last time we saw many open books for $S^3$ but the exist for all $M^3$

**Thm (Alexander):**

Every 3-manifold has an open book decomposition.

**Proof:**

- Alexander proved any closed oriented 3-manifold $M$ is a branched cover of $S^3$ with branch locus some link $L$

$$p : M \rightarrow S^3$$

such that

$$p : (M - p^{-1}(L)) \rightarrow S^3 - L$$

is a covering map
We know any link can be braided about the unknot $U$.

That is $L$ can be isotoped to be transverse to the pages of the open book $(U, \pi)$ of $S^3$.

Set $B = \rho^{-1}(U)$

**Exercise:**

$\pi' = \pi \circ \rho : (M - B) \to S^1$

is a fibration.
In 2000 Giroux made the following definition: a contact structure \( \xi \) on \( M \) is **supported** by an open book \((L, \tau)\) if there is a 1-form \( \alpha \) s.t.:

1) \( \mathfrak{I} := \ker \alpha \)

2) \( \alpha(v) > 0 \) \( \forall v \) pos tangent to \( L \)

3) \( \tau^*(d\theta) \wedge d\alpha > 0 \) where \( \theta \) is the coord on \( S^1 \) (i.e. \( d\alpha \) is an area form on each page)

**Example:**

1) \((U, \bar{\tau})\) supports the standard contact structure on \( S^3 \)

2) \((H_+, \tau)\) does too

**Thm** (Thurston-Winkelnkemper 1975)

Every open book supports a contact structure
Remark 1) This shows all three manifolds have contact structures
2) Not hard to show supported contact str is unique

To prove this theorem we need to reinterperate open books from the **monodromy** perspective.

Given an open book \((L, \pi)\) of \(M\), let \(\Sigma = \pi^{-1}(\Theta)\) for some \(\Theta \in \mathcal{S}'\).

**Note:**
\[
\begin{align*}
(M - L) \setminus \Sigma & \cong \Sigma \times [0,1] \\
\downarrow & \downarrow \\
S' \setminus \{\Theta\} & \cong [0,1]
\end{align*}
\]

so we recover \(M \setminus L\) by gluing \(\Sigma \times \{0\}\) to \(\Sigma \times \{1\}\) by some diffeomorphism \(\Phi: \Sigma \to \Sigma\) (that is the identity on \(\partial \Sigma\)).
write $V_\phi = \Sigma \times [0,1]/(0, \phi(x)) \sim (1, x)$

morphism torus of $\phi$

note: for each component of $\partial \Sigma$

$V_\phi$ has a torus boundary component

we recover $M$ from $V_\phi$ by collapsing the circles $\rho \times [0,1]/\sim$ to points

$M = V_\phi/\{\rho \times [0,1]/\sim\}$
Given any surface $\Sigma$ and diffeo. $\phi: \Sigma \to \Sigma$ (=$\text{id}$ on $\Sigma$) we get a manifold

$$M_\phi = \bigvee \frac{\{p \times [0,1]/\sim\}}{\phi}$$

with $L = \{p \times [0,1]/\sim\} \subseteq M_\phi$ binding of 0.65

So we see that we could have defined open books of $M$ to be a pair $(\Sigma, \phi)$ (and an identification of $M$ with $M_\phi$)

$\phi$ is called the **monodromy** of the open book

**Example:**

$(D^2, \phi = \text{id})$ gives $S^3$ with open book given by $U < \text{unknot}$

$$V_\phi = \begin{array}{c}
\text{cylinder} \\
\text{id}
\end{array} = \begin{array}{c}
\text{solid torus}
\end{array}$$
1) \((S^1 \times [0,1], \text{id})\) gives an open book for \(S^2 \times S^1\)

\[
\begin{array}{c}
\text{\(S^2 \times S^1\)}
\end{array}
\]

2) \(\Sigma\) a surface of genus \(g\) with \(n\) boundary components

\[
\phi = \text{id}_\Sigma
\]

Show \(M_\phi = \#_{2g+n-1} S^2 \times S^1\)

3) \(\Sigma = S^1 \times \mathbb{E}_{0,1}\)

\[
\phi = \text{right handed Dehn twist}
\]

Show \(M_\phi = S^3\) and binding of the open book is Hopf link
Proof of Thurston-Winkelnkemper:

- First we need a little room near binding so given \((\Sigma, \phi)\) let \(\Sigma' \subset \Sigma\) such that \(\Sigma \setminus \Sigma'\) is a nbhd of \(\partial \Sigma\) and isotope \(\phi\) so \(\phi|_{\Sigma \setminus \Sigma'} = \text{id}\).

\[
\text{note: } \frac{(\Sigma \setminus \Sigma')/\sim_{\phi}}{\{p \times [0,1]/\sim\}} \subset \mathcal{M}_\phi
\]

\[
\frac{\Sigma}{\{p \times [0,1]/\sim\}} = \text{is a nbhd of binding}
\]
So $M_\phi = \bigvee \phi|_\Sigma$,

we construct a contact structure on each piece.

