V Open Book Decomposition (continued)

last time we saw many open books for S^3 but the exist for all M^3

Thm (Alexander):

> Every 3-manifold has an open book decomposition.

Proof:

- Alexander proved any closed oriented 3-manifold M is a branched cover of S^3 with branch locus some link L

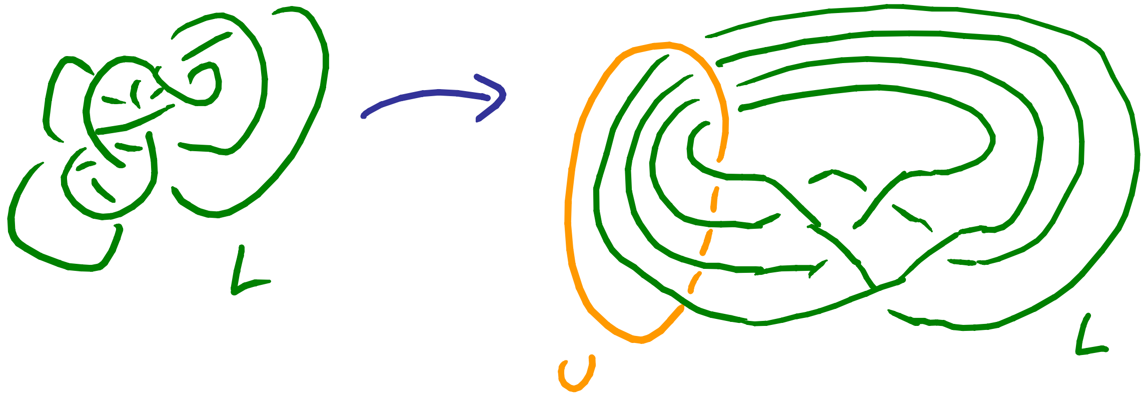
$$p: M \to S^3$$

such that

$$p: (M - p^{-1}(L)) \to S^3 - L$$

is a covering map

- We know any link can be braided about the unknot U.



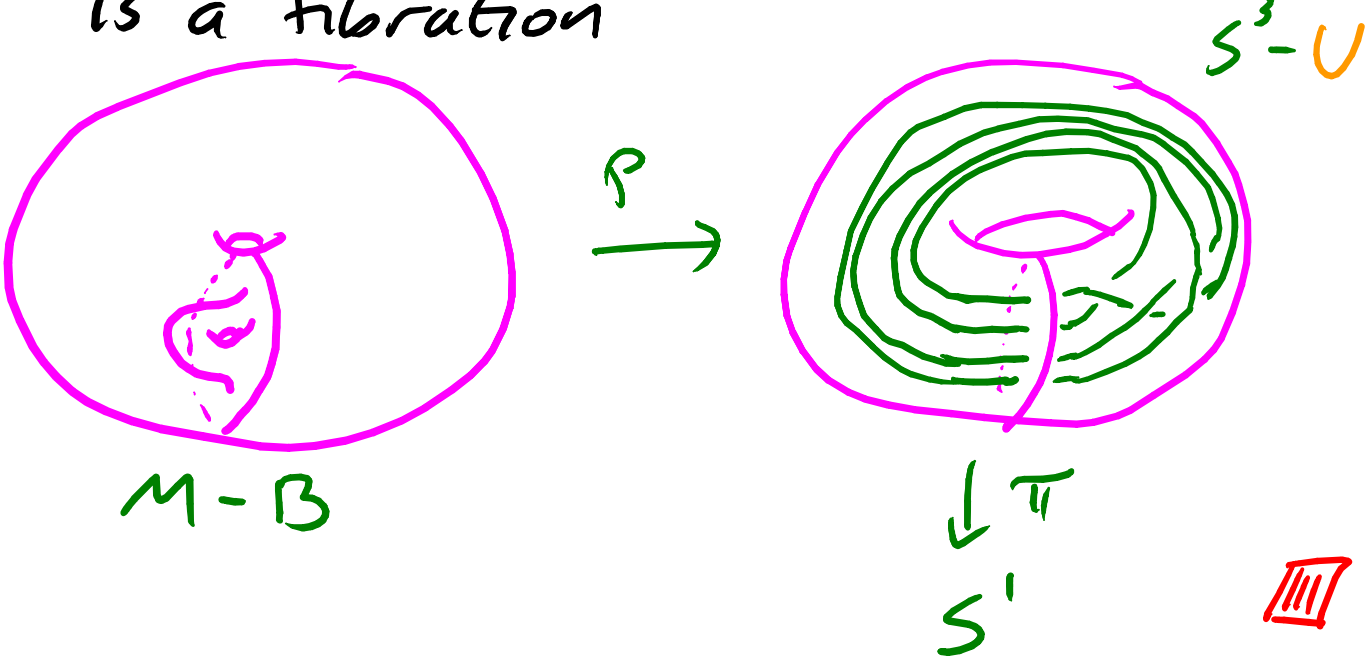
that is L can be isotoped to be transverse to the pages of the open book (U, π) of S^3

- Set $B = p^{-1}(U)$

<u>exercise:</u>

$$\pi' = \pi \circ p : (M - B) \to S^1$$

is a fibration



In 2000 Giroux made the following definition: a contact structure ξ on M is <u>supported</u> by an open book (L, π) if there is a 1-form α s.t.

