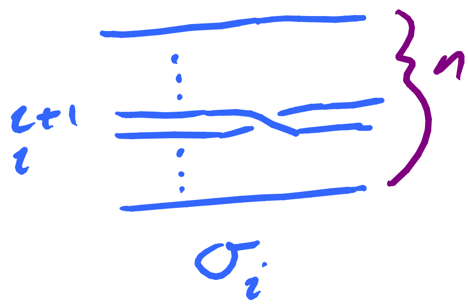
III Positivity in the Braid Group

The standard generators of the n-strand braid group $B(n)$ are:

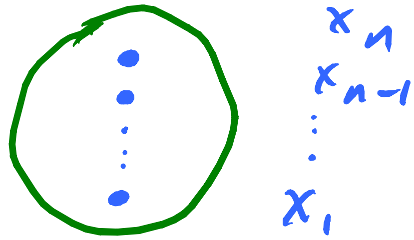


so any braid is a word in $\sigma_1 \dots \sigma_{n-1}, \sigma_1^{-1} \dots \sigma_{n-1}^{-1}$ and a braid is called **positive** if it is a word in $\sigma_1 \dots \sigma_{n-1}$

Note: this notion of positive depends on the generators we chose for $B(n)$. But are the σ_i the most "natural" generators?

Recall that a braid can be thought of as a loop in the configuration space of n points in D^2: $C(D^2, n)$

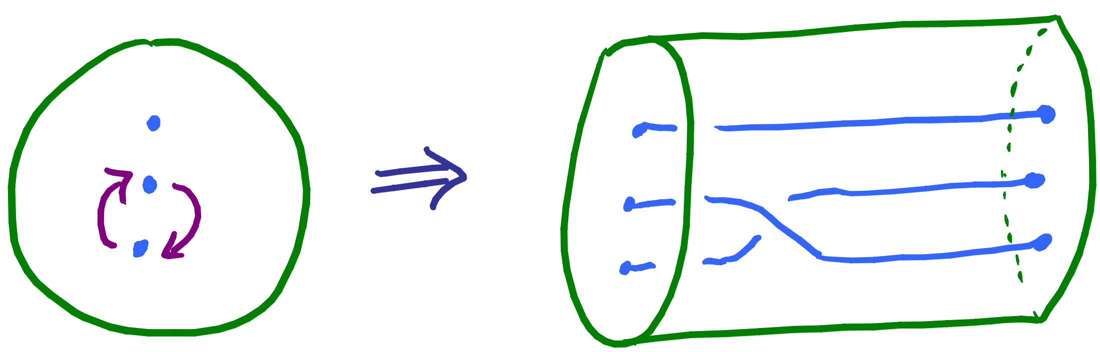
Indeed, consider loops based at



given such a loop we get a braid by thinking of the loop as an isotopy of $\{x_1 \dots x_n\} \xrightarrow[f_t]{} D^2$ and looking at the trace of the isotopy

$$\text{image}\{\phi : \{x_1 \dots x_n\} \times [0,1] \to D^2 \times [0,1]\}$$
$$\phi(x,t) = (f_t(x), t)$$

example:

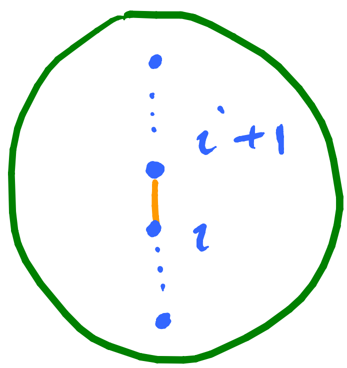


f_t = half twist exchanging x_1, x_2

trace is the braid σ_1

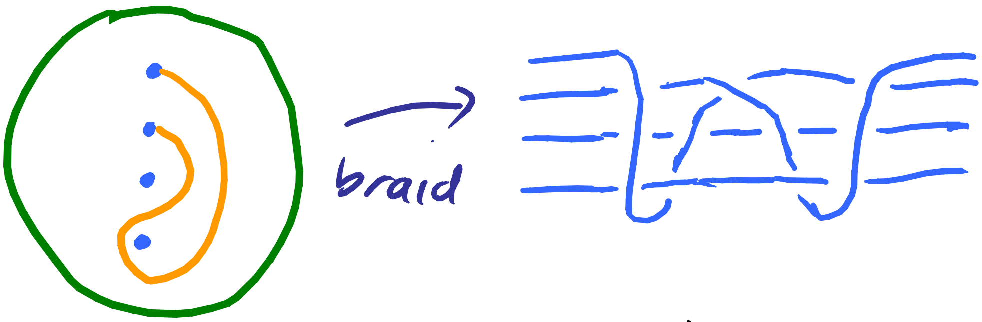
Similarly given a braid B think of it as sitting in $D^2 \times [0,1]$ then each $(D^2 \times \{t\}) \cap B$ is an element in the configuration space $C(D^2, n)$