- For $\bigvee \phi|_\Sigma$,

  exercise: there is a 1-form $\beta$ on $\Sigma'$ such that
  1) $d\beta$ an area form on $\Sigma'$
  2) $\beta = r \, d\theta$ near $\partial \Sigma'$

set

$$\lambda_{(x,t)} = t \beta_x + (1-t) (\phi^* \beta)_x$$

exercise: $\alpha_K = \lambda_{(x,t)} + K \, dt$ is a contact form on $\bigvee \phi|_\Sigma$

for large $K$
For solid tori, the 1-form
\[ \alpha = \beta \rho \partial_r \times f(r) \, d\psi \]
where \[ f(r) \]
will extend \( \alpha_k \) over solid tori.

Note: The theorem (plus uniqueness remark) gives a well-defined function
\[ \Phi: \{ \text{open books on } M \} \rightarrow \{ \text{contact structures on } M \} \]
Thm (Giroux 2000):

1) $\Phi$ is onto.

2) If $\Phi((\mathcal{L}, \mathcal{P})) = \Phi((\mathcal{L}', \mathcal{P}'))$ then $(\mathcal{L}, \mathcal{P})$ and $(\mathcal{L}', \mathcal{P}')$ are related by positive stabilization.

Remark: This theorem has been a cornerstone of contact geometry since 2000 and is the key to the braid theory results from yesterday.

Given an open book $(\Sigma, \Phi)$ and any arc $\gamma$ properly embedded in $\Sigma$. 

\[
\begin{array}{c}
\Sigma \\
\end{array}
\]
Let $\Sigma' = \Sigma \cup 1$-handle (attached along $\partial \Sigma$)

and $\hat{\gamma} = \gamma \cup$ core of 1-handle

set $\phi' = D_{\hat{\gamma}} \circ \phi$

We say $\left( \Sigma', \phi' \right)$ is a \underline{positive} stabilization of $\left( \Sigma, \phi \right)$

\textbf{Exercise:}

1) Show $M_\phi = M_{\phi'}$

2) Determine how the binding changes. (Hint: Hopf plumbing)

3) Show supported contact structures are the same
Recall we are trying to prove

\textbf{Thm} (E-Van Horn-Morris 2010):

let \( K \) be
1) fibered
2) strongly quasi-positive

link in \( S^3 \)

and \( \Sigma \) is associated Seifert surface

Then there is a unique transverse knot in \((S^3, \xi_{\text{std}})\) in the knot type of \( K \) with \( SL = -\chi(\Sigma) \).

First we need

\textbf{Thm} (E-VM):

let \( M \) be an atoroidal 3-manifold

and \( \xi \) a tight contact structure

and \( L \) a fibered knot in \( M \)

with the fiber a Seifert surface \( \Sigma \n
Then \( L \) supports \( \xi \)

\[ \Leftrightarrow \]

\exists \text{ transverse } K \text{ isotopic to } L \text{ with } \text{SL}(K) = -\chi(\Sigma) \]
1) Bennequin bound on $\text{sl}$ is not sharp for (nicely) fibered links unless the link supports the contact structure!

2) $S^3$ is atoroidal and $K$ strongly quasi-positive

$\Rightarrow \text{sl}(K) = -\chi(\Sigma)$

$K$ fibered then $\Sigma$ a fiber so $K$ supports $\mathfrak{f}_{\text{std}}$

3) if $M = S^3$, Hedden proved that using Heegaard-Floer Theory

4) From the construction of $\mathfrak{f}$ associated to a fibered $K$ one may easily check all singularities of $\Sigma_f$ are positive so $\Rightarrow$ in theorem is clear.
Sketch of Proof (of $\leq$ uses convex surfaces):

- Suppose $K$ is isotopic to $L$ with $\ell(K) = \chi(L)$
- Let $\Sigma_0, \Sigma_{\nu_2}$ be two fibers in fibration

\[ \Sigma \times [\nu_2, 1] \]

\[ \Sigma \times [0, \nu_2] \]

\[ \Sigma_0 \]

\[ \Sigma_{\nu_2} \]

\[ S^3 \]

