III Positivity in the Braid Group

The standard generators of the \( n \)-strand braid group \( \mathcal{B}(n) \) are:

\[
\sigma_i
\]

so any braid is a word in \( \sigma_i, \sigma_i^{-1}, \sigma_{i+1}, \sigma_{i+1}^{-1} \)

and a braid is called \textbf{positive} if it is a word in \( \sigma_1, \ldots, \sigma_{n-1} \).

\textbf{Note}: this notion of positive depends on the generators we chose for \( \mathcal{B}(n) \). But are the \( \sigma_i \) the most “natural” generators?

Recall that a braid can be thought of as a loop in the configuration space of \( n \) points in \( \mathbb{D}^2 : C(D^2, n) \).
Indeed, consider loops based at given such a loop we get a braid by thinking of the loop as an isotopy of \( \{x_1 \ldots x_n\} \xrightarrow{f_+} D^2 \) and looking at the trace of the isotopy

\[
\text{image} \{ \phi : \{x_1 \ldots x_n\} \times [0,1] \to D^2 \times [0,1] \} \\
\phi(x, \tau) = (f_+(x), \tau)
\]

**Example:**

\[f_+ = \text{half twist exchanging } x_1, x_2\]

trace is the braid \( \sigma_1 \)
Similarly given a braid $B$ think of it as sitting in $D^2 \times [0,1]$, then each $(D^2 \times \{t\}) \cap B$ is an element in the configuration space $C(D^2, n)$.

In this language the $\sigma_i$ are isotopies exchanging the $i$ and $(i+1)$ points by a right handed twist in a nbd of an arc connecting them.

Quasi-Positive Generators: a more "natural" generating set would be the set of right handed twists in a neighborhood of any arc connecting any points!
Example: \(n=4\)

\[
\begin{align*}
\text{note: this can be written} & \quad (\sigma_3 \sigma_2 \sigma_1 \sigma_2^{-1}) \sigma_3 (\sigma_2 \sigma_1^{-1} \sigma_1 \sigma_2^{-1}) \\
& = w \sigma_3 w^{-1} \\
\text{where } w & = \sigma_3 \sigma_2 \sigma_1 \sigma_1 \sigma_2^{-1}
\end{align*}
\]

- So these generators are just conjugates of the standard generators.
- We call a braid **quasi-positive** if it can be written as a product of conjugates of the \(\sigma_i\).
- You might think this is a more natural notion of positive since it does not rely on choosing special arcs to twist along.
• Of course one draw back to this is the quasi-positive generating set is infinite...

• Quasi-positive has another geometric interpretation

Think of $S^3$ a sphere of some radius in $\mathbb{C}^2$

let $\Sigma$ be a complex curve in $\mathbb{C}^2$

(for example the zeros of a two variable polynomial)

if $\Sigma \sqsubset S^3$, then $K = \Sigma \cap S^3$

is called a transverse $\mathbb{C}$-link

(note this includes links of singularities, like torus knots, but is a much bigger class of knots)
A link $K$ is called **quasi-positive** if it is the closure of a quasi-positive braid.

**Theorem:**

\[
\{\text{transverse } \mathbb{C} \text{-links}\} = \{\text{quasi-positive}\} \text{ links}
\]

so quasi-positivity has a geometric meaning!

**Remark:**  $\geq$ by Rudolph 1983

$\leq$ by Boileau-Overkamp 2001

Overkamp has a method to use quasi-positivity to study Hilbert's 16th problem: the configurations of real algebraic curves in $\mathbb{R}^2$.

\[\text{ok} \\ \text{not ok} (\text{for } n\text{-sextics})\]
In his study Orevakov asked two questions:

Given two quasi-positive braids representing a fixed link, are they related by positive Markov moves and conjugation?

Given a quasi-positive link, is any minimal braid index representative of the link quasi-positive?

We will give partial answers to these questions using contact geometry.

But first we want to discuss another type of positivity and surfaces that closures of braids bound.
Using the generators $\sigma_1 \ldots \sigma_{n-1}$ for $B(n)$ we can easily construct a Seifert surface for the closure of a braid.

**Example**: $n=2$, $B=\sigma_1 \sigma_1 \sigma_1$

Start with:

Then add half-twisted band for each $\sigma_i$.