In this language the σ_i are isotopies exchanging the i and $(i+1)$ points by a right handed twist in a nbhd of an arc connecting them



Quasi-Positive Generators: a more "natural" generating set would be the set of right handed twists in a neighborhood of any arc connecting any points!

example: $n=4$



note: this can be written

$$(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2^{-1})\sigma_3(\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})$$

$$= w\sigma_3 w^{-1}$$

where $w=\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2^{-1}$

- so these generators are just conjugates of the standard generators
- we call a braid <u>quasi-positive</u> if it can be written as a product of conjugates of the σ_i
- you might think this is a more natural notion of positive since it does not rely on chosing specal arcs to twist along

- Of course one draw back to this is the quasi-positive generating set is infinite...
- Quasi-positive has another geometric interpretation

Think of S^3 a sphere of some radius in $\mathbb{C}^2$

let Σ be a complex curve in $\mathbb{C}^2$ (for example the zeros of a two variable polynomial)

if $\Sigma \pitchfork S^3$, then $K = \Sigma \cap S^3$ is called a <u>transverse $\mathbb{C}$-link</u>

(note this includes links of singularities, like torus knots, but is a much bigger class of knots)

A link K is called <u>quasi-positive</u> if it is the closure of a quasi-positive braid.

<u>Thm</u>:

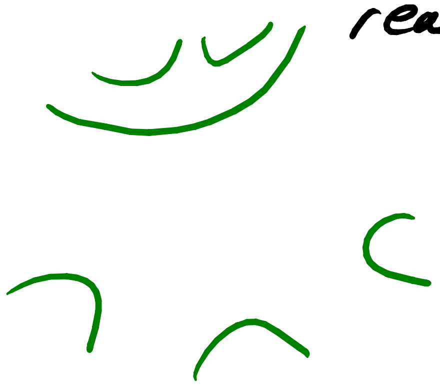
$$\boxed{\{\text{transverse } \mathbb{C}\text{-links}\} = \{\text{quasi-positive links}\}}$$

So quasi-posositivity has a geometric meaning!

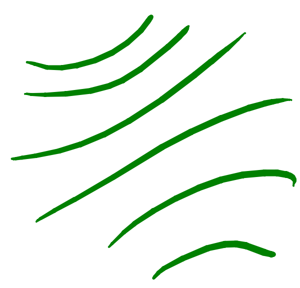
<u>Remark</u>: $\supseteq$ by Rudolph 1983

$\subseteq$ by Boileau-Orevkov 2001

Orevkov has a method to use quasi-positivity to study Hilbert's 16^{th} problem: the configurations of real algebraic curves in $\mathbb{R}^2$



ok



not ok ("for M-sextics")

In his study Orevkov asked two questions

> Given two quasi-positive braids representing a fixed link, are they related by <u>positive</u> Markov moves and conjugation?

> Given a quasi-positive link, is any minimal braid index representative of the link quasi-positive?

We will give partial answers to these questions using contact geometry.

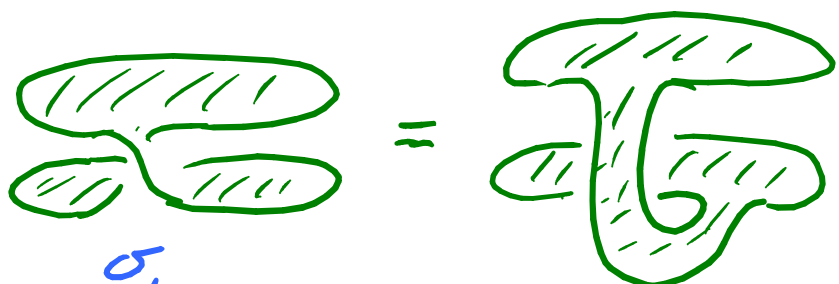
But first we want to discuss another type of positivity and surfaces that closures of braids bound.