**Note:** $V_0, V_1$ give a Heegaard splitting of $S^3$

Let $S = \partial V_0 = \partial V_1 = \Sigma_0 \cup \Sigma_{\nu_2}$

- A surface $F$ is called **convex** if there is a vector field $v$ transverse to $F$ whose flow preserves $\lambda$

Given $F$ and $v$, let  \[ \Gamma = \{ p \in F : v(p) \in T_p F \} \] be the dividing set
\[ \text{Th}^m \text{(Torisu 2000)} \]

Given \( M, T, K, S \) as above

\[ K = \Gamma \text{ supports } \exists \]

\[ S \text{ can be made convex with } \Gamma = K \]

\( \Rightarrow \) almost obvious

\( \Leftarrow \) not hard to show \( \exists ! \) contact structure satisfying hypotheses

- Easy to make \( \Sigma = \Sigma_0 \) convex

\[ \text{and can take } V_0 \text{ an invariant nbhd of } \Sigma \text{ so } S = \Sigma_0 \cup \Sigma_{1/2} \]
• Get rid of extra dividing curves using bypasses

If you push a convex surface past a bypass, it changes as follows.
**Note:** If we find a bypass

then push $\Sigma$ over bypass to get

so with enough bypasses we can get

so $\Sigma$ and we are done by Torisu.

- So where to find bypasses?
Answer: Disks in $V_1$

recall $\partial V_1 = \partial \left( \Sigma \times \{\frac{1}{2}, 1\} \right)$

= $\Sigma_{\frac{1}{2}} \cup \Sigma_1$

\[\text{id} \downarrow \downarrow \downarrow \phi\]

$S = \partial V_0 = \Sigma_{\frac{1}{2}} \cup \Sigma_0$

If $\gamma$ any arc properly embedded in $\Sigma$ then $D_\gamma = \gamma \times \{\frac{1}{2}, 1\}$

is a disk in $V_1$

on $S$ we see $\partial D = \gamma \cup \phi(\gamma)$

on $\Sigma_{\frac{1}{2}}$ on $\Sigma_0$
• Now if two convex surfaces meet (along curve tangent to \( K \)) then dividing curves \underline{interlace}

• So on \( D_3 \), we see

\[ \text{we can use the boundary parallel dividing curves to find bypasses and if they are not attached to an arc hitting } K \text{ can use as above.} \]

\textbf{Exercise:} Where does a toroidal come in?
Not hard to generalize the above to show

**Thm (E-VM):**

Suppose $L$ is any fibered link in an atoroidal 3-manifold $M$ with fiber the Seifert surface $\Sigma$.
Let $\xi$ be the supported contact structure.

Then given any two transverse links $K_1, K_2$ topologically isotopic to $L$ with $sL = -\chi(\Sigma)$ there is a contactomorphism $f : M \to M$ s.t.

$$f(K_1) = K_2$$

And our main result about strongly quasi-positive knots in $S^3$ clearly follows from this.
VI Monoids and Geometry

Recall a monoid is a set $G$ with a multiplication that is associative and has identity (that is a “group without inverses”)

It is easy to see that the following subsets of the braid group $B(n)$:

1) $P = \{\text{positive braids}\}$
2) $QP = \{\text{quasi-positive braids}\}$
3) $SQP = \{\text{strongly quasi-positive braids}\}$

We have seen monoids are associated with interesting geometric properties (they also give interesting algebraic structures)
Some other interesting monoids are associated to contact geometry.

Given a surface $\Sigma$ with boundary, let $\text{Map}^+(\Sigma)$ be the mapping class group of orientation-preserving diffeos of $\Sigma$ that are the identity on $\partial \Sigma$.

We know from above that

$\phi \in \text{Map}^+(\Sigma) \mapsto \{ \phi \in \text{Map}^+(\Sigma) : \Gamma_\phi \text{ is a contact structure} \}$

$\phi \in \text{Map}^+(\Sigma) \mapsto \{ \phi \in \text{Map}^+(\Sigma) : \Gamma_\phi \text{ is a 3-manifold} \}$

Given a property $\mathcal{P}$ of contact structures, let

$\text{Map}_\mathcal{P}(\Sigma) = \{ \phi \in \text{Map}^+(\Sigma) : \Gamma_\phi \text{ has property } \mathcal{P} \}$

Is $\text{Map}_\mathcal{P}(\Sigma)$ a monoid?