$B=\sigma_1 \sigma_1 \sigma_1$

call this surface $\Sigma_B$
Lemma:
If $K$ is the closure of a positive braid $B$ then $\Sigma_B$ is minimal genus.

Proof: recall Bennequin's $\leq$ says

\[ sL(K) \leq -\chi(\Sigma) = 2g(\Sigma) - 1 \]

for all $\Sigma$ with $\partial \Sigma = K$ so

\[ g(\Sigma) \geq \frac{sL(K) - 1}{2} = \frac{\alpha(B) - n(B) - 1}{2} \]

but $\chi(\Sigma_B) = n(B) - \ell(B)$

and since $B$ is positive $\ell = \alpha$

so

\[ g(\Sigma_B) = \frac{\alpha(B) - n(B) - 1}{2} \]
Fact (Stallings 1978):

If \( K \) is the closure of a positive braid then \((S^3-K)\) fibers over \( S^1 \) and \( \Sigma_B \) is the fiber.

\[
\begin{align*}
\dot{\Sigma}_B & \to (S^3-K) \\
\downarrow & \pi \\
S^1 & 
\end{align*}
\]

**Exercise:** Prove this.

**Note:** When constructing \( \Sigma_B \), we started with \( n(B) \) disks and added a half-twisted band, whose core was the arc we twisted along in the configuration space description of the braid.
Example:

\[ \sigma_1 \sigma_2^{-1} \]

We can do the same thing for a braid written in terms of quasi-positive generators.
On the good side: This surface has much less genus than the one constructed using ordinary generators $\sigma_1 \ldots \sigma_{n-1}$.

On the bad side: This surface is not embedded!

Note: this surface only has ribbon singularities:

\[ \text{no clasp singularities!} \]

so this surface can be perturbed in $\mathbb{B}^4$, rel boundary, to be embedded.
Generalizing Bennequin's ≤ we have

**Theorem (Rudolph 1995):**

If \( K \) is a transverse knot in \( S^3 \) and \( \Sigma \) is an embedded surface in \( B^4 \) with \( \partial \Sigma = K \) then

\[
\text{sl}(K) \leq -\chi(\Sigma)
\]

The proof uses Gauge Theory and has been generalized further by Lisca-Matic (1993) to Stein fillings.

The inequality gives a bound on the slice genus of \( K \)

\[
g_4(K) \geq \frac{\text{sl}(K) + 1}{2}
\]

We now have:

**Lemma:**

If \( \Sigma \) is an embedded surface in \( B^4 \) coming from a quasi-positive presentation of \( K = \partial \Sigma \) then \( g_4(K) = g(\Sigma) \)
We now define the strongly quasi-positive generators of the braid group $B(n)$: let $\sigma_{ij}$ be

\[ \sigma_{ij} = (\sigma_i \ldots \sigma_{j-2}) \sigma_{j-1} (\sigma_i \ldots \sigma_{j-2})^{-1} \]

so these are also quasi-positive generators.

- a link is strongly quasi-positive if it is the closure of a braid that can be written as a word in the $\sigma_{ij}$.
- the surface for $K$ that comes from a strongly gp-presentation is embedded in $S^3$!
so if $K$ is strongly $qp$ then any strongly $qp$ braid representing $K$ can be used to construct a surface $\Sigma$ s.t. $g(\Sigma) = g_4(K) = g_3(K)$

**note:** unlike for closures of positive braids, a strongly $qp$ knot does not need to be fibered.

to summarize:

$$\left\{\text{closure of positive braids}\right\} \subseteq \left\{\text{strongly } qp \text{ links}\right\} \subseteq \left\{\text{links}\right\} = \left\{\text{transv. C-links}\right\}$$
IV Contact Geometry and Positivity

Thm (E-Van Horn-Morris 2010):

Let $K$ be 1) fibered
2) strongly quasi-positive

link in $S^3$

and $\Sigma$ is associated Seifert surface

Then there is a unique transverse knot in $(S^3, \xi_{\text{std}})$ in the knot type of $K$ with $\sigma = -\chi(\Sigma)$.

We discuss the proof of this later but for now explore the consequences.