Using the generators $\sigma_1 \ldots \sigma_{n-1}$ for $B(n)$ we can easily construct a Seifert surface for the closure of a braid

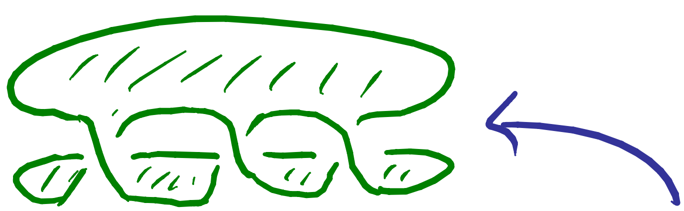
<u>Example</u>: $n=2$, $B=\sigma_1\sigma_1\sigma_1$

start with:

then add half-twisted band for each σ_i



σ_1



$B=\sigma_1\sigma_1\sigma_1$

call this surface Σ_B

lemma:

$\boxed{\text{If } K \text{ is the closure of a positive braid } B \text{ then } \Sigma_B \text{ is minimal genus}}$

Proof: recall Bennequin's $\leq$ says

$$sl(K) \leq -\chi(\Sigma) = 2g(\Sigma) - 1$$

for all Σ with $\partial\Sigma = K$

so

alg length

$$g(\Sigma) \geq \frac{sl(K)-1}{2} = \frac{a(B) - n(B) - 1}{2}$$

braid index

but $\chi(\Sigma_B) = n(B) - \ell(B)$

length

and since B is positive $\ell = a$

so $$g(\Sigma_B) = \frac{a(B) - n(B) - 1}{2}$$

∎

Fact (Stallings 1978):

If K is the closure of a positive braid then (S^3-K) fibers over S^1 and $\mathring{\Sigma}_B$ is the fiber.

i.e.

$$\begin{tikzcd} \mathring{\Sigma}_B \arrow[r, hook] & (S^3-K) \arrow[d, "\pi"] \\ & S^1 \end{tikzcd}$$