1) $\xi = \ker \alpha$
2) $\alpha(v) > 0$ $\forall v$ pos tangent to L
3) $\pi^*(d\theta) \wedge d\alpha > 0$ where θ is the coord on S^1 (i.e. $d\alpha$ is an area form on each page)

<u>example:</u>

1) (U, π) supports the standard contact structure on S^3
2) (H_+, π) does too

<u>Thm (Thurston-Winkelnkemper 1975)</u>

$\boxed{\text{Every open book supports a contact structure}}$

Remark 1) This shows all three manifolds have contact structures

2) Not hard to show supported contact str is unique

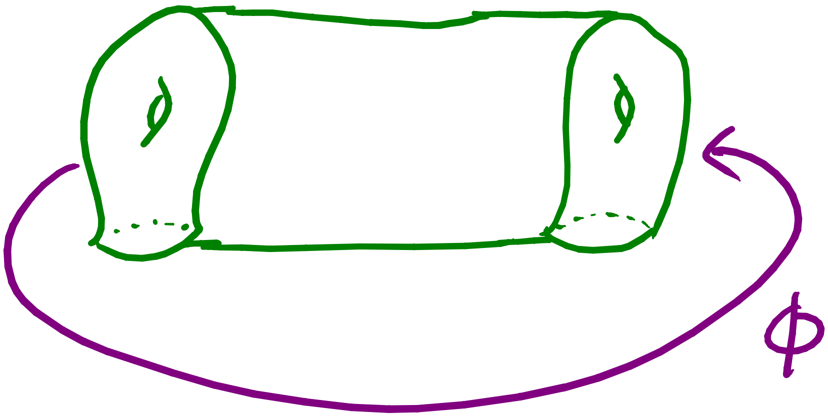
To prove this theorem we need to reinterperote open books from the monodromy perspective

Given an open book (L, π) of M let $\Sigma = \overline{\pi^{-1}(\theta)}$ for some $\theta \in S^1$

note:

$$\begin{array}{ccc} ((M-L) \setminus \Sigma) & & \Sigma \times [0,1] \\ \downarrow & = & \downarrow \\ S^1 \setminus \{\theta\} & & [0,1] \end{array}$$

so we recover $M \setminus L$ by gluing $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ by some diffeomorphism $\phi : \Sigma \to \Sigma$ (that is the identity on $\partial\Sigma$)

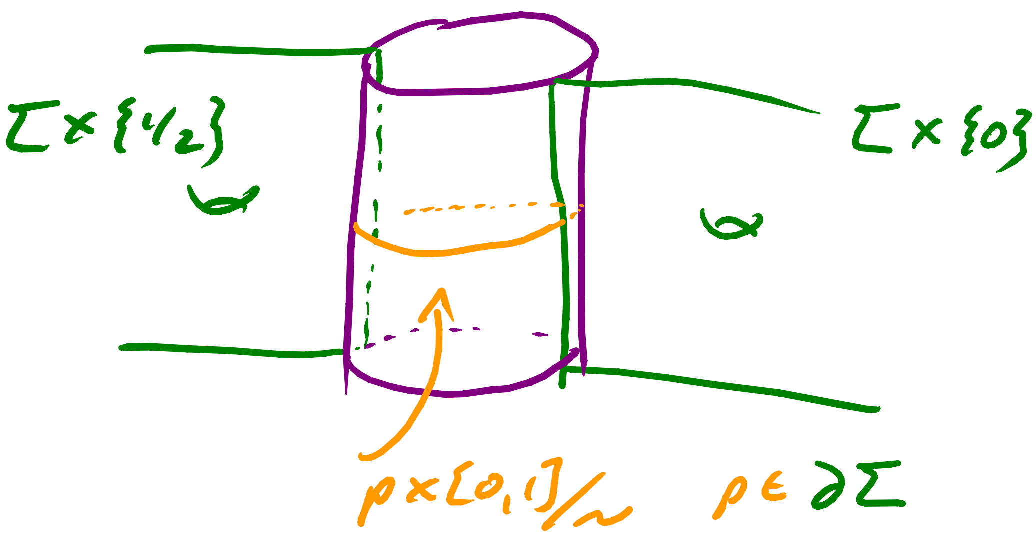


write $V_\phi = \Sigma \times [0,1] / (0,\phi(x)) \sim (1,x)$

mapping torus of ϕ

note: for each component of $\partial\Sigma$

V_ϕ has a torus boundary component



we recover M from V_ϕ by collapsing the circles $p \times [0,1]/\sim$ to points

$$M = V_\phi / \{p \times [0,1]/\sim\}$$

note: Given any surface Σ and diffeo. $\phi: \Sigma \to \Sigma$ ($= id$ on $\partial\Sigma$) we get a manifold

$$M_\phi = V_\phi \Big/ \{p \times [0,1]/\sim\}$$

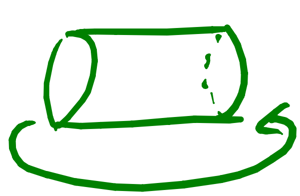
with $L = \{p \times [0,1]/\sim\} \subseteq M_\phi$ binding of O.B.