Sometimes yes, sometimes no.
Examples:

1) $P = \text{tight}$
   denote $\text{Map}_P(\Sigma)$ by $\text{Tight}(\Sigma)$

2) $P = \text{Stein fillable}$
   denote $\text{Map}_P(\Sigma)$ by $\text{Stein}(\Sigma)$

   $\exists$ complex 4-mfld.
   $(\Sigma,J)$ that
   properly embeds in
   $\mathbb{C}^n$
   s.t.
   $M = \partial X$
   $\exists = T \partial X \cap J T \partial X$

3) $P = \text{universally tight}$
   denote $\text{Map}_P(\Sigma)$ by $\text{UT}(\Sigma)$

   $\exists$ pulled back
to the universal
cover of $M$ is
tight
4) \( P = \textit{strongly fillable} \) denote \( \text{Map}_P(\Sigma) \) by \( \text{Strong}(\Sigma) \)

\[
\exists \text{ a symplectic mfd.} \quad (X, \omega) \quad \text{and a vector field} \quad v \quad \text{such that} \quad \forall x \in M \quad \text{st.} \quad L_v \omega = \omega \\
\exists \gamma = \ker(L_v \omega)_{|_M}
\]

5) \( P = \textit{weakly fillable} \) denote \( \text{Map}_P(\Sigma) \) by \( \text{Weak}(\Sigma) \)

\[
\exists \text{ a symplectic mfd.} \quad (X, \omega) \quad \text{such that} \quad \omega|_{\gamma} \quad \text{non-degenerate}
\]

Let \( \text{Dehn}^+(\Sigma) = \) compositions of positive Dehn twists
It is known that

\[ \text{Dehn}^+ \cong \text{Stein} \cong \text{Strong} \cong \text{Weak} \cong \text{Tight} \]

\[ UT \cong \]

\[ \text{Th}^\text{m} (\text{Baker-E-Van Horn-Morris 2010}) \]

\[ \text{Stein, Strong, Weak are monoids} \]

\[ UT \text{ is not a monoid} \]

**Major Open Question:** Is Tight a monoid? \((\Leftrightarrow \text{Legendrian surgery preserves tightness})\)

**Other Questions:**

1) Can you characterize when a given \( \phi \) is in one of the above monoids? \((\text{e.g. } \Sigma \text{ planar then } \text{Dehn}^+ = \text{Stein} = \text{Strong Word})\)

2) Are there other monoids in \( \text{Map}^+(\Sigma) \)? Do they correspond to anything interesting in the contact world?
Recall, we say a knot $K$ in $S^3$ \red{braided} if it is transverse to the pages of the open book $(U,\overline{U})$.

In general, given any open book $(L,\overline{L})$ of $M$ we say $K$ is \red{braided about $L$} if $K\cap L = \emptyset$ and $K$ is transverse to the pages of $(L,\overline{L})$.

We have the following generalization of Alexander + Markov.
Thm (Skora 1992, Sundheim 1993):

- any knot $K$ in $M$ can be braided about any open book $(L, \pi)$ of $M$.
- two "braids" are isotopic as knots (⇒ they are related by "braid isotopy" and Markov moves).

We also have

Thm (Pavelescu 2008):

- let $\xi$ be the contact structure supported by the open book $(L, \pi)$ for $M$.
- any transverse knot $K$ can be braided about $(L, \pi)$.
- two braids are isotopic through transverse knots (⇒ they are related by braid isotopy and positive Markov moves).
These theorems prompt a huge number of interesting questions. For example:

1) on $S^3$ all the positive torus knots $K_{pq}$ are fibered.

So given any $K$ we can braid it about any $K_{pq}$ and it has a $(pq)$-braid index $b_{pq}(K)$.

Exercise: $b_{pq}(K) = 1$ for some $pq$.

Question: how good of a knot invariant is the set of all $b_{pq}(K)$?
Since all the $K_{g,q}$ ($g \geq 0$) support the standard contact str on $S^3$ we can ask the same question for transverse knots.

2) Given any $(\mathcal{L}, \overline{\mu})$ for $\mathcal{M}$ and a sequence of stabilizations:

$$(\mathcal{L}, \overline{\mu}) \rightarrow (\mathcal{L}_1, \overline{\mu}_1) \rightarrow (\mathcal{L}_2, \overline{\mu}_2) \rightarrow \ldots$$

we again get an infinite sequence of braid indices $b_k$ for a knot.

**Exercise:** $b_k(K) = b_{k+1}(K)$

**Question:** Is $b_k(K)$ always 1 for some $k$? How good of an invariant are these?
3) Understand the algebraic structure of the generalized braid groups, and how they relate to knots.

4) Understand how braid stabilization and open book stabilization interact.

5) Try to generalize Bonnepin's arguments to prove certain contact structures are tight without using the hard analysis that is currently needed.