is a fibration

exercise: prove this

note: when constructing Σ_B

we started with $n(B)$ disks

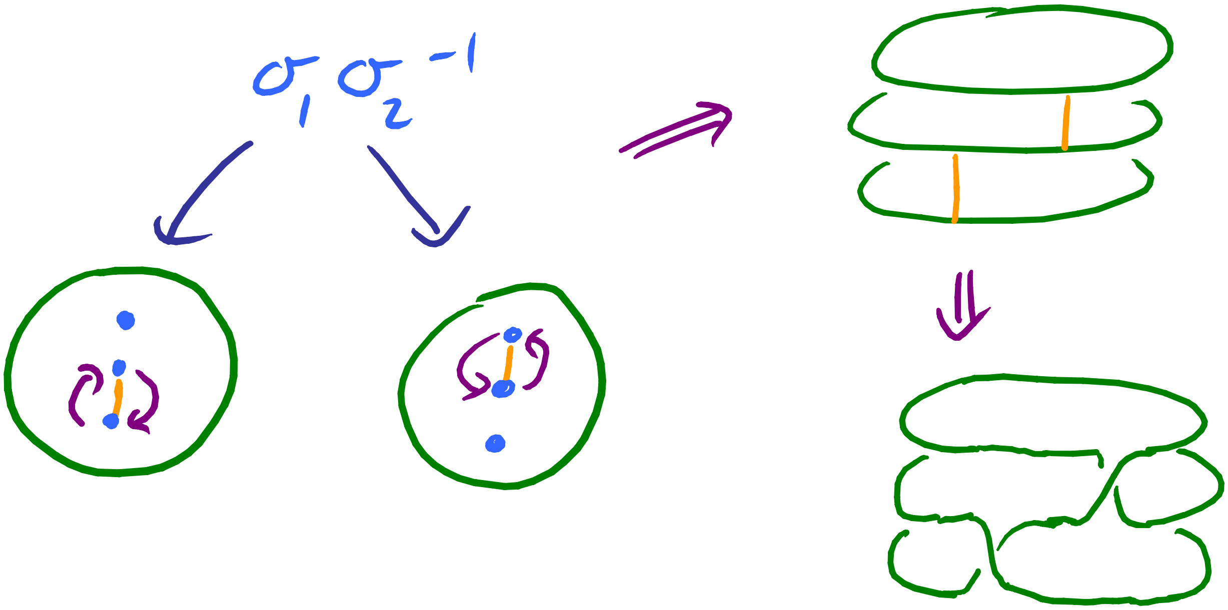
and added a half twisted band

whose core was the arc we twisted

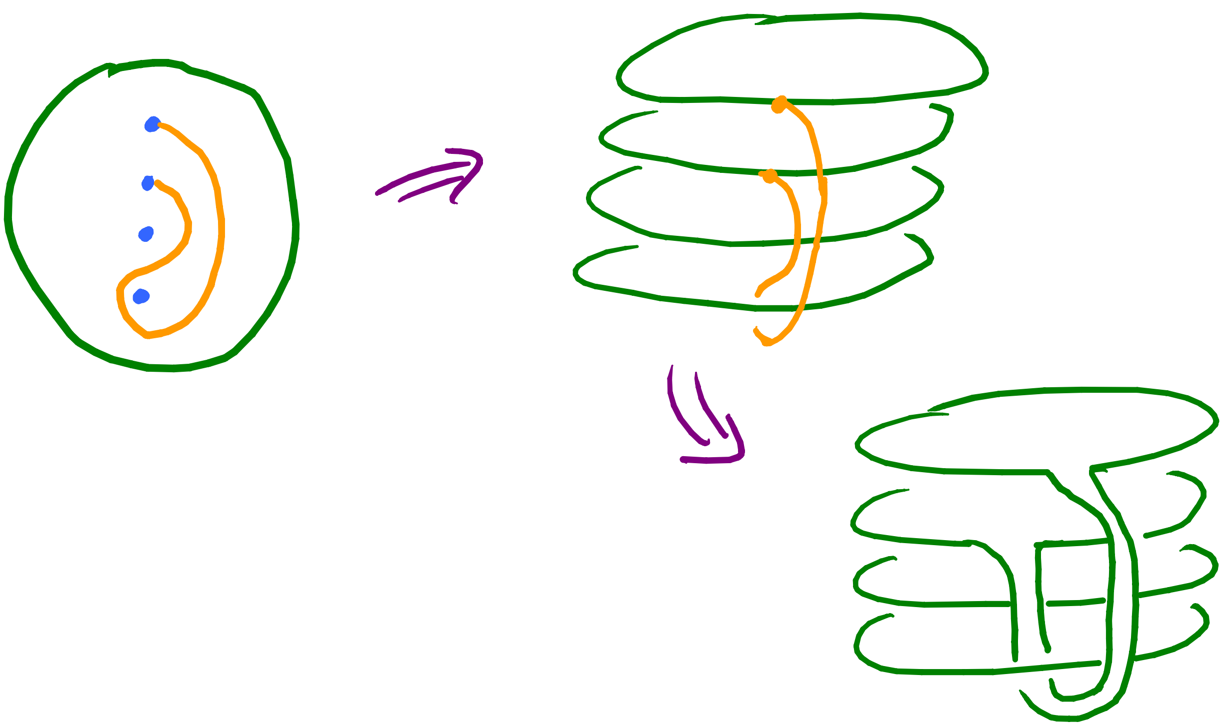
along in the configuration space

description of the braid

example:



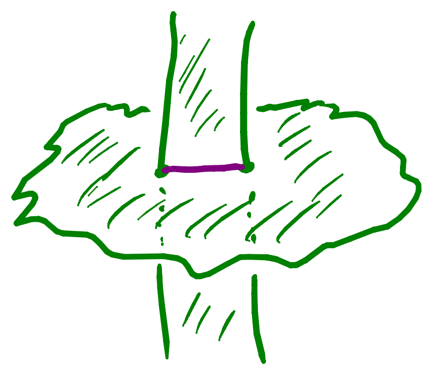
We can do the same thing for a braid written in terms of quasi-positive generators



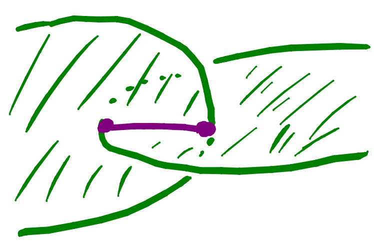
On the good side: This surface has much less genus than the one constructed using ordinary generators $\sigma_1 \dots \sigma_{n-1}$

On the bad side: This surface is not embedded!

Note: this surface only has ribbon singularities:



no clasp singularities:



so this surface can be perturbed in B^4, rel boundary, to be embedded.

Generalizing Bennequin's ≤ we have

Thm (Rudolph 1995):

If K is a transverse knot in S^3 and Σ is an embedded surface in B^4 with $\partial\Sigma = K$ then

$$sl(K) \leq -\chi(\Sigma)$$

the proof uses Guage Theory and has been generalized further by Lisca–Matic (1993) - to Stein fillings.