So we see that we could have defined open books of M to be a pair (Σ, ϕ) (and an identification of M with M_ϕ)

ϕ is called the monodromy of the open book

example:

$(D^2, \phi = id)$ gives S^3 with open book given by U ← unknot

$V_\phi =$ 

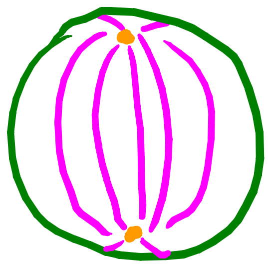
id

$=$ 

solid torus

exercisé!

1) $(S^1\times[0,1], id)$ gives an open book for $S^2\times S^1$

S^2  $\times S^1$

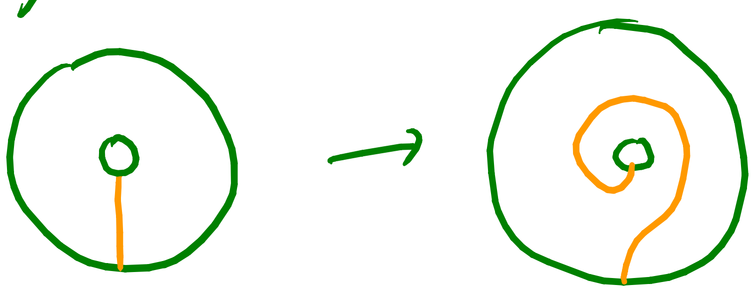
2) Σ a surface of genus g with n boundary components

$\phi = id_\Sigma$

show $M_\phi = \#_{2g+n-1} S^2\times S^1$

3) $\Sigma = S^1\times[0,1]$

ϕ = right handed Dehn twist



show $M_\phi = S^3$ and binding of the open book is Hopf link

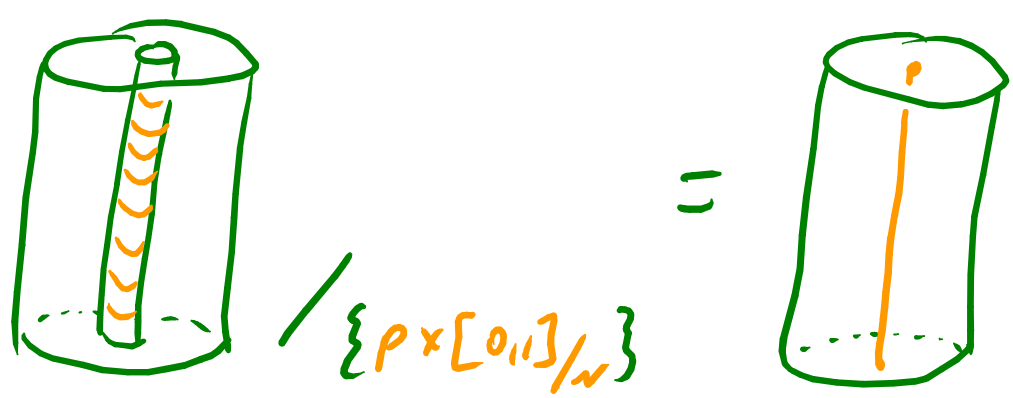
Proof of Thurston-Winkelnkemper:

- First we need a little room near binding so given (Σ, ϕ) let $\Sigma' \subset \Sigma$ such that

  $\Sigma \setminus \Sigma'$ is a nbhd of $\partial \Sigma$

  and isotope ϕ so $\phi|_{\Sigma \setminus \Sigma'} = id$

<u>note</u>: $\underbrace{\left((\Sigma \setminus \Sigma')/\sim_\phi\right)}/\{p \times [0,1]/\sim\} \subset M_\phi$



is a nbhd of binding

so $M_\phi = V_{\phi|_{\Sigma'}} \cup$ solid tori

we construct a contact structure on each piece.

- For $V_{\phi|_{\Sigma'}}$

<u>exercise</u>: there is a 1-form β on Σ' such that

1) $d\beta$ an area form on Σ'
2) $\beta = r\,d\theta$ near $\partial\Sigma' = \{1\} \times S^1$

set

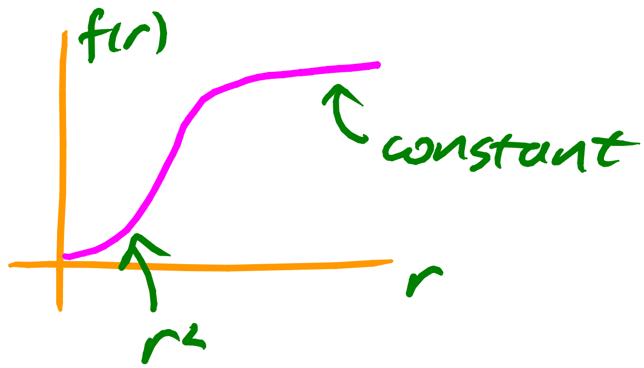
$$\lambda_{(x,t)} = t\beta_x + (1-t)(\phi^*\beta)_x$$

<u>exercise</u>: $\alpha_K = \lambda_{(x,t)} + K\,dt$ is a contact form on $V_{\phi|_{\Sigma'}}$ for large K

