Fibered knots and the Bennequin Bound

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Perspectives in Analysis, Geometry, and Topology

Outline

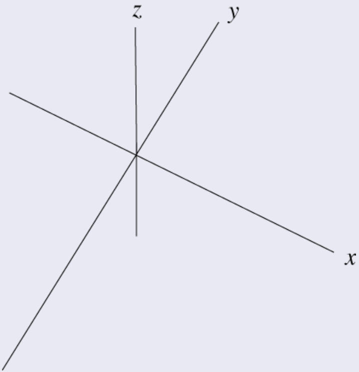
- Let M be closed oriented 3-manifolds.
- An oriented hyperplane field ξ on M is a contact structure if there is a 1-form α such that

$$\xi_x = \ker(\alpha_x : T_x M \to \mathbb{R})$$

for all $x \in M$ and

$$\alpha \wedge d\alpha > 0$$

Example



On $\mathbb{R}^3$
with coordinates (x, y, z)

$$\xi_{std} = \ker(\underbrace{dz - y\,dx}_{\alpha})$$

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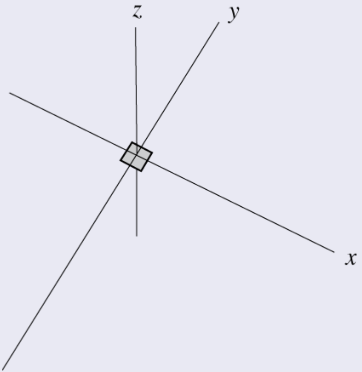

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at $(0,0,0)$ we have the span of

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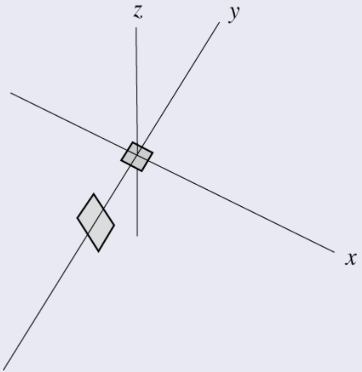

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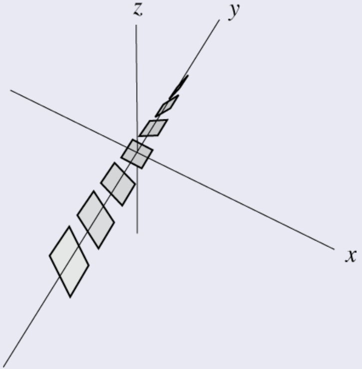

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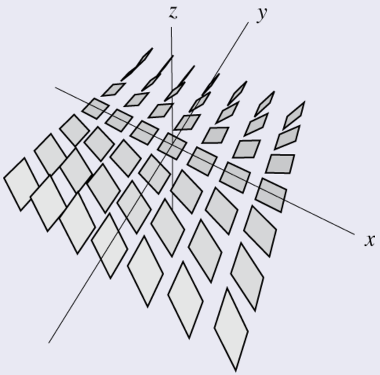

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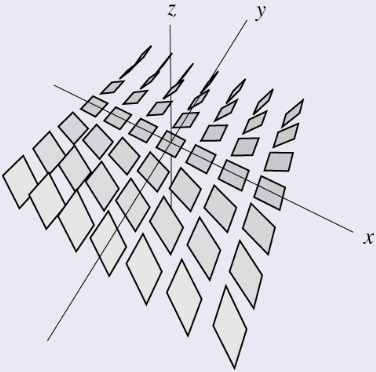

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Darboux: All contact structures are locally diffeomorphic to this one.

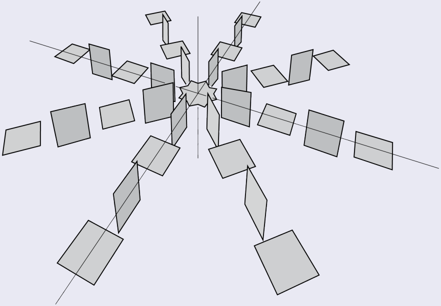
- An overtwisted disk in ξ is an embedded disk D such that

$$T_x D = \xi_x,$$

for all $x \in \partial D$.

- We call ξ overtwisted if there is an overtwisted disk in ξ otherwise ξ is called tight.