The inequality gives a bound on the slice genus of K

$$g_4(K) \geq \frac{sl(K)+1}{2}$$

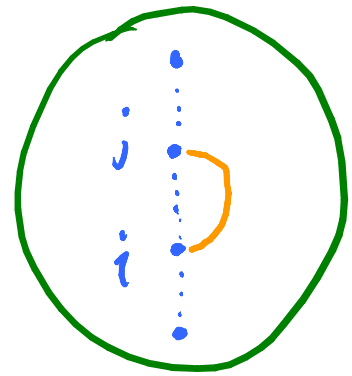
We now have:

Lemma:

If Σ is an embedded surface in B^4 coming from a quasi-positive presentation of $K = \partial\Sigma$ then $g_4(K) = g(\Sigma)$

We now define the strongly quasi-positive generators of the braid group $B(n)$: let σ_{ij} be

in terms of the config space



Note:
- there are a finite number of these
- $\sigma_{ij} = (\sigma_1 \cdots \sigma_{j-2})\,\sigma_{j-1}\,(\sigma_1 \cdots \sigma_{j-2})^{-1}$

  so these are also quasi-positive generators
- a link is strongly quasi-positive if it is the closure of a braid that can be written as a word in the σ_{ij}.
- the surface for K that comes from a strongly qp-presentation is embedded in S^3!

so if K is strongly qp then any strongly qp braid representing K can be used to construct a surface Σ s.t. $g(\Sigma) = g_4(K) = g_3(K)$

<u>note</u>: unlike for closures of positive braids, a strongly qp knot does not need to be fibered.

to sumarize:

$$\left\{\begin{matrix}\text{closure of}\\ \text{positive}\\ \text{braids}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{strongly}\\ \text{qp links}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{qp}\\ \text{links}\end{matrix}\right\} = \left\{\begin{matrix}\text{transv.}\\ \mathbb{C}\text{-links}\end{matrix}\right\}$$

IV Contact Geometry and Positivity

Thm (E-Van Horn-Morris 2010):

let K be 1) fibered
2) strongly quasi-positive
link in S^3
and Σ is associated Seifert surface
Then there is a unique transverse knot in (S^3, ξ_{std}) in the knot type of K with $sl = -\chi(\Sigma)$.

we discuss the proof of this later but for now explore the consequences

Cor 1 (a recognition result):

let K be a fibered, strongly qp link and B a braid representing K
Then
B is quasi-pos. $\Leftrightarrow$ $a(B) = n(B) - \chi(K)$

For the proof we need

Thm (Orevkov 2000):

a braid B is quasi-pos. iff any positive Markov stabilization of B is quasi-pos.

the proof of this result is indirect and uses Gromov's Theory of holomorphic curves

Also recall

Fact (Orevkov - Shevchishin, Wrinkle '03):

to closed braids representing in $(\mathbb{R}^3, \xi_{std})$ are isotopic as transverse knots $\Longleftrightarrow$ they are related by

1) conjucation in braid group (i.e. braid isotopy)

2) positive Markov moves

Proof of Cor 1:

We are assuming there is a strongly quasi-positive braid B' whose closure $K_{B'}$ is isotopic to K

let $\Sigma_{B'}$ be the surface constructed from B'

$$\chi(\Sigma_{B'}) = n(B') - \ell(B')$$

length of B' in the generators σ_{ij}

also $K_{B'}$ is a transverse knot with

$$sl(K_{B'}) = -\chi(\Sigma_{B'})$$

---

now assume B is a braid with K_B isotopic to K and $a(B) = n(B) - \underbrace{\chi(K)}_{= \chi(\Sigma_{B'})}$

- so K_B is a transverse knot with
$$sl(K_B) = -\chi(\Sigma_{B'})$$
- so by the main theorem we know K_B and $K_{B'}$ are transversely isotopic
- so by the Fact above we see
$K_B, K_{B'}$ are related by positive Markov moves.
- Orevkov's Th$\underline{\text{m}}$ now says K_B is quasi-positive!

Conversely assume B is quasi-positive
- From Bennequin we know
$$a(B) \leq n(B) - \chi(\Sigma_{B'})$$
- So if we do not have equality then
$$a(B) < n(B) - \chi(\Sigma_{B'})$$
- let Σ be the "ribbon" surface constructed from qp-presentation of B

- $\chi(\Sigma) = n(B) - a(B)$
- By Orevkov we can positively Markov stabilize B and B' if necessary so that $n(B) = n(B')$
- Thus $\chi(\Sigma) = n(B) - a(B)$
$$\geq n(B') - a(B') = \chi(\Sigma_{B'})$$
- This contradicts Bennequin
$$a(B') - n(B') = sl(K_{B'}) \leq -\chi(\Sigma)$$
- Thus $a(B) = n(B) - \underbrace{\chi(\Sigma_{B'})}_{= \chi(K)}$

$\boxed{\text{QED}}$

On to Orevkov's Questions we start with

> Given two quasi-positive braids representing a fixed link, are they related by <u>positive</u> Markov moves and conjugation?

Cor 2:

Let K be a strongly quasi-pos. fibered knot
Any two quasi-positive braids are related by positive Markov moves (and conjugation)

Note: 1) In particular, two positive braids represent the same knot $\Longleftrightarrow$ they are related by positive Markov moves.