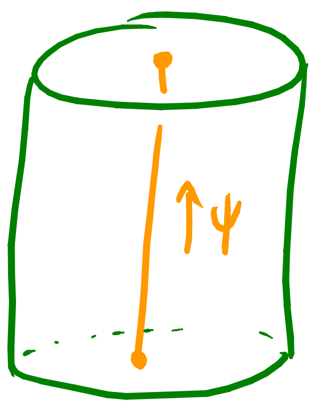
- For solid tori the 1-form

$$\alpha = \beta|_{\partial\Sigma'} \times f(r)\, d\psi$$

where



will extend α_K over solid tori



Note: The theorem (plus uniqueness remark) gives a well-defined function

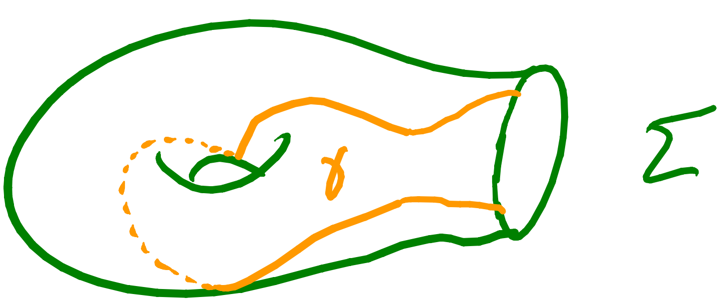
$$\boxed{\Phi : \left\{ \begin{array}{l} \text{open books} \\ \text{on } M \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{contact} \\ \text{structures} \\ \text{on } M \end{array} \right\}}$$

Thm (Giroux 2000):

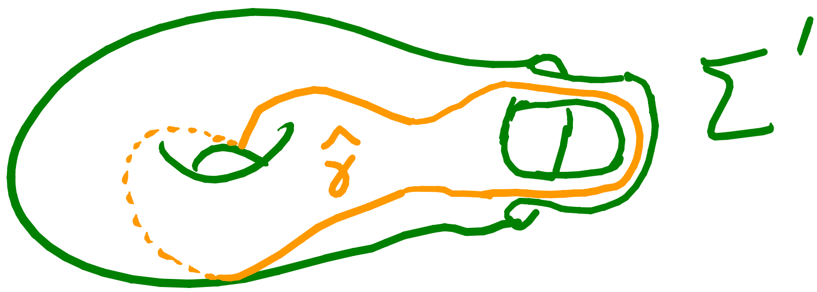
1) Φ is onto.
2) If $\Phi(L,\pi) = \Phi(L',\pi')$ then (L,π) and (L',π') are related by positive stabilization.

Remark: This theorem has been a cornerstone of contact geometry since 2000 and is the key to the braid theory results from yesterday.

Given an open book (Σ,ϕ) and any arc γ properly embedded in Σ



let $\Sigma' = \Sigma \cup$ 1-handle
(attached along $\partial\gamma$)



and $\hat{\gamma} = \gamma \cup$ core of 1-handle

set $\phi' = D_{\hat{\gamma}} \circ \phi$

right handed Dehn twist about $\hat{\gamma}$

We say (Σ', ϕ') is a <u>positive stabilization</u> of (Σ, ϕ)

<u>exercise:</u>

1) Show $M_\phi = M_{\phi'}$
2) Determine how the binding changes. (hint: Hopf plumbing)
3) Show supported contact strs are the same

Recall we are trying to prove

Thm (E-Van Horn-Morris 2010):

let K be 1) fibered
2) strongly quasi-positive
link in S^3
and Σ is associated Seifert surface
Then there is a unique transverse knot in (S^3, ξ_{std}) in the knot type of K with $sl = -\chi(\Sigma)$.

First we need

Thm (E-VM):

let M be an atoroidal 3-manifold
and ξ a tight contact structure
and L a fibered knot in M
with the fiber a Seifert surface Σ
Then L supports ξ

$\Longleftrightarrow$

$\exists$ transverse K isotopic to L with
$sl(K) = -\chi(\Sigma)$

note:

1) Bennequin bound on sl is not sharp for (nicely) fibered links unless the link supports the contact structure!

2) S^3 is atoroidal and K strongly quasi-positive
$\Rightarrow sl(K) = -\chi(\Sigma)$
K fibered then Σ a fiber so K supports ξ_{std}