Example



On $\mathbb{R}^3$
with coordinates (x, y, z)

$$\xi_{ot} = \ker(\cos r\, dz + r \sin r\, dx)$$

Then $D = \{r \leq \pi, z = 0\}$ is tangent to ξ_{ot}, so ξ_{ot} is overtwisted.

Theorem [Bennequin 1983]

$(\mathbb{R}^3, \xi_{std})$ is tight.

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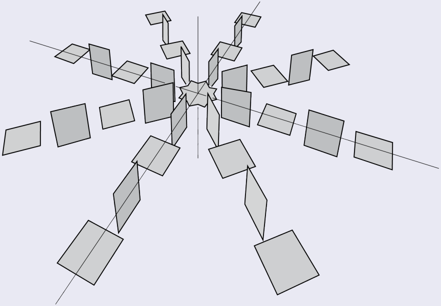

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$(\mathbb{R}^3, \xi_{std})$ is tight.

A transverse knot in a contact manifold (M, ξ) is an embedded circle $K \subset M$ that is transverse to ξ for all $x \in K$,

$$T_x K \oplus \xi_x = T_x M.$$

(We orient K so that the above equality is as oriented vector spaces.)

Transverse knots in $(\mathbb{R}^3, \xi_{std})$

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TRANSVERSE KNOTS IN $(\mathbb{R}^3, \xi_{std})$

If $t \mapsto (x(t), y(t), z(t))$ parameterizes a transverse knot in

$$\xi_{std} = \ker(dz - ydx)$$

then we must have

$$z'(t) - y(t)x'(t) > 0.$$

Transverse Knots
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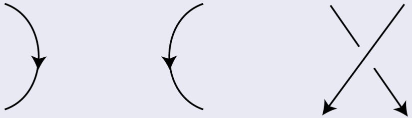
Transverse knots in $(\mathbb{R}^3, \xi_{std})$

So projections to the xz-plane must not have regions like



any other regions OK.

- Let K is a null-homologous knot in M, so there is an oriented surface $\Sigma \subset M$ such that $\partial\Sigma = K$.
- Suppose K is transverse to ξ.
- The ξ restricted to Σ is a trivial bundle

$$\xi|_\Sigma = \Sigma \times \mathbb{R}^2.$$

- Choose a non-zero vector v in $\xi|_\Sigma$.
- The self-linking number of $K, sl(K)$, is the difference between the framings of K given by v and Σ.
- Notice that this number might depend on the relative homology class of Σ, so we should denote the invariant $sl(K, [\Sigma])$.

In the xz-projection of a transverse knot in $(\mathbb{R}^3, \xi_{std})$ the self-linking number is given by

$$sl(K) = \text{writhe}(K).$$

Example

$$sl(K) = writhe(K) = -1$$

Transverse Knots

Examples in $\mathbb{R}^3$

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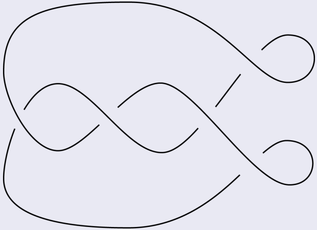
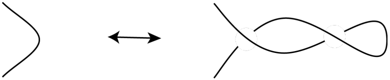


$$sl(K) = writhe(K) = -1$$

We can modify the *xz*-projection of a knot K by:



This results in a transverse knot $S(K)$ in the same knot type but with

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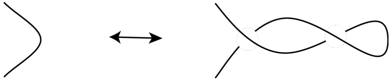

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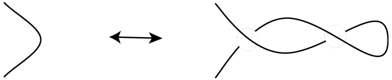

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Theorem (Bennequin '82 and Eliashberg '92)

If (M, ξ) is a tight contact manifold and $K = \partial\Sigma$ is a transverse knot then

$$sl(K) \leq -\chi(\Sigma)$$

- Bennequin proved this for knots in $(\mathbb{R}^3, \xi_{std})$.
- Eliashberg, after defining the notion of tight, proved this for a general tight manifold (need result of Eliashberg and Gromov to prove $(\mathbb{R}^3, \xi_{std})$ is tight independent of Bennequin's result).
- This result was in some real sense the birth of contact topology! It establishes subtle connections between topology (eg. genus of a knot) and contact geometry (eg. the self-linking number).