2) So all questions about knots represented by positive braids can be answered in the Positive Braid Monoid

3) Answer to Orevkov is
Yes for fibered strongly q.p.
No in general...

recall Birman-Menasco say,

for example,

$$\sigma_1^{2p+1}\sigma_2^{2r}\sigma_1^{2q}\sigma_2^{-1}$$
$$= \sigma_{1,2}^{2p+1}\sigma_{2,3}^{2r-1}\sigma_{1,3}^{2q}$$

and

$$\sigma_1^{2q}\sigma_2^{2r}\sigma_1^{2p+1}\sigma_2^{-1}$$
$$= \sigma_{1,2}^{2q}\sigma_{2,3}^{2r-1}\sigma_{1,3}^{2p+1}$$

give two transverse knots with the same self-linking number that are not transversly isotopic so they are not related by pos. Markov stab.

(note they are strongly quasi-positive, so the fibered assumption in Cor 2 is clearly necessary!)

Proof of Cor 2:

- let B_1 and B_2 be two qp-braids representing K, a strongly-qp fibered knot.
- by Cor 1 we know the closures of the braids K_{B_1}, K_{B_2} are transverse knots with

$$sl(K_{B_1}) = sl(K_{B_2}) = -\chi(K)$$

- $\therefore$ by the main theorem we see K_{B_1}, K_{B_2} are transversely isotopic
- So they are related by positive Markov moves! $\blacksquare$

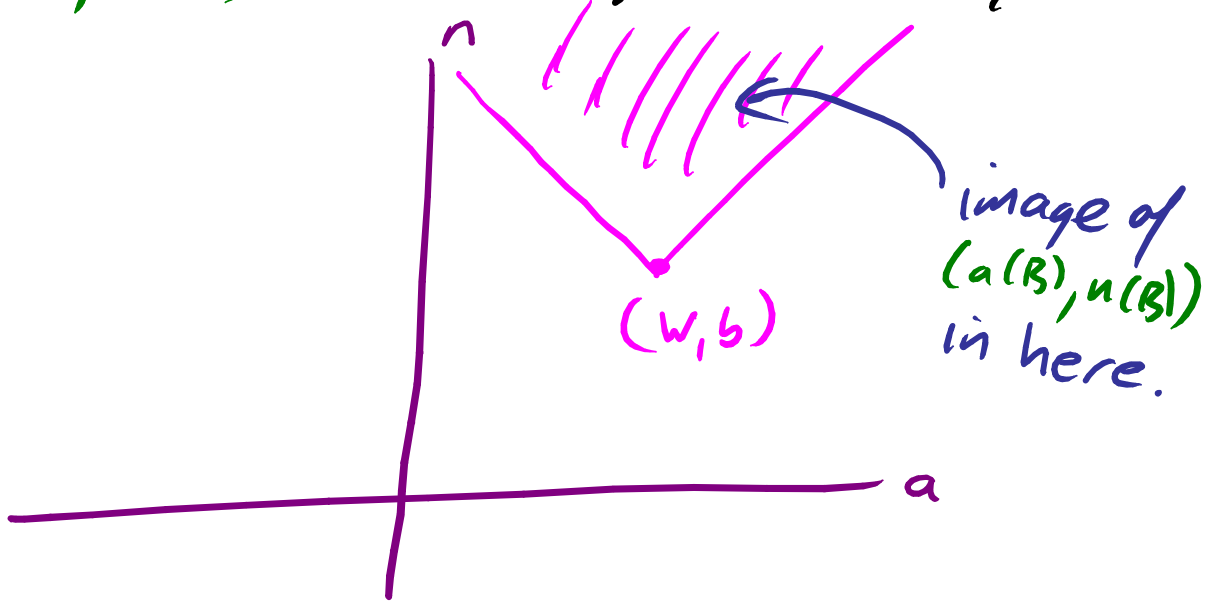
Now for Orevkov's 2nd question

> Given a quasi-positive link, is any minimal braid index representative of the link quasi-positive?

For this we recall Kawamura's Braid Geography Conjecture: given a knot type K, there are numbers $b>0, w$ such that for any braid β representing K we have

$$\boxed{b+|a(\beta)-w| \leq n(\beta)}$$

Graphically, if we plot all the pairs $(a(\beta), n(\beta))$ the conjecture says



note 1) this conjecture generalizes the Jones's Conjecture that there is a unique value for $a(B)$ among braids representing K with minimal braid index.