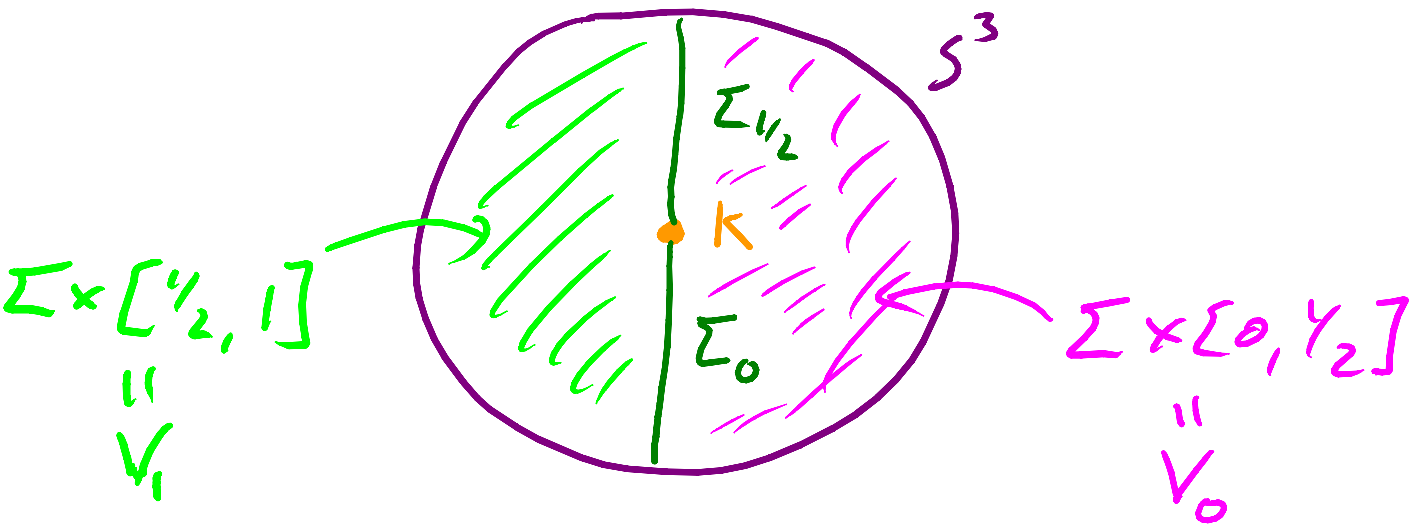
3) if $M = S^3$, Hedden proved thm using Heegaard-Floer Theory

4) From the construction of ξ associated to a fibered K one may easily check all singularities of Σ_ξ are positive so $\Rightarrow$ in theorem is clear.

Sketch of Proof (of $\Leftarrow$ uses convex surfaces):

- Suppose K is isotopic to L with $sl(K) = -\chi(\Sigma)$

- let $\Sigma_0, \Sigma_{1/2}$ be two fibers in fibration



note: V_0, V_1 give a Heegaard splitting of S^3

let $S = \partial V_0 = \partial V_1 = \Sigma_0 \cup \Sigma_{1/2}$

- A surface F is called convex if there is a vector field v transverse to F whose flow preserves ξ

- Given F and v let

$$\Gamma = \{p \in F : v(p) \in \xi_p\} \leftarrow \text{dividing set}$$

Thm (Torisu 2000)

Given M, ξ, K, S as above

$K = L$ supports ξ

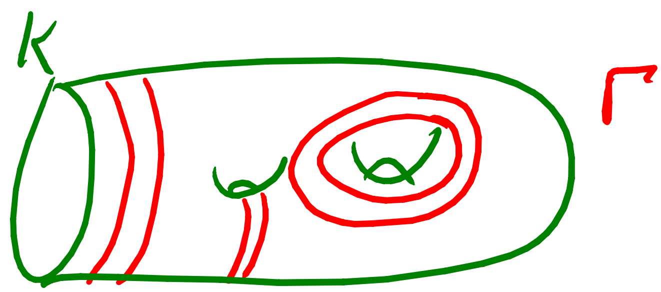
$\Longleftrightarrow$

S can be made convex with $\Gamma = K$

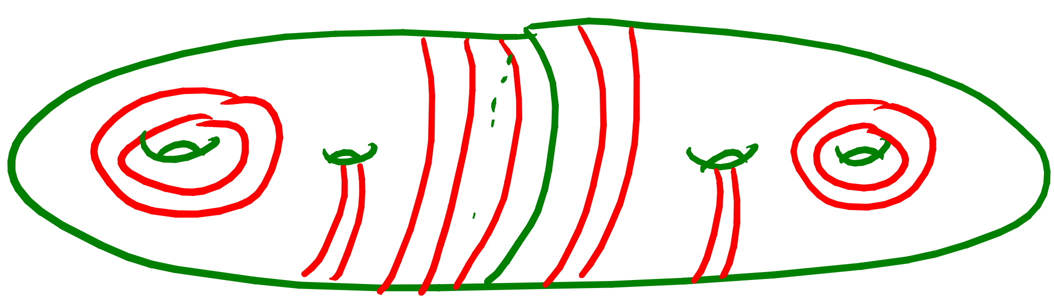
($\Longrightarrow$ almost obvious

$\Longleftarrow$ not hard to show $\exists!$ contact structure satisfying hypothesis)

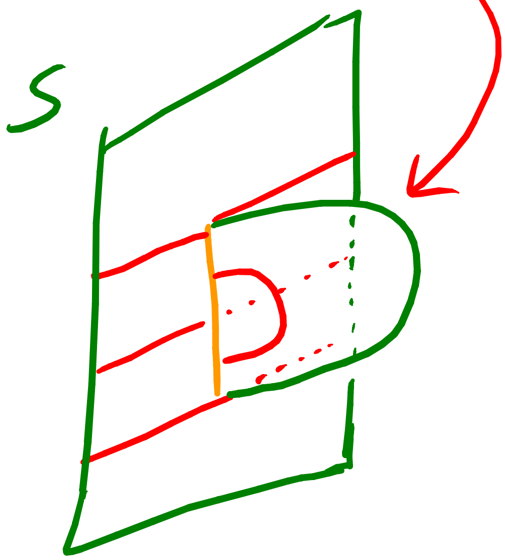
- Easy to make $\Sigma = \Sigma_0$ convex