Cor 1 (a recognition result):

Let $K$ be a fibered, strongly qp link and $B$ a braid representing $K$

Then $B$ is quasi-pos. $\iff \alpha(B) = n(B) - \chi(K)$
For the proof we need

\textbf{Thm (Orevkov 2000):}

A braid $B$ is quasi-pos. iff any positive Markov stabilization of $B$ is quasi-pos.

The proof of this result is indirect and uses Gromov's Theory of holomorphic curves.

Also recall

\textbf{Fact (Orevkov - Shevchishin, Wrinkle '03):}

to closed braids representing in $(\mathbb{R}^3, \mathbb{R}^3_{std})$ are isotopic as transverse knots \iff they are related by

1) conjugation in braid group (i.e. braid isotopy)

2) positive Markov moves
Proof of Cor 1:

We are assuming there is a strongly quasi-positive braid $B'$ whose closure $K_{B'}$ is isotopic to $K$

let $\Sigma_{B'}$ be the surface constructed from $B'$

$$\chi(\Sigma_{B'}) = n(B') - \ell(B')$$

length of $B'$ in the generators $\sigma_i, \sigma_j$

also $K_{B'}$ is a transverse knot with

$$\delta(\Sigma_{B'}) = -\chi(\Sigma_{B'})$$

now assume $B$ is a braid with $K_B$ isotopic to $K$

and $\alpha(B) = n(B) - \chi(K)$

$$= \chi(\Sigma_{B'})$$
• so \( K_B \) is a transverse knot with \( \text{sl}(K_B) = -\chi(\Sigma_B) \).

• so by the main theorem we know \( K_B \) and \( K_{B'} \) are transversely isotopic.

• so by the Fact above we see \( K_B, K_{B'} \) are related by positive Markov moves.

• Orevkov's Thm now says \( K_B \) is quasi-positive!

Conversely assume \( B \) is quasi-positive.

• From Bennequin we know

\[ a(B) \leq n(B) - \chi(\Sigma_B) \]

• So if we do not have equality then

\[ a(B) < n(B) - \chi(\Sigma_B) \]

• let \( \Sigma \) be the “ribbon” surface constructed from gp-presentation of \( B \).
• \( \chi(\Sigma) = n(B) - a(B) \)
• By Orevkov we can positively Markov stabilize \( B \) and \( B' \) if necessary so that \( n(B) = n(B') \)
• Thus \( \chi(\Sigma) = n(B) - a(B) \geq n(B') - a(B') = \chi(\Sigma_B') \)
• This contradicts Bennequin \( a(B') - n(B') = \text{sl}(K_B) \leq -\chi(\Sigma) \)
• Thus \( a(B) = n(B) - \chi(\Sigma_B') = \chi(K) \)

On to Orevkov's Questions we start with

Given two quasi-positive braids representing a fixed link, are they related by positive Markov moves and conjugation?
Cor 2: Let $K$ be a strongly quasi-positively fibered knot.

Any two quasi-positive braids are related by positive Markov moves (and conjugation).

**Note:**
1) In particular, two positive braids represent the same knot $\iff$ they are related by positive Markov moves.

2) So all questions about knots represented by positive braids can be answered in the *Positive Braid Monoid*.

3) Answer to Orevkov is
   - Yes for fibered strongly gp.
   - No in general...
recall Birman-Menasco say, for example,
\[
\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{2p+1} \sigma_2^{-2r} \sigma_1^{-2q} \sigma_2^{-1} = \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_1 \sigma_2^{2q} \sigma_2^{-1} \sigma_1^{-2q} \sigma_2^{-1} \\
\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{2p+1} \sigma_2^{-2r} \sigma_1^{-2q} \sigma_2^{-1} = \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_1 \sigma_2^{2q} \sigma_2^{-1} \sigma_1^{-2q} \sigma_2^{-1}
\]
and give two transverse knots with the same self-linking number that are not transversely isotopic so they are not related by pos. Markov stub.
(note they are strongly quasi-positive, so the fibered assumption in cor2 is clearly necessary !)
Proof of Cor 2:

• let $B_1$ and $B_2$ be two qp-braids representing $K$, a strongly qp fibered knot.

• by Cor 1 we know the closures of the braids $K_{B_1}, K_{B_2}$ are transverse knots with

$$sl(K_{B_1}) = sl(K_{B_2}) = -\chi(K)$$

• i.e. by the main theorem we see $K_{B_1}, K_{B_2}$ are transversely isotopic

• so they are related by positive Markov moves! \[\blacksquare\]
Now for Orevkov's 2nd question

Given a quasi-positive link, is any minimal braid index representative of the link quasi-positive?