2) The conjecture is true for

- pos braids that contain a full twist
- alternating fibered knots
- many other knots

Cor 3:

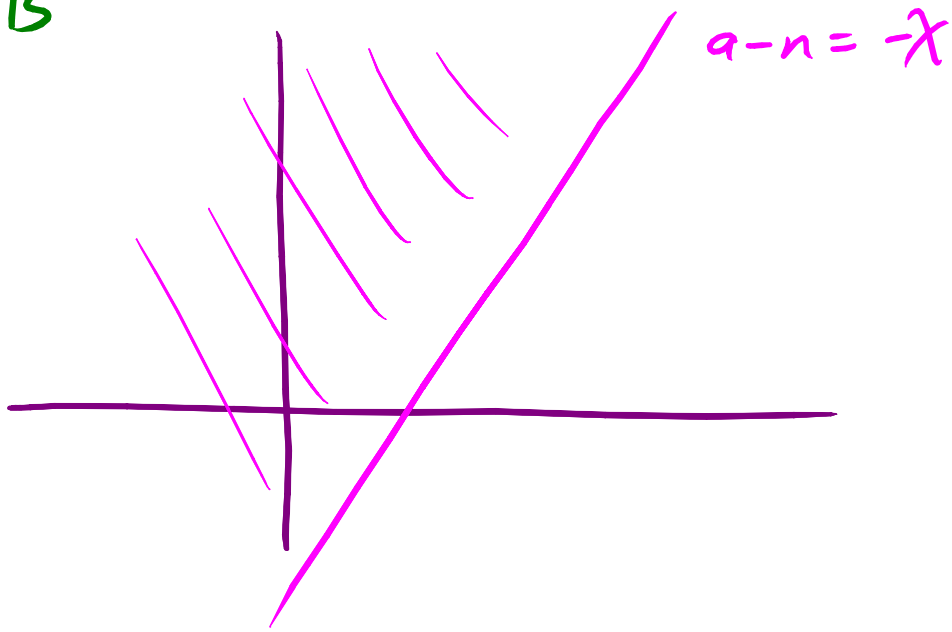
$\boxed{\text{If } K \text{ is a fibered strongly quasi-pos knot that satisfies the braid geography conjecture then any minimal braid index representative of } K \text{ is quasi-positive}}$

Proof:

- from the Bennequin ineq we have

$$a(B) - n(B) \leq -\chi(K)$$

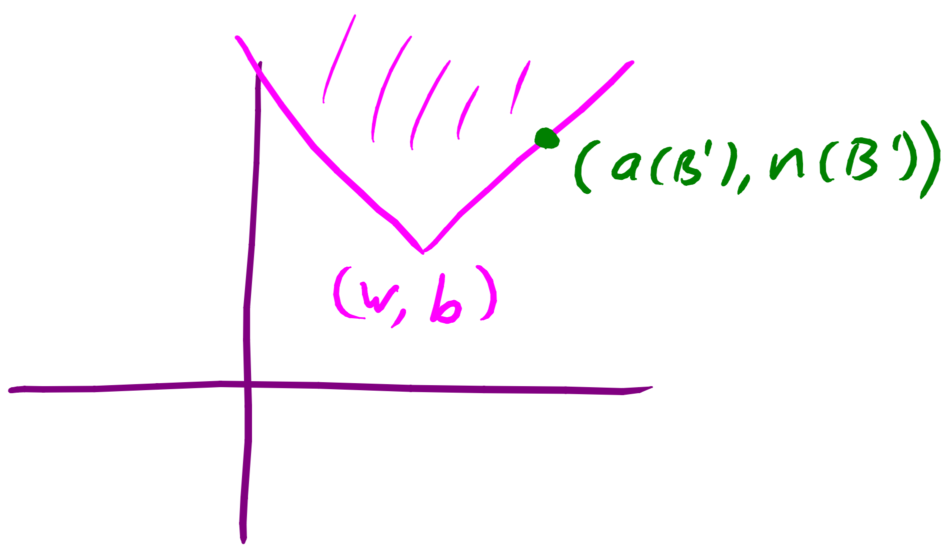
for any B



- If B' is a strongly qp braid rep. K

then $sl(K_{B'}) = a(B') - n(B') = -\chi(K)$

- so



- Thus if B is a braid with minimal braid index $sl(K_B) = a(B) - n(B)$
$$= sl(K_{B'})$$
- So the main theorem says K_B, $K_{B'}$ are transversely isotopic
- So B, B' related by positive Markov moves.
- So Orevkov says B is quasi-pos!

We now turn to the proof of the main theorem.

For this we need <u>Open Book Decompositions</u> ...