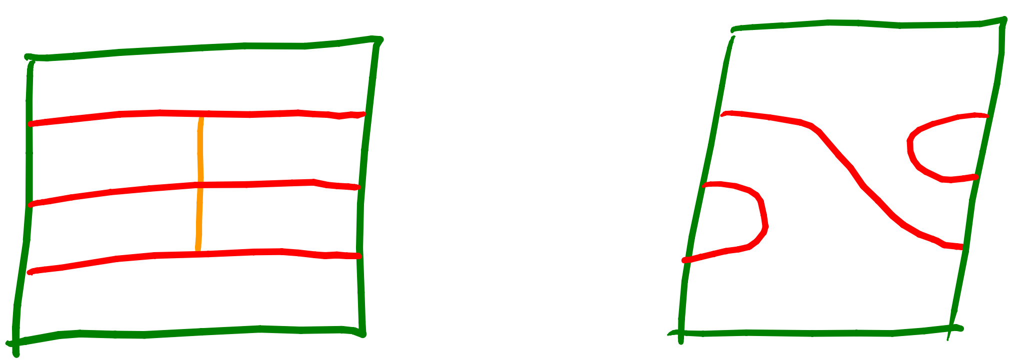
and can take V_0 an invariant nbhd of Σ so $S = \Sigma_0 \cup \Sigma_{1/2}$



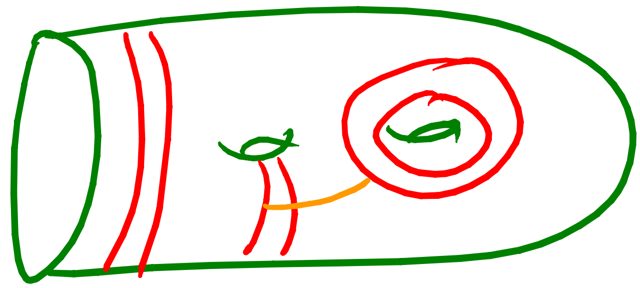
- Get rid of extra dividing curves using bypasses



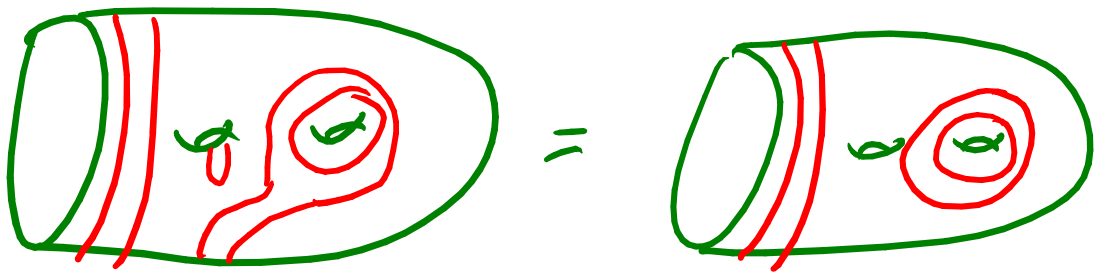
If you push a convex surface past a bypass Γ changes as follows



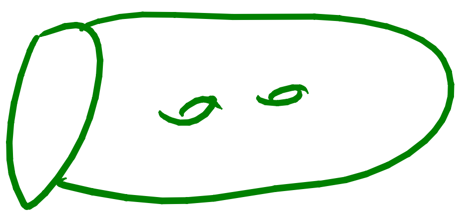
note: if we find a bypass

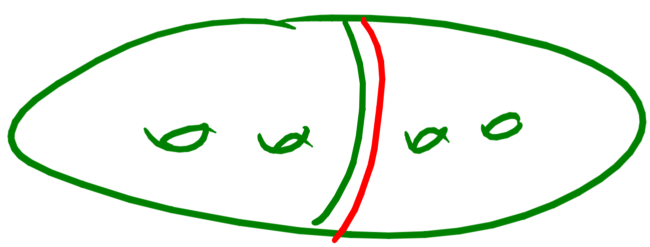


then push Σ over bypass to get



so with enough bypasses we can get

Σ 

so S 

and we are done by Torisu.

- So where to find bypasses?