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- In 1997, Fuchs and Tabachnikov showed for knots in $(\mathbb{R}^3, \xi_{std})$

$$sl(K) \leq d_{P_K},$$

where d_{P_K} is the lowest degree in the variable z for the multi-variable Jones polynomial P_K satisfying

$$\frac{1}{v} P_{K_+} - v P_{K_-} = z P_{P_0}.$$

Thus the Bennequin bound is not always sharp, eg for the left handed trefoil we see

$$sl \leq -5.$$

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- In 2006, Ng gave a bound (on a "Legendrian" analog of self-linking) using Khovanov homology that is sharp for all alternating knots.
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An open book decomposition of a closed 3-manifold M is a pair (B, π) where

- B is an oriented link in M and
- $\pi : (M \setminus B) \to S^1$ is a fibration of the complement of B such that

$$\Sigma_\theta = \overline{\pi^{-1}(\theta)}$$

has boundary B.

We call B the binding of the open book and any Σ_θ a page of the open book.

- Let U be the unknot in S^3 then $S^3 \setminus U = S^1 \times \mathbb{R}^2$ and $\partial(\overline{\theta \times \mathbb{R}^2}) = U$.
- If H is the Hopf link in S^3 then $S^3 \setminus H = T^2 \times \mathbb{R}$ which can be fibered by $(1,1)$-curves in T^2 times $\mathbb{R}$.
- Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial that vanishes at $(0,0)$ and has no critical points inside S^3 except possibly $(0,0)$. Then $B = f^{-1}(0) \cap S^3$ gives an open book of S^3 with fibration

$$\pi_f : S^3 \setminus B \to S^1 : (z_1, z_2) \mapsto \frac{f(z_1, z_2)}{|f(z_1, z_2)|}.$$

This is called the Milnor fibration of the hypersurface singularity $(0,0)$.

Fact [Alexander]

All closed oriented 3-manifold have (many) open book decompositions.

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A contact structure ξ on M is supported by an open book decomposition (B, π) if there is a contact form α for ξ such that

- $\alpha(v) > 0$ for all $v \in T_x B$ pointing in the positive direction and
- $\pi^*(d\theta) \wedge d\alpha > 0$ where θ is the coordinate on S^1.

Theorem [Thurston-Winkelnkemper 1975]

Every open book decomposition of M supports a contact structure.

It is easy to prove the supported contact structure is unique.

Theorem [Giroux 2000]

Every contact structure is supported by some open book decomposition.
In fact there is a one to one correspondence between

$$\{\text{oriented contact structures up to isotopy}\}$$

and

$$\{\text{open book decompositions up to isotopy and positive stabilization}\}$$

We call a link L in M fibered if it is the binding of an open book.
Notice that this is a slightly more restricted definition of fibered that usual.

THEOREM [E – VAN HORN-MORRIS]

Let M be a closed 3-manifold and ξ a tight contact structure on M.

A fibered link (L, Σ) realizes the Bennequin bound in (M, ξ)
if and only if
ξ is supported by the open book (L, Σ) or
ξ is obtained from $\xi_{(L,\Sigma)}$ by adding Giroux torsion.

Giroux torsion is an embedding of

$$(T^2 \times [0,1], \ker(\cos 2n\pi t\, dx + \sin 2n\pi t, dy))$$

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COROLLARY

Let M be a closed atoroidal 3-manifold and ξ is a tight contact structure on M.

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COROLLARY

If the "enhanced Milnor invariant" (a.k.a. "Hopf invariant") of a fibered link L in S^3 does not vanish then the Bennequin bound is not sharp for links transverse to ξ_{std} in the link type L.

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Theorem [E – Van Horn-Morris]

If K is a fibered knot type in (S^3, ξ_{std}) and there is a transverse representative of K with $sl = -\chi(K)$ then all transverse knots in the knot type K with $sl = -\chi(K)$ are transversely isotopic.

Notice that it is still hard to classify such fibered knots.
For example the $(2, 3)$-cable of the $(2, 3)$-torus knot has a unique transverse representative with $sl \neq 3$ (note must be ≤ 7) and exactly two representatives with $sl = 3$.

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Work In Progress [E]

If M is an atoroidal oriented 3-manifold and ξ is an overtwisted contact structure then there are many fibered knot types K such that the transverse knots in this knot type (up to contactomorphism) with $sl = -\chi(K)$ are in one to one correspondence with $\mathbb{Z} \cup \{p\}$.
The knot corresponding to p has overtwisted complement and the rest have tight complement.
(Only one has non-trivial Heegaard-Floer invariant by a result of Vela-Vick).

