\theta$) and so is $\Sigma$ so to each singular point we can assign a sign

+ if orientations agree
- if they disagree

We get an induced orientation of $I_x$ and $\Sigma$

**Exercise:** if an elliptic point is + then

![Diagram](image) if - then

![Diagram](image)
given \( \Sigma \) let
\[
\begin{align*}
\tau_+ = & \# \left\{ \text{elliptic moduli in } \Sigma \right\} \\
\tau_- = & \# \left\{ \text{hyperbolic moduli in } \Sigma \right\}
\end{align*}
\]

Lemma:

If \( K = \partial \Sigma \) then
\[
(1) \quad s\ell(K) = -(\tau_+ - \tau_-) + (\tau_- - \tau_+)
\]

Exercise: prove this

Hint:

Let \( w \) be a vector field along \( \Sigma \) that is tangent to \( \Sigma \) and points out of \( \Sigma \) along \( \partial \Sigma \)

Note \( \text{link}(K, K + \epsilon w) = 0 \)

Show the difference between \( v \) and \( w \) along \( K \) is given in terms of \( \tau_+, \tau_- \).
notice that \( w \) is a vector field tangent to \( \Sigma \) and pointing out \( \partial \Sigma \) so by Poincaré–Hopf we have

\[ \chi(\Sigma) = e_+ + e_- - h_+ - h_- \]  

(2)

if we add (1) and (2) we get

\[ \mathfrak{s} \mathfrak{l}(K) + \chi(\Sigma) = 2(e_- - h_-) \]

thus if we can show that \( \Sigma \) can be isotoped, rel \( \partial \Sigma \), so that \( e_- = 0 \) then we will have shown

\[ \text{Th}^{\mathfrak{m}}(\text{Benoquin '82}): \]

\[ \mathfrak{s} \mathfrak{l}(K) \leq -\chi(\Sigma) \]

to do this we need...
Step 3: Braid Foliations

Let $K$ be a closed braid $\Sigma$ a surface s.t. $\partial \Sigma = K$

Recall $\mathbb{R}^3 - (z\text{-axis})$ is foliated by $H_a = \{(r, \theta, z) : \theta = a\}$ $a \in \mathbb{R}$

This induces a singular foliation on $\Sigma$ called the braid foliation and denoted $\mathcal{F}_3$
Examples:

1) 

2) 

Note/exercise:

1) near $\Sigma$

2) can assume $\Sigma$ (z-axis) so finitely many points like
3) Can orient \( F_\Sigma \) (as we did \( \Sigma \)) then if \( z \)-axis positively transverse to \( \Sigma \) then also near \( \partial \Sigma \) otherwise

4) Can perturb \( \Sigma \) so that the only singularities of \( F_\Sigma \) (away from \( z \)-axis) are hyperbolic center
5) If you keep expanding a center singularity you see (more or less)

replace

with

\[ \mathbb{R} \]
so we can isotopy so that there are no center singularities!

6) You can isotopy so that
\[ \mathcal{Z}_2 \text{ is very close to } \mathcal{Z}_3 \]
thus you can read off \( e_\pm h_\uparrow \) from \( \mathcal{Z}_2 \)
and prove the inequality by isotoping \( \Sigma \) so that it does not \( \land \) \( z \)-axis negatively!

From now on let \( K \) be an unknot and \( \Sigma \) be a disk it bounds

consider a negative singularity so \( \mathcal{Z}_3 \)

let \( \mathcal{B}_s = \{ \text{all leaves of } \mathcal{Z}_3 \text{ limiting to } s \} \)
note: $B_s$ is disjoint from $\Sigma$

(since we can't have $\exists 2e$

so $B_s$ is

or

or

or...

if the "sack" is empty then replace $\Sigma$ with

evaluating $s$!
If the "sack" is not empty then you can empty it via exchange moves.
note: this does not change the braid index or algebraic length of the braid.
so does not change $sl(K)$

exercise: show this can be done without changing $\gamma_s$
we can then eliminate $s$

exercise: think about more general $B_s$ (can always reduce to the above)

Continuing this analysis we will prove that for any transverse unknot $K$
we can isotope a disk $D$ with $\partial D = K$
so that $e_-= 0$
thus $SL(K) \leq -\chi(D) = -1$
Note: let \( D = \{ (r, \theta, z): \theta \leq \pi + \varepsilon \} \)
\[ z = 0 \]
\[ K = \partial D \]
\[ \ln I_{ot} = \ker \{ \cos r d r d \theta + r \sin r d \theta \} \]
the knot \( K \) is transverse

Perturb \( D \) to get

so \( \xi_c(K) = 1 \)

Thus we see \( I_{std} \) and \( I_{ot} \) are indeed different contact stras! with more work you can extend the Bennequin inequality for unknots in \( I_{std} \) to any knot (or link)
There are 2 ways to continue the above work:

I. Contact geometry
tight vs. overtwisted
(Elashberg)

II. Braid theory
(Birman-Menasco)

we first discuss contact geometry

**Definition:** We call a contact structure \( I \) overtwisted if there is a disk \( D \) such that \( T_x D = \mathbb{R}^2 \) for all \( x \in \partial D \) otherwise call \( I \) tight (this definition due to Eliashberg)
Note: If $I$ is overtwisted then (by extending $D$ as above) we get a transverse unknot $K$ with $\mathfrak{s}l(K) = 1$, so Bennequin's bond does not hold.

Theorem (Eliashberg +3, '92): Let $(M, \xi)$ be a contact manifold. The following are equivalent:

I. $I$ is tight

II. $\mathfrak{s}l(K) \leq -\chi(\Sigma)$ for all $K$ and $\Sigma = K$

III. $\mathfrak{s}l(K)$ is bounded above for all knots

IV. $\mathfrak{s}l(K)$ is bounded above for all knots in a fixed topological knot type (for example unknots)
In braid theory Birman–Menasco have proven many things using the braid foliation analysis.

a sample of results:

3-braids: A link $L$ that is the closure of a 3-braid has a unique 3-braid representative (up to conjugation) except for

1) Unknot: $\sigma_1 \sigma_2, \sigma_1 \sigma_2^{-1}, \sigma_1 \sigma_2^{-1}$

2) $(2,1)$ torus knot: $\sigma_1^2 \sigma_2, \sigma_1^2 \sigma_2^{-1}$

3) Links that are the closure of $\sigma_1 \sigma_2^2 \sigma_1 \sigma_2^{-1}, \sigma_1 \sigma_2^2 \sigma_1 \sigma_2^{-1}$

where $p,q,r$ distinct, abs vol $\geq 2$

$s = \pm 1$

Markov Theorem Without Stabilizations:

"For each $n$ there are a finite set of "moves" so if two braids of index $\leq n$ represent the same link then you can get from one to the other with these moves"
Transverse "Non-simple" knots:

Let $K_1$ and $K_2$ be the closures of the 3-braids

\[
\begin{array}{c}
\begin{array}{c}
2p+1 \\
2q
\end{array}
\begin{array}{c}
2r
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
2q
\end{array}
\begin{array}{c}
2r
\end{array}
\begin{array}{c}
2p+1
\end{array}
\end{array}
\]

where $p+1 \neq q \neq r$, $p, q, r > 1$

then $K_1$ and $K_2$ are transverse knots that are

1) topologically isotopic

2) have same SL

3) are not transversely isotopic

Note: these were the first such examples!

(see also E- Honda... )