- Answer: Disks in V_1

recall $\partial V_1 = \partial(\Sigma \times [1/2, 1])$

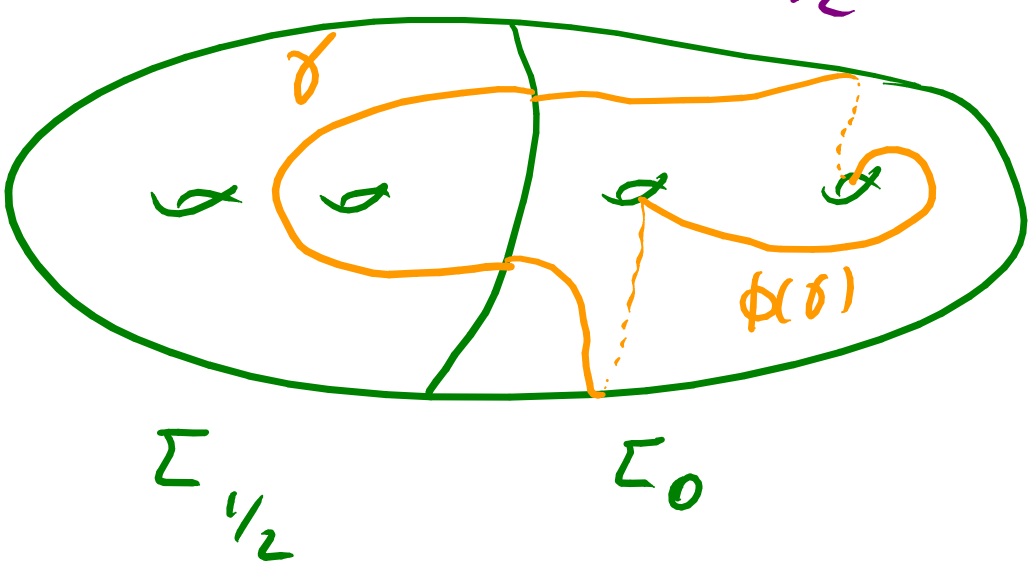
$$\begin{array}{ccccc} & = & \Sigma_{1/2} & \cup & \Sigma_1 \\ & & \downarrow \text{id} & & \downarrow \phi \\ S = \partial V_0 & = & \Sigma_{1/2} & \cup & \Sigma_0 \end{array}$$

monodromy

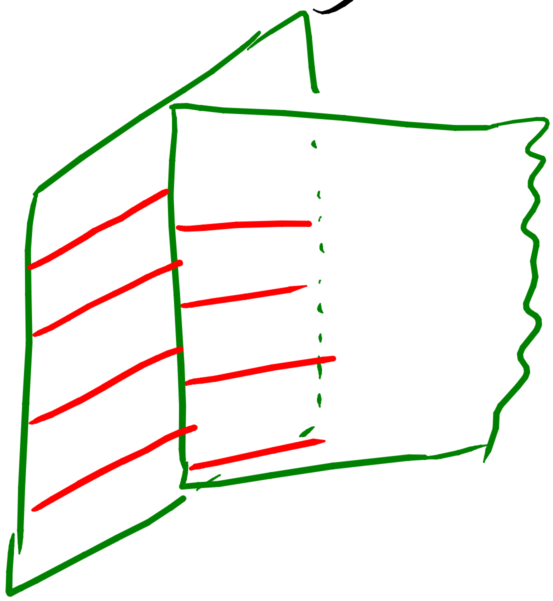
if γ any arc properly embedded in Σ then $D_\gamma = \gamma \times [1/2, 1]$ is a disk in V_1

on S we see $\partial D = \gamma \cup \phi(\gamma)$

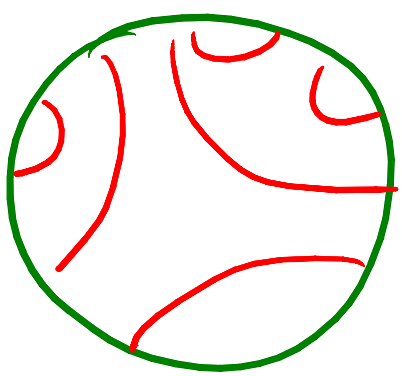
on $\Sigma_{1/2}$ (for γ), on Σ_0 (for $\phi(\gamma)$)



- Now if two convex surfaces meet (along curve tangent to ξ) then dividing curves interlace



- So on D_γ we see



we can use the boundary parallel dividing curves to find bypasses and if they are not attached to an arc hitting K can use as above.

exercise: Where does atoroidal come in?

Not hard to generalize the above to show

Thm (E-VM):

Suppose L is any fibered link in an atoroidal 3-manifold M with fibe the Seifert surface Σ let ξ be the supported ct str

Then given any two transverse links K_1, K_2 topologically isotopic to L with $s\ell = -\chi(\Sigma)$ there is a contactomorphism $f: M \to M$ s.t. $f(K_1) = K_2$

And our main result about strongly quasi-positive knots in S^3 clearly follows from this.

VI Monoids and Geometry

Recall a monoid is a set G with a multiplication that is associative and has identity

(that is a "group without inverses")

It is easy to see that the following subsets of the braid group $B(n)$

1) $P = \{$positive braids$\}$
2) $QP = \{$quasi-positive braids$\}$
3) $SQP = \{$strongly qp-braids$\}$

We have seen monoids are assocated with interesting geometric properties

(they also give interesting algebraic structures)

Some other interesting monoids are associated to contact geometry.

Given a surface Σ with boundary let $Map^+(\Sigma)$ be the mapping class group of orientation pres. diffeos of Σ that are the identity on $\partial\Sigma$

We know from above that

3-manifold

$$\phi \in Map^+(\Sigma) \rightsquigarrow \begin{cases} M_\phi \\ \xi_\phi \end{cases}$$

contact structure

Given a property P of contact structures, let

$$Map_P(\Sigma) = \{\phi \in Map^+(\Sigma) : \xi_\phi \text{ has property } P\}$$

Is $Map_P(\Sigma)$ a monoid?

sometimes yes, sometimes no

examples:

1) $\mathcal{P}$ = tight denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $\mathrm{Tight}(\Sigma)$

2) $\mathcal{P}$ = Stein fillable denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $\mathrm{Stein}(\Sigma)$