For this we recall Kawamura's Braid Geography Conjecture: given a knot type $K$, there are numbers $b > 0, w$ such that for any braid $B$ representing $K$ we have

$$b + 1 - a(B) - w| \leq n(B)$$

Graphically, if we plot all the pairs $(a(B), n(B))$ the conjecture says

[Diagram showing a graph with axes labeled 'a' and 'n', with a region highlighted indicating the inequality $b + 1 - a(B) - w| \leq n(B)$]
Note 1) This conjecture generalizes the Jones Conjecture that there is a unique value for \( a(\beta) \) among braids representing \( K \) with minimal braid index.

2) The conjecture is true for

- pos braids that contain a full twist
- alternating fibered knots
- many other knots

Cor 3:

If \( K \) is a fibered strongly quasi-pos knot that satisfies the braid geography conjecture, then any minimal braid index representative of \( K \) is quasi-positive.
Proof:

- from the Bennequin ineq we have
  \[ a(B) - n(B) \leq -X(K) \]
  for any \( B \)

- If \( B' \) is a strongly gp braided rep. \( K \) then
  \[ 5\mathbb{E}(K_{B'}) = a(B') - n(B') = -X(K) \]

- so
  \[ (a(B'), n(B')) \]
  \[ (w, b) \]
• Thus if $B$ is a braid with minimal braid index $\text{sl}(K_B) = a(B) - n(B) = \text{sl}(K_B')$

• So the main theorem says $K_B$ and $K_B'$ are transversely isotopic.

• So $B, B'$ related by positive Markov moves.

• So Orevkov says $B$ is quasi-pos!

We now turn to the proof of the main theorem.

For this we need Open Book Decompositions ...
V. Open Book Decompositions

An open book decomposition of a 3-manifold (closed, oriented) $M$ is a pair $(L, \pi)$ where

1) $L$ is a link in $M$ called the binding and

2) $\pi : (M - L) \to S^1$ is a fibration

so that $\Sigma_\theta = \pi^{-1}(\theta)$ is a Seifert surface for the link $L$.

the $\Sigma_\theta$ are called pages

Examples:

1) let $U$ be the unknot in $S^3$

\[
(S^3 - U) = (\mathbb{R}^3 - z\text{-axis}) \quad (r, \theta, z)
\]

\[
\downarrow \pi
\]

\[
S^1
\]

\[
\theta
\]
so \((U, \pi)\) is an open book of \(S^3\)

**Note:** a link \(K\) in \(S^3\) is a closure of a braid (or "braided") if \(\pi|_K : K \to S^1\) is a covering map!

2) let \(H_+\) be the Hopf link

\[
\begin{align*}
\text{note } (S^3 - H_+) &= (S^3 - U) - U' \\
&\cong S^1 \times R^2 \quad S^1 \times \{\text{pt}\} \\
&= T^2 \times (0,1)
\end{align*}
\]
\[ T^2 \times [0,1) = \] 

Identify top to bottom and front to back

\[ \pi' : T^2 \to S' \]

This fibers \( S^3 - H_+ \) with fiber alternately, let \( S^3 \subset C^2 \) be the unit sphere

\[ H_+ = \{ (z_1, z_2) \in S^3 \mid z_1 z_2 = 0 \} \]

\[ \pi : (S^3 - H_+) \to S' : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|} \]
3) If \( p : \mathbb{C}^2 \to \mathbb{C} \) is any polynomial that vanishes at \((0,0)\) and has no critical points (except possibly \((0,0)\)) inside \(S^3\), then

\[
K_p = \{ (z_1, z_2) \in S^3 : p(z_1, z_2) = 0 \}
\]

is fibered with fibration

\[
\pi_p : (S^3 - K_p) \to S^1 : (z_1, z_2) \mapsto \frac{f(z_1, z_2)}{1 + |f(z_1, z_2)|}
\]

so \((K_p, \pi_p)\) is an open book of \(S^3\)

Exercise: find a polynomial so that \(K_p\) is 
\[
\text{\(\bigcirc\)}
\]