The Proof

- It is easy to see that if a link L is the binding of an open book supporting ξ then it is naturally a transverse knot with $sl = -\chi(L)$.
- So we prove the other direction. Suppose that L is a fibered, transverse link in (M, ξ) with $sl(L) = -\chi(L)$.
- Let Σ be a fiber of the fibration of $M \setminus L$ (used to compute sl).
- Now a Heegaard splitting of M is given by

$$M = V_0 \cup V_1$$

where $V_0 = \Sigma \times [0, 1]$ is a neighborhood of Σ and $V_1 = \Sigma \times [1, 2]$ is the closure of $M \setminus V_0$.
- We focus on $S = \partial V_0$.

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- It is easy to see that if a link L is the binding of an open book supporting ξ then it is naturally a transverse knot with $sl = -\chi(L)$.
- So we prove the other direction. Suppose that L is a fibered, transverse link in (M, ξ) with $sl(L) = -\chi(L)$.
- Let Σ be a fiber of the fibration of $M \setminus L$ (used to compute sl).
- Now a Heegaard splitting of M is given by

$$M = V_0 \cup V_1$$

  where $V_0 = \Sigma \times [0, 1]$ is a neighborhood of Σ and $V_1 = \Sigma \times [1, 2]$ is the closure of $M \setminus V_0$.
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Convex surfaces

- A surface S is called convex if there is a vector field v transverse to it such that its flow preserves ξ.
- The dividing set of a convex surface is

$$\Gamma = \{p \in S : v_p \in \xi_p\}.$$

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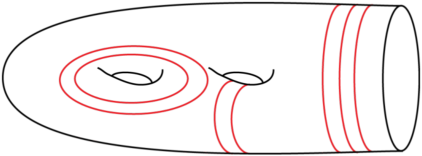

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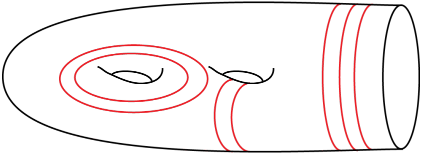

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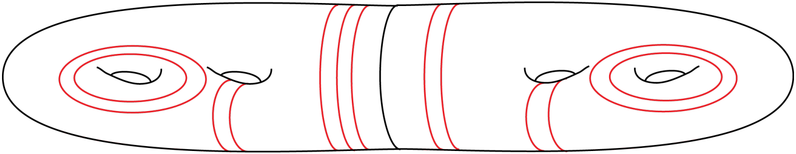

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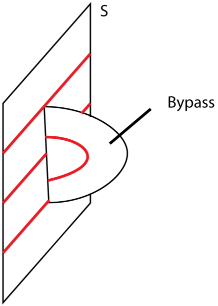

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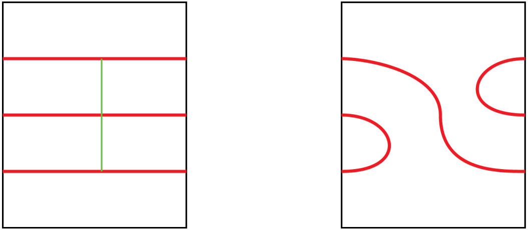

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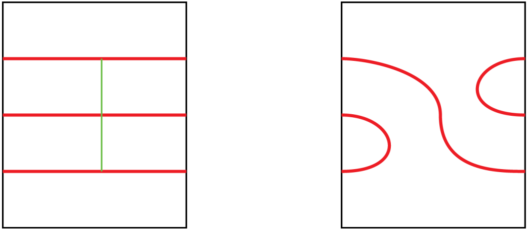

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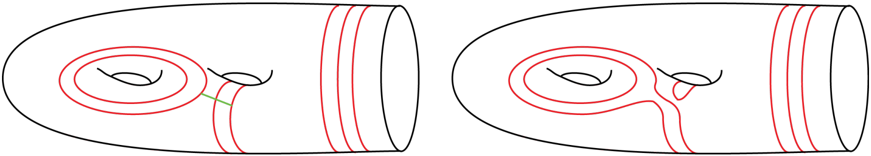
- With work we can use these bypasses to reduce the number of dividing curves on Σ. For, a very easy, example



The End

Thank You!