$\exists$ complex 4-mfd. (X, J) that properly embedds in $\mathbb{C}^N$ s.t. $M = \partial X$, $\xi = T\partial X \cap J T\partial X$

3) $\mathcal{P}$ = universally tight denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $UT(\Sigma)$

ξ pulled back to the universal cover of M is tight

4) P = strongly fillable $\iff$ [$\exists$ a sympletic mfd. (X, ω) and a vector field $v \pitchfork \partial X = M$ s.t. $\mathcal{L}_v \omega = \omega$, $\xi = \ker(\iota_v \omega)|_{\partial M}$]

denote $Map_P(\Sigma)$ by $Strong(\Sigma)$

5) P = weakly fillable $\iff$ [$\exists$ a symplectic mfd (X, ω) such that $\omega|_\xi$ non-degenerate]

denote $Map_P(\Sigma)$ by $Weak(\Sigma)$

Let $Dehn^+(\Sigma)$ = compositions of positive Dehn twists

It is known that

$$\text{Dehn}^+ \subsetneq \text{Stein} \subsetneq \text{Strong} \subsetneq \text{Weak} \subsetneq \text{Tight}$$
$$\text{UT} \subsetneq$$

Thm (Baker-E-Van Horn-Morris 2010, Baldwin):

$\boxed{\text{Stein, Strong, Weak are monoids; UT is not a monoid}}$

Stein, Strong, Weak are monoids
UT is not a monoid

Major Open Question: Is Tight a monoid? ($\Leftrightarrow$ Legendrian surgery preserves tightness)

Other Questions:

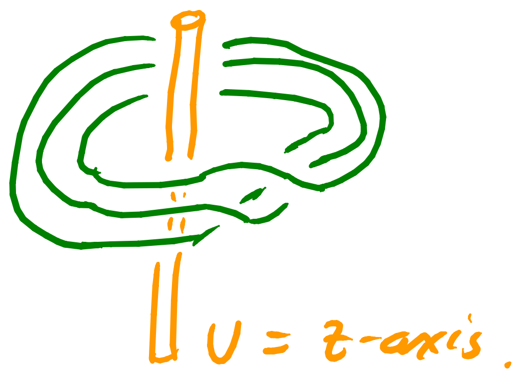
1) Can you characterize when a given ϕ is in one of the above monoids?

   e.g. Σ planar then $\text{Dehn}^+ = \text{Stein} = \text{Strong}$ Wendl

2) Are there other monoids in $\text{Map}^+(\Sigma)$? Do they correspond to anything interesting in the contact world?

VII Generalized Braids:

Recall, we say a knot K in S^3 **braided** if it is transverse to the pages of the open book (U, π)



In general, given any open book (L, π) of M we say K is **braided about L** if $K \cap L = \emptyset$ and K is transverse to the pages of (L, π).

We have the following generalization of Alexander + Markov

Thm (Skora 1992, Sundheim 1993)

- any knot K in M can be braided about any open book (L, π) of M.
- two "braids" are isotopic as knots $\Longleftrightarrow$ they are related by "braid isotopy" and Markov moves

We also have

Thm (Pavelescu 2008):

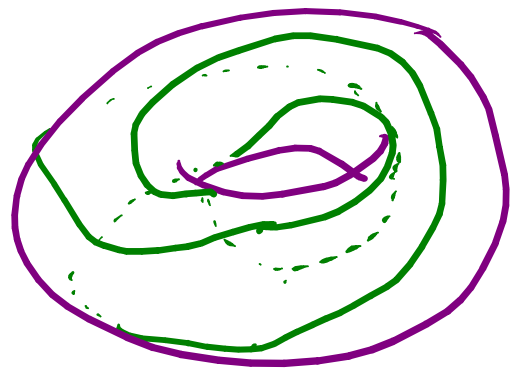
let ξ be the contact structure supported by the open book (L, π) for M

- any transverse knot K can be braided about (L, π)
- two braids are isotopic through transverse knots $\Longleftrightarrow$ they are related by braid isotopy and positive Markov moves

These theorems prompt a huge number of interesting questions.

For example:

1) on S^3 all the positive torus knots $K_{p,q}$



are fibered.

So given any K we can braid it about any $K_{p,q}$ and it has a <u>(p,q)-braid index</u> $b_{p,q}(K)$

<u>exercise:</u> $b_{p,q}(K) = 1$ for some p,q.

<u>question:</u> how good of a knot invariant is the set of all $b_{p,q}(K)$?

Since all the $K_{p,q}$ $(p,q>0)$ support the standard contact str on S^3 we can ask the same question for transverse knots

2) Given any (L,π) for M and a sequence of + stabilizations

$$(L,\pi) \underset{+\text{ stab}}{\to} (L_1,\pi_1) \underset{+\text{ stab}}{\to} (L_2,\pi_2) \to \cdots$$

we again get an infinite sequence of braid indices b_k for a knot

exercise: $b_k(K) \geq b_{k+1}(K)$

question: Is $b_k(K)$ alway 1 for some k?

How good of an invt are these?

3) Understand the algebraic structure of the generalized braid groups, and how they relate to knots.

4) Understand how braid stabilization and open book stabilization interact.

5) Try to generalize Bennequin's arguments to prove certain contact structures are tight without using the hard analysis that is currently needed.

Thanks

for

Your

Attention!