V Open Book Decompositions

An open book decomposition of a 3-manifold (closed, oriented) M is a pair (L, π) where

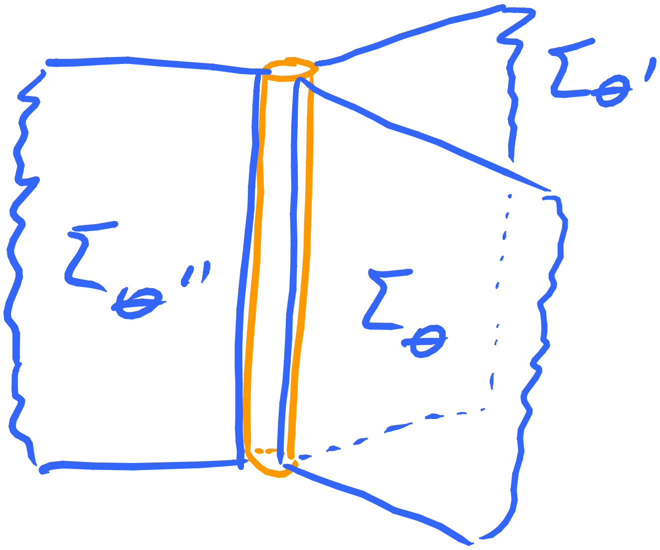
1) L is a link in M called the binding and
2) $\pi : (M - L) \to S^1$ is a fibration so that $\Sigma_\theta = \overline{\pi^{-1}(\theta)}$ is a Seifert surface for the link L

the Σ_θ are called pages

Examples:

1) let U be the unknot in S^3

$$
\begin{array}{ccc}
(S^3 - U) = (\mathbb{R}^3 - z\text{-axis}) & & (r, \theta, z) \\
\big\downarrow \pi & & \big\downarrow \\
S^1 & & \theta
\end{array}
$$

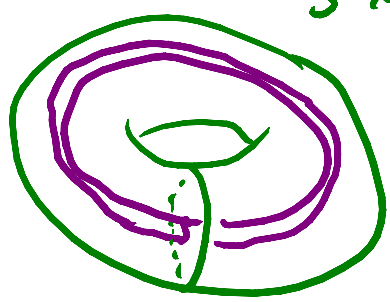


so (U, π) is an open book of S^3

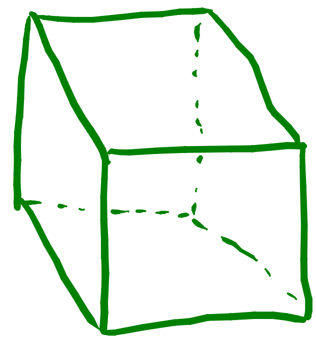
note: a link K in S^3 is a closure of a braid (or "braided") if $\pi|_K : K \to S^1$ is a covering map!

2) let H_+ be the Hopf link

note $$(S^3 - H_+) = \underbrace{(S^3 - U)}_{S^1 \times \mathbb{R}^2} - \underset{\substack{\uparrow \\ S^1 \times \{pt\}}}{U'}$$

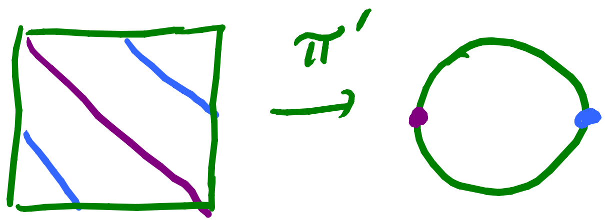
$=$  $= T^2 \times (0,1)$

$T^2 \times (0,1) =$



identify top to bottom and front to back

let $\pi' : T^2 \to S^1$



$$\pi : T^2 \times [0,1] \to S^1 : (p,t) \mapsto \pi'(p)$$

this fibers $S^3 - H_+$ with fiber

alternately, let $S^3 \subseteq \mathbb{C}^2$ be the unit sphere

$$H_+ = \{(z_1, z_2) \in S^3 \mid z_1 z_2 = 0\}$$

$$\pi : (S^3 - H_+) \to S^1 : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$$

3) If $p: \mathbb{C}^2 \to \mathbb{C}$ is any polynomial that vanishes at $(0,0)$ and has no critical points (except possibly $(0,0)$) inside S^3, then

$$K_p = \{(z_1, z_2) \in S^3 : p(z_1, z_2) = 0\}$$

is fibered with fibration

$$\pi_p : (S^3 - K_p) \to S^1 : (z_1, z_2) \mapsto \frac{f(z_1, z_2)}{|f(z_1, z_2)|}$$

so (K_p, π_p) is an open book of S^3

<u>exercise</u>: find a polynomial so